THE DESIGN OF NEW DIGITAL ARTEFACTS AS KEY FACTOR TO INNOVATE THE TEACHING AND LEARNING OF ALGEBRA: THE CASE OF ALNUSET

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The integration of CAS systems into school practices of algebra is marginal. To integrate effectively digital technology in the teaching and learning of algebra, it is necessary to go beyond the experience of CAS and of their instrumented techniques and to face the design of new artefacts. In this paper we discuss design problems faced in the development of a new digital artefact for teaching and learning of algebra, the Alnuset system. We present the key ideas that have oriented its design and the choices we have worked out to instrument its incorporated algebraic techniques. We compare the quantitative, symbolic and functional instrumented techniques of Alnuset with those of CAS highlighting crucial differences in the teaching and learning of algebra.

Keywords: *Alnuset, Instrumented technique, CAS, Algebraic learning*

INTRODUCTION

In the last 15 years a scientific debate on the role of technology in supporting teaching and learning processes in the domain of algebra has been going on. This debate originates from research studies carried out in different countries with the purpose of studying the use of Computer Algebra Systems (CAS) in school contexts. In particular, near benefits (Heid, 1988, Kaput, 1996, Thomas, Monaghan and Pierce, 2004) obstacles and difficulties have been identified in using this technology by students and teachers (Mayes, 1997, Drijvers, 2000, Drijvers, 2002, Guin & Trouche, 1999). Results of these research works (Artigue, 2005) highlight that the integration of CAS systems into the school practice of algebra remains marginal due to different reasons. CAS expands the range of possible task-solving actions. As a matter of fact, techniques involved in a CAS (instrumented techniques) are in general different from those of the paper and pencil environment. Managing the complexity of CAS instrumented techniques and highlighting the potential offered by the machine to the student is hard work. As shown by some experiments (Artigue, 2005), CAS use may cause an explosion of techniques which remain in a relatively simply-crafted state. Moreover, any technique that goes beyond a simple, mechanically learnt gesture, should be accompanied by a theoretical discourse. For the paper and pencil techniques this discourse is known and can be found in textbooks. For instrumented techniques it has to be built and its elaboration raises new, specific difficulties. Even if the use of CAS seems fully legitimate in the class, in general, instrumented techniques cannot be institutionalised in the same way as paper and pencil ones (Artigue, 2005).

THE RATIONALE

To frame the results carried out by these research studies and the complexity of the processes involved in the educational use of CAS, some French researchers (Lagrange 2000, Artigue, 2002, Lagrange, Artigue, Guin and Trouche, 2003) have elaborated a theoretical framework, named 'instrumental approach', integrating both the ergonomic theory (Rabardel, 1995) and the anthropological theory (Chevallard, 1992). The 'instrumental approach' provides a frame for analyzing the processes of instrumental genesis both in their personal and institutional dimensions, and the effect of instrumentation issues on the integration of CAS in the educational practice. Using this framework, Artigue observes that CAS are extremely effective from a pragmatic standpoint and for this reason professionals (mathematicians, engineers..) are willing to spend time to master them (Artigue, 2002). At pragmatic level the effectiveness often comes with the difficulty to justify, at a theoretical level, the instrumented techniques used. In particular, this is true for users who do not fully master mathematical knowledge and techniques involved in the solution of the task. As a consequence, the epistemic value of the instrumented technique can remain hidden. This can constitute a problem for the educational context where technology should help not only to yield results but also to support and promote mathematical learning and understanding. In educational practice, techniques should have an epistemic value contributing to the understanding of objects involved. "*Making technology legitimate and mathematically useful from an educational point of view, whatever be the technology at stake, requires modes of integration that provide a reasonable balance between the pragmatic and the epistemic values of instrumented techniques*" (Artigue, 2007, p. 73). These results might account for the marginalization of CAS integration into the school algebraic practices. For some researchers, to integrate digital technology effectively in the domain of algebra, it is necessary to go beyond the experience of CAS and of their instrumented techniques and to face the design of new artefacts. As underlined by Monaghan (2007) up to now CAS-in-education workers have paid little attention to design issues, preferring, in general, to work with the design supplied by CAS designers (Monaghan, 2007). Moreover, it should be noted that no comparison between the design of CAS and of technological tools for education has been developed so far. This article aims at pointing out design issues that can effectively support teaching and learning processes in algebra. This goal will be pursued considering the design of ALNUSET (ALgebra on the NUmerical SETs), a system developed to improve teaching and learning of crucial topics involved in the mathematical curricula such as algebra, functions and properties of numerical sets. In particular, in this article we compare design aspects of Alnuset and of CAS and we highlight the relevance of differences in their instrumented techniques for the teaching and learning of algebra.

