INTERPRETING STUDENTS' REASONING THROUGH THE LENS OF TWO DIFFERENT LANGUAGES OF DESCRIPTION: INTEGRATION OR JUXTAPOSITION?

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This contribution exemplifies the interpretation of a common set of data by using two languages of description originating from different theoretical perspectives. One account uses categories from a psychological and the other from a sociological perspective. The interpretations result in different explanations for the students' struggles with sense making. However, the results cannot be integrated into a combined insight, but only be juxtaposed.

INTRODUCTION

The role of theory in mathematics education research has many facets so that comparisons of outcomes of research carried out within different perspectives remain a challenging and complex task (Silver & Herbst, 2007; Radford, 2008). The observed diversity of theories, paradigms, and frameworks in the field has called for serious efforts of understanding, comparing, contrasting, coordinating, combining, synthesising, or integrating different perspectives (Prediger, Bikner-Ahsbahs, & Arzarello, 2008). In line with this work, this paper, by way of an example, sets out the task to construct two *accounts of* a transcript from a video taped problem solving session for the purpose of comparing and contrasting different *accounts for* it (Mason, 2002), based on two languages of description stemming from two different theoretical traditions. In the session pairs of students were working on tasks on limits of functions, a topic where most of the research about students' sense making has been done from a cognitive psychology approach (Artigue, Batanero, & Kent, 2007). For an alternative account, we have chosen a sociological approach, which is rather uncommon but has the potential of overcoming deficit orientated interpretations of students' struggles.

Much of the research that aims at accounting for the problems students have, focuses on a distinction between "intuitive" and "formal conceptions" of limits (e.g. Harel and Trgalova, 1996, pp. 682-686). The notion of limits of functions is conceived as one where intuitive conceptions of infinity may prove insufficient or even contradictory to a formal mathematical treatment (Núñez et al, 1999). As an exemplary of approaches that account for students' problems with limits of functions in terms of the individual's cognition, we produce an account of the data that draws on the work of Alcock and Simpson (2004, 2005). Their conceptualisation describes an interplay between modes of representations and beliefs about oneself and the role of algebra in reasoning about limits.

Starting from a sociological perspective, in a second attempt, we outline an account of the students' productions in terms of the dilemma they face when participating in different types of discourses. This interpretation draws on a language of description developed in the context of studies of recontextualisation that represent a structuralist tradition (Bernstein, 1996). In drawing on Bernsteins's theory, a successful student can be described as being able to realize in which context she participates and produces what is expected in this context, that is, the student must have access to "recognition rules" and "realisation rules" in order to produce "legitimate text". The ultimate agenda of such an approach is to explain how the students' access to these rules is distributed unevenly with respect to their different backgrounds. For our account of the empirical text from the problem solving sessions, we use categories of expression and content of mathematical problems from the perspective of recontextualisation of different types of discourses about limits of functions.

THE INTERVIEW SITUATION

Six beginning engineering students from a first semester calculus course volunteered to participate in the video study, where they were working in pairs to solve problems on limits of functions. Each session lasted for about 45 minutes. After an introductory question about the concept of a limit and its definition, the students were asked to investigate the limits of functions. The type of problems chosen were similar to the ones they encountered in the course: to find the limits as $x \to \infty$ and as $x \to 0$ for the three functions $f(x) = \frac{2x}{x^2 + \sin x}$, $g(x) = \frac{1}{x} - \frac{1}{x^2}$, and $h(x) = \frac{\ln(1+x^2)}{x}$.

For our accounts presented below, we used the transcribed protocol from the work of two pairs (A and B) of students on the function $h(x)$ and on the introductory question.

At the time of the interview the lectures had covered the definitions and basic properties of limits and continuity, and introduced and proved theorems about standard limits such as $\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$, as well as worked examples. The textbook provided an exposition of an introductory calculus course based on the standard $\varepsilon - \delta$ definition of limits and continuity. In particular, standard limits were proved within this theory and used as theoretical tools to investigate the limits of functions given in algebraic form. Other techniques taught include removing dominating factors, extension by the conjugate expression, and change of variable. The approach was algebraic and non-numerical. Occasionally, diagrams were used. The teacher of the course sets out his agenda as follows (see Bergsten, 2007, p. 63):

> I want to present, to make things seem true, the most important I think is that students believe they understand better what a concept means. To exemplify what you can handle practically, to illustrate the standard way of doing things.