PROBLEMS OF DESIGN IN DEVELOPING NEW DIGITAL ARTEFACTS

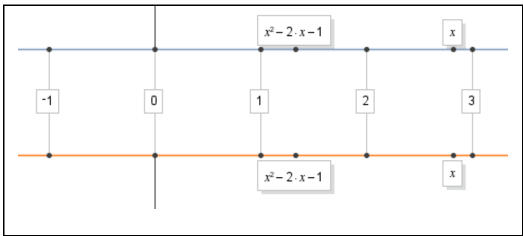
Going beyond the design of CAS requires new creative ideas to instrument techniques for mathematical activity different from those of CAS. The advent of both the dynamic geometrical artefacts and of spreadsheets has evidenced that even a single creative idea can determine a new typology of innovative artefacts. This can occur when new creative ideas allow to instrument mathematical techniques characterizing them with new operative and representative dimensions such as the drag of the variable point of a geometrical construction, as in the case of dynamic geometrical software, or the automatic re-computation of formulas of the table, as in the case of spreadsheet. Moreover, when a technique must be instrumented on the basis of an idea, various types of design problems emerge. They regard the way tasks and responsibilities have to be distributed between user and computer and the management of the interactivity, namely the operative modalities of the input by the user, the representation of the result by the computer (output), the visualisation of specific feedback to support the user action or to accompany the presentation of the result. Moreover, problems of design regard also the way in which the instrumented techniques have to be connected between each other. The way these problems are solved affects the accessibility of techniques, their usefulness for the task to be solved, the meaning that the instrumented technique evidences in the interaction, the discourse that can be developed about it. Hence, the way these problems are solved affects the balance between pragmatic and epistemic values of instrumented techniques within the didactical practice and this can affect mathematics teaching and learning. The anthropological framework is the theoretical tool used to analyse the way in which techniques are implemented and their effectiveness on the educational level. Ideas are evaluated on the base of this framework. We discuss these general assumptions in the domain of algebra referring to Alnuset System.

ALNUSET: IDEAS AND CHOICE OF DESIGN

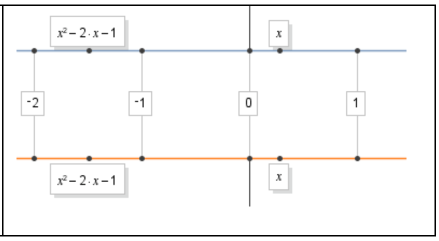
ALNUSET is a system designed, implemented and experimented within the ReMath (IST - 4 - 26751) EC project that can be used to improve the teaching and learning of algebra at lower and upper secondary school level. The design of ALNUSET is based on some ideas that have oriented the realisation of the three, strictly integrated components: the Algebraic Line component, the Algebraic Manipulator component, and the Function component. These three components make available respectively techniques of quantitative, symbolic and functional nature to support teachers and students in developing algebraic objects, processes and relations involved in the algebraic activity. In the following we present the main ideas that have oriented the realisation of the three components of Alnuset and illustrate the choices and decisions taken to instrument algebraic techniques so that an appropriate balance between their epistemic and pragmatic values can emerge when used in the educational practice.

Algebraic line component

The main idea in the design of the Algebraic line component is the representation of algebraic variables on the number line through mobile points associated to letters, namely points that can be dragged on the line with the mouse. In this component the user can edit expressions to operate with. The computer automatically computes the value of the expression on the basis of the value of the variable on the line and it places a point associated to the expression on the algebraic line. When the user drags the mobile point of a variable, the computer refreshes the positions of the points corresponding to the expressions containing such a variable in an automatic and dynamic manner. This is possible only thanks to the digital technology that allows to transform the traditional number line into an algebraic line. The following two figures report the representation of a variable and of an algebraic expression on the lines of this component. Note that the presence of two lines is motivated by operative necessities regarding the use of the algebraic editor based on geometrical models that is available in this component. This editor is not considered in this report.