In the lecture the teacher made some efforts to integrate formal algebraic treatment with non-formal ideas about limits and behaviour of elementary functions (ibid.).

ACCOUNT 1: INDIVIDUALS' BELIEFS AND PREFERENCES

The style of work of Anne and Adam is dominated by algebraic manipulations across all tasks, where the observed notations are used mainly as keys for performing procedures that hopefully will lead to a possibility to apply a standard limit. This is done immediately when starting a new task, without prior discussion about how to attack the problem or what can be "seen" by considering properties of the functions involved. In the transcript, when discussing the case where x tends to infinity for the function $h(x)$, Anne immediately suggests making a change of variables:

Anne: Change of variables.

Adam: ...yes ... I think you get ... the logarithm can be rewritten, the function inside.

Anne: No, we can't touch the function inside [writes, Adam looks at her seemingly puzzled] there is no expression for LN X plus LN Y equal to LN X plus Y.

Adam: Yes yes but you can write it as LN one plus X ... that part [points] one plus X square can be written as ... one plus ... one minus X.

Anne: Yes, equal to LN [inaudible, Anne writes].

Adam: It does not help much in this case.

Anne: No [erases what she wrote].

While solving this task no diagram is drawn or point made on properties of the functions involved that could lead the process forward. Standard limits and comparison tables are recalled as incitements and as clues to continued algebraic manipulation. Uncertainty in recalling these facts correctly does not prevent them from proceeding the algebraic explorations, possibly thinking it will eventually lead to a result:

Anne: I must elaborate further on that one and see if it works.

The work goes on along the same lines in all tasks, trying to remember what one can do and trying out different algebraic methods, sometimes ending up in what could be called an algebraic mess, using expressions like "this is just impossible". In the following excerpt the students substitute $1+x^2$ by t.

Adam: If we in the original expression extend with ... the square root of minus one ... T minus one in the denominator, LN T the square root of T minus one ... that one was not much better [looks at Anne].

Anne: [writing] This is also unnecessary because we can't do this, it is the same shit ... doesn't matter ... than we have that this one moves this one moves and then this one moves.

Adam: Yes all tend to infinity.

Anne: To be honest, I think that infinity is the answer, as ... when I changed variables.

The last sentence indicates a weak "internal authority", as she cannot find a method that works, and on another occasion (on problem f) this is directly expressed:

Anne: The question is if it is correct. Now I just want to know the right answer.

Interviewer: You don't feel confident with the result?

Adam: I can't say it should be another result, but this is a kind of task where I feel I could easily make a mistake.

Anne: Yes, me too.

Adam: By some change of variable it can be possible to make it tend to zero. /…/

Anne: I think it is zero in both cases. What was the answer?

This predominantly algebraic way of working seems to be in contrast to the response to the opening question on the meaning of a limit, where they initially describe it verbally as a dynamic process using words like "approaching" but then prefer to make a drawing and add gestures when talking about it. However, as these images do not seem to have a link to their subsequent work on the problems they may lack a sufficient generality to justify their reasoning (cf. Alcock & Simpson, 2004).

Also the students in pair B describe the mathematical notion of limit as a dynamic process of 'approaching' but seem to accept both a potential and actual infinity, as when they discuss the arrows commonly used to denote limits:

Bob: Yes I would maybe miss a little arrow ...

Ben: Yes.

Bob: ... in front of A [i.e. the limit], tends to A, but I don't know if ...

Ben: it gets so very close, yes goes to A.

Bob: Yes, you usually don't have those arrows like that. But the function attains the value A when X is infinitely large, is a very very large number, don't know if I need to add more.

Interviewer: Do you agree?

Ben: Yes.

They also state that it is more easy to explain when using a diagram. However, their diagram is more elaborated and seems to support their thinking during the work with the problems. For pair B this work proceeds in quite a different manner from pair A, dominated by more informal reasoning about the size of the quantities of the different parts of the given functions. They frequently use the expressions "a very small number" and "a very large number". In 'simple' cases this way of reasoning is functional but in the case $\lim_{x\to\infty} h(x)$, this kind of intuitive method proves insufficient to find the limit even after 15 minutes of work:

Bob: Zero times infinity is ok, almost zero times infinity is more tricky, it is not really zero but only tends to it. So it can be almost anything. Do we get anywhere? [looking at Ben]

Ben: No [Bob laughing].

Bob: Yes, but which one goes more, does that one go more to zero than that one to infinity? No it goes more to infinity than to zero, I think. [silence]

It seems as if algebraic methods, shown in the lectures, here are tried only when the conceptual approach does not produce an answer. However, when it does these students do not feel any need to verify the solution formally by the use of proven theorems on standard limits. They rely on "internal authority".

Internal authority is also evident by the use of the words "I think we are done" in the case $\lim_{x\to 0} h(x)$, after identifying a standard limit and applying it after expanding the term by x. But again no algebraic manipulations are performed on $\lim_{x\to 0} g(x)$, where they reason about approaching zero from the right or from the left. They conclude, after testing a numerical value, drawing a diagram and comparing infinities, that $g(x)$ tends to negative infinity. However, Bob is not fully satisfied:

Ben: So this [i.e. when approaching zero from the right] must also be negative infinity, don't you think so?

Bob: Yes, but it is kind of delicate when you take infinity minus infinity, it is kind of vague. But if we accept this way of reasoning with infinities of different size, then we have found that, if it is correct.

Thus, relying on internal authority might have prompted questioning the bases of their arguments and imply an uncertainty about the correctness of the result.

ACCOUNT 2: WEAKLY / STRONGLY INSTITUTIONALISED DISCOURSE

For the purpose of analyzing the recontextualisation of domestic practices in school mathematics texts, Dowling (2007) introduces a "relational space" of domains of action that differentiates between content and expression of a text, both being weakly or strongly institutionalised (see Table 1). Esoteric domain text refers to the conventional institutionalised mathematical language and its strongly classified specific meanings. In descriptive domain text, the expression is conventional mathematical language though its object of reference is not institutionalised mathematics. In expressive domain text, a mathematical concept or procedure etc. is expressed via signifiers that are not or weakly institutionalised (in an extreme case via non-mathematical signifiers). Public domain text is text with both weakly institutionalised forms of expressions and content.

The following interpretation employs these notions. As the context is a university lecture in calculus, public domain text cannot be expected to be found. The oral discourse in the lecture analysed in Bergsten (2007) included metaphorical language

	Content (signifieds)	
Expression (signifiers)	strong institutionalisation	weak institutionalisation
strong institutionalisation	*esoteric domain*	*descriptive domain*
weak institutionalisation	*expressive domain*	*public domain*

Table 1: Domains of Action (Dowling, 2007, p. 5; layout adjusted)

and gestures describing graphs of functions in terms of motion and direction as well as hints about what to do when applying standard procedures.

The written discourse focused on algebraic representations. The topics were presented with very detailed formalisations, very much in line with the textbook (co-authored by the lecturer), that is, as esoteric domain text drawing on strongly classified and institutionalised language and meanings. So it is the oral discourse that is situated in another domain, a domain of visuo-spatial and movement metaphors that are used for describing the Cartesian graphs, "the behaviour", of functions and their limits (in terms of shape, growth, getting bigger and smaller and approaching). The meanings in this discourse are weakly classified, as are the modes of expressions. In the course of establishing the esoteric discourse, this discourse is re-contextualised from the perspective of an algebra of functions and their limits, and in doing so the first is subordinated to the latter. The students attempts to solve the tasks in the interview situation can be interpreted as a struggle to produce a legitimate text, that is an esoteric text. However, if they discussed with their peers and approached the solutions in terms of the weakly classified oral discourse, they were faced with a problem of recontextualisation. However, in the introductory question of the interview, they were asked to explain the concept of limit, which is a quite different challenge. The interviewer shows to the students a piece of technical language from the course: "$\lim_{x \to \infty} f(x) = A$" and asks:

Interviewer: Imagine you have a friend who just started such a course in calculus and has never seen this. How would you explain to him what this means?