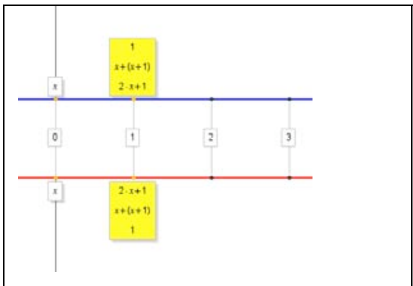
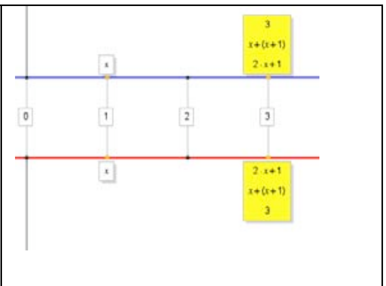


The drag of the variable x determines the dynamic movement of the expression containing it

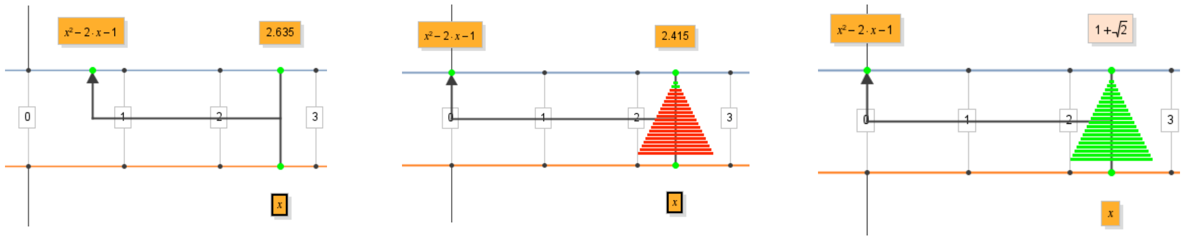


Through its visual feedback, this technique can be used either to explore what an expression indicates in an indeterminate way or to compare expressions. The design of this component is associated to every point represented on the line by a post-it. The computer automatically manages the relation among expressions, their associated points and post-it. The post-it of a point contains all the expressions constructed by the user that denote that point. By dragging a variable on the line, dynamic representative events can occur in a post-it. They might be very important for the development of a discourse concerning the notions of equality and equivalence between expressions. As a matter of the fact, the presence of two expressions in a post-it may mean:

- A relationship of equality, if taking place at least for one value of the variable during its drag along the line
- A relationship of equivalence, if taking place for all the values assumed by the variable when it is dragged along the line.
- A relationship of equivalence with restrictions, if taking place for every value of the variable when it is dragged along the line, but for one or more values, for which one of the two expressions disappears from the post-it and from the line.

	The expressions x+(x+1) and 2*x+1 are equivalent, because they refer to the same point on the Algebraic line and they are contained in the same post-it whatever the value of the x variable is during the drag.	

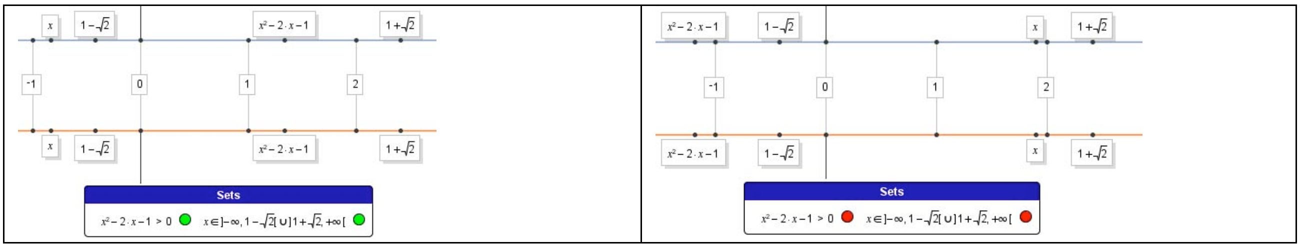
Moreover, the algebraic line component has been designed to provide two very important instrumented techniques for the algebraic activity, i.e. for finding the roots of polynomial with integer coefficients and for identifying and validating the truth set of algebraic propositions. The root of a polynomial can be found dragging the variable on the algebraic line in order to approximate the value of the polynomial to 0. When this happens, the exact root of the polynomial is determined by a specific algorithm of the program and it is represented as a point on the line.



This technique, that can be controlled by the user through his visual and spatial experience, is effective not only at a pragmatic level but also at an epistemic level, because it can concretely support the development of a discourse on the notion of root of a polynomial, as value of the variable that makes the polynomial equal to 0. The truth set of a proposition can be found through the use of a specific graphical editor. Let us consider the inequation x^2-2x-1>0, that once edited, is visualised in a specific window of this component named “Sets”. Once the root of the polynomial associated to the inequation has been represented on the line, a graphic editor can be used to construct its truth set (see the figure).

Two open intervals on the line, respectively on the right and on the left side of the roots of the polynomial x^2-2x-1, have been selected with the mouse. The system has translated the performed selection into the formal language.