The students are faced with the problem to recognize what a legitimate text in this interview situation would be. Into which domain has the expression to be translated for this imaginary friend?

Anne and Adam interpret this question as a task to produce expressive domain text. They first have to establish this new domain and start negotiating the translation and eventually agree that this new domain includes drawings of examples of functions. The technical language comprises "x", "function", "A" (which remains untranslated), "LN-function". The expression "$\lim_{x \to \infty} f(x)$" is translated into "the limit".

Anne: This is an expression for the limit. One looks at how a function behaves when X tends to infinity…and when X tends to infinity and the function approaches a constant which is called capital A, so it is convergent, as one calls it. This means that one can say the function then approaches a value if it does not go on …

Adam: It approaches a finite value then, so it is bounded, a bounded function.

Anne: This is a little hard without drawing it.

Adam: Yes, this is hard to explain, it is more easily explained with a figure, I think.

After two comments of Adam who talks about the value going "closer and closer", the interviewer interferes by asking them whether they would want "to draw a figure for that friend":

Anne: I think the friend should get a clearer picture in any case [Adam draws quietly, Anne watches] … yes [approves the figure and holds up the paper to the friend and smiles].

Adam: This is a function that approaches but never really reaches [illustrates with a gesture].

Anne: A bit like LN one can say

Adam: Yes, LN-function.

Anne: This looks like an LN [both laugh].

In their conversation while solving the tasks $\lim_{x\to 0} h(x)$ and $\lim_{x\to\infty} h(x)$, they focus on associating it with a standard limit they have encountered in the lecture. They eventually solve the version for x approaching zero by expanding the expression by x and substituting $x^2 = t$, that is, by producing esoteric domain text. However, they do not explicitly refer to the "multiplication rule" for limits from the lecture to justify their conclusion. They are not successful in their attempt to solve the second part of the task. As they adhere to a strategy to formalize their informal approaches and employ some methods suggested in the lecture, this can be seen as a production of descriptive domain text, which in parts, switches into the esoteric domain when they are trying out different algebraic transformations. Anne several times refers to "writing" it down properly, which indicates that she recognizes what type of text they are usually supposed to produce. The episode, in which the pair tries to solve the second part of the question, ends with a remark about the criteria for producing legitimate text:

Anne: Now you have made a writing mistake. You have to write X, or T, T goes towards infinity…They will like that at the mathematics department … also when we are very detailed.

The second pair also takes the interviewer's question as a prompt to produce expressive domain text by describing the meaning in terms of the weakly institutionalised oral discourse. Bob refers to the limit as "the value A when X is infinitely large, is a very very large number, don't know if I need to add more" and talks about "the little arrow" (see the transcript from the first account). After another prompt of the interviewer, they expand their explanation:

Ben: Yeah, the function value A as X tends to infinity, or? [silence ...] Then we have drawn [moving his hand as if he is drawing], have we not? [glancing at Bob]

Bob: Yes, it gets like that, x tends to infinity, it is very simple if you make a sketch [raising his hand with the pencil but does not draw, making drawing gestures while talking]. If we have A at a certain part of the y-axis we can say, we get such a horizontal line. The function starts at zero maybe and then goes up, kind of approaching A all the time, getting thinner, the bigger the x-value the closer you get ... and ... I don't know if I should bring that in too, you can always get closer than you already are, that is this thing with limits. That is the whole point, as in this case it will finally be as close as … you can't say as close as you can because you can always get closer but ...

They solve the task $\lim_{x \to 0} h(x)$ by reducing it to a standard limit, talking about substituting x^2 and decide about a solution. However, Ben seems unsure about the status of the solution produced by Bob (who does not refer to "multiplication rule"):

Bob: And this her goes towards zero, that X goes towards zero. One times zero.

Ben: One times a very small number next to zero.

Bob: This is what I also would like to say, indeed one times zero becomes zero.

Ben: I think we are clear with this one.

The last remark indicates that they do not adhere to the criteria for legitimate text established in the lecture. In the course of the solution of the second task, they remain in oral discourse and use visuo-spatial and movement metaphors for describing the shapes of standard functions and the "limit" as "approaching and coming closer". However, they are not successful in re-contextualising this discourse from the perspective the formal algebraic discourse. However, as the other pair, they seem to know the criteria for legitimate text, as Bob says at one occasion: "You can't do it like this mathematically /…/ It can be done, there is a method".

None of the pairs interpreted the first question of the interviewer as an invitation to establish the meaning for a novice by introducing her into the technical language and its institutionalized meanings, that is, to come up with a definition. Both pairs seem to realize that the legitimate text for successful participation in the course is located in the esoteric domain.

DISCUSSION

One goal of this exercise has been to see whether both interpretations can in combination produce useful insights about the students' reasoning about limits in the context of a university calculus course.

The first interpretation pictures those students showing an external sense of authority as the ones who tend to use the mathematical notations as keys to apply algebraic procedures. A conclusion could be that they lack an "intuitive feeling" for the mathematical objects involved, which should form the basis for using algebraic techniques. The second pair is pictured as showing an internal sense of authority and a preference for an "intuitive" approach. They often "know" by informal reasoning what the limit is and occasionally express a need to use algebraic representations. A conclusion could be that they lack an ability to use algebraic representations to formalise their reasoning. As the first approach focuses on the individuals' cognition it does not include the relation of their preferences to the context, in which these arose, as a specific research question.

The second interpretation shows that both pairs were, for different reasons, not able to produce solutions that would satisfy the criteria for legitimate text established in the lectures. The first pair did not have full access to the technical language and its institutionalized meanings, which they tried to employ, the second did not recontextualise their own productions from a formal algebraic perspective. This account draws attention to the structural complexities that relate to the ways in which the recontextualisation by means of formal algebra of the oral discourse about functions and limits employed in the lecture operates. It includes the establishment of a link of the students' productions to the discourse, in which they participate, as a *paradigmatic research question* (Radford, 2008) by conceptualising it in terms of their possession of recognition and realisation rules for producing legitimate text.

The two approaches also differ in terms of the methodology. While within the first framework the interview situation is a method for gaining insights into the students' beliefs and preferences, the second interpretation takes into account that the conversation during the problem solving sessions can also be conceived as a situation, in which the students are faced with the challenge of producing legitimate text. However, the students can neither have recognition nor reproduction rules for such a situation because it is the first time they participate in a study like this. They seem to have interpreted the interview situation differently, as more (Anne and Adam) or less (Bob and Ben) identical with the context of the course they were attending and thus more or less identifying the researcher with the official side of the university course. This interpretation would account for the fact that the second pair did not spend so much effort to translate their versions into a formal algebra as the first one and that they were mostly convinced that their solutions are reasonable, perhaps because of recognizing the context as informal. The first pair, in contrast to their following productions, engaged in weakly institutionalized discourse only as a response to the introductory question, perhaps recognizing the story about the imaginary friend as not belonging to the esoteric domain. In contrast, the first interpretation takes the students' explanations that follow the introductory question as an indication of their understanding of the concept of limit, or alternatively as an indicator of whether they know a definition in formal algebraic terms.

From the second perspective, "understanding" can be framed as having access to both of the discourses identified, as well as to the principle by which the oral discourse can be recontextualised from the perspective of the written one. The "intuitive" approach is only represented in the oral discourse. Both interpretations suggest a tension between these discourses that cannot easily be resolved.

It remains a highly questionable undertaking to look for combined insights stemming from interpretations that use languages of description, which stem from different theoretical traditions, particularly if issues of validity are at stake (cf. Gellert, 2008). The two interpretations presented here illustrate their points by selecting different episodes from the transcript. Considering that the research situation is re-interpreted in the second account (and thus taking the interviewer's questions as a piece of data), one could say that the two accounts are not interpretations of the same "data". In addition, different background information about the course has been used.

The outcomes of this interpretational exercise do not result in conflicting readings of the data. However, the results cannot be integrated into a combined insight, but only be juxtaposed. This is because the basic principles of the theories from which the approaches originate have established two different "universes of discourse" (Radford, 2008) in which the paradigmatic research questions are formulated.

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