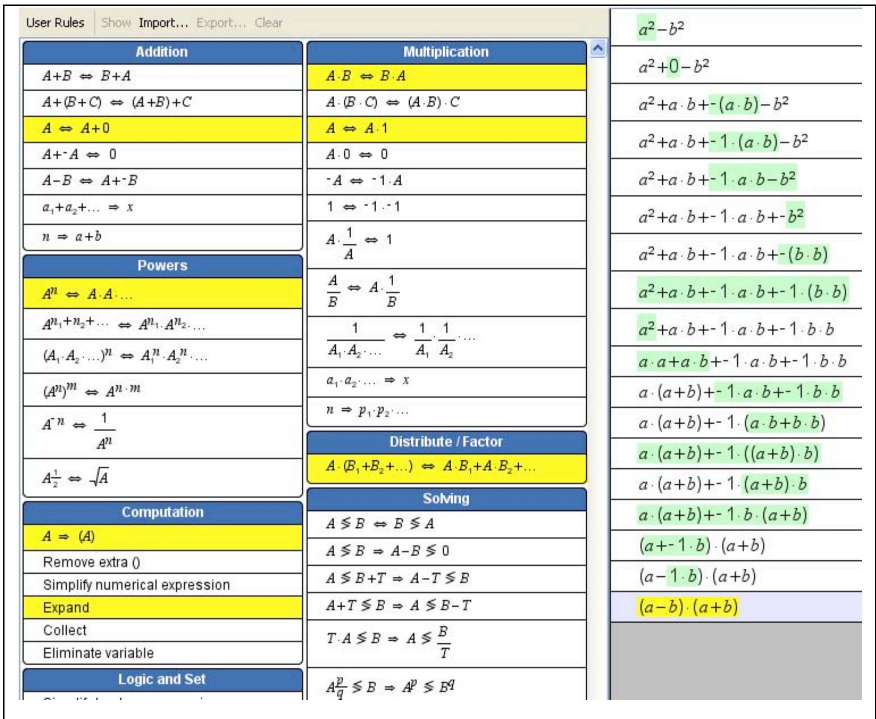
Once the truth set of a proposition has been edited, it can be validated using a specific feedback of the system. In the set window propositions and numerical sets are associated to coloured (green/red) markers that are under the control of the system. The green/(red) colour for the proposition means that it is true/(false) while the green/(red) colour for the numerical set means that the actual variable value on the line is/(is not) an element of the set. Through the drag of the variable on the line, colour accordance between proposition marker and set marker allows the user to validate the defined numerical set as truth set of the proposition (see figure below). The validation process is supported by the accordance of colour between the two markers and by the quantitative feedback provided by the position of variable and of the polynomial on the algebraic line during the drag.



This feedback offered by the system during the drag of the variable is important to introduce the notions of truth value and of truth set of an algebraic proposition and to develop a discourse on their relationships. All the described instrumented techniques that are specific of the Algebraic line component make a quantitative and dynamic algebra possible.

Algebraic manipulator component

The interface of this component has been divided into two distinct spaces: a space where symbolic manipulation rules are reported and a space where symbolic transformation is realised.



This figure shows a part of the commands available with the interface and an example of algebraic transformation.

The figure shows a characteristic of the interactivity of this manipulator: the selection of a part of an expression determines the activation of the commands of the interface that can be applied on it. This characteristic can help students to explore the systems of rule for the algebraic transformation and the effects they produce

The main idea characterizing the design of the Algebraic Manipulator component is the possibility to exploit pattern matching procedures of computer science to transform algebraic expressions and propositions through a structured set of basic rules that are deeply different from those of the CAS. In CAS pattern matching procedures are exploited according to a pragmatic perspective oriented to produce a result of symbolic transformation that could be also very complex, as in the case of command like factor or solve. As a consequence, the techniques of transformation can be obscure for a not expert user. In the Algebraic Manipulator component of Alnuset pattern matching procedures have been exploited according to three specific pedagogical necessities. The first necessity is to highlight the epistemic value of algebraic transformation as formal proof of the equivalence among algebraic forms. To this aim we have designed this manipulator with a set of basic rules that correspond to the basic properties of addition, multiplication and power operations, to the equality and inequality properties between algebraic expressions, to basic logic operations among propositions and among sets. Every rule produces the simple result of transformation that is reported on the icon of its corresponding command on the interface, and this makes the control of the rule and the result easy to control. Moreover a fundamental function of this component allows the student to create a new transformation rule (user rule) once this rule has been proved using the rules of transformation available on the interface. For example, once the rule of the remarkable product a^2-b^2= (a+b)*(a-b) has been proved, it can be added as new user rule in the interface a^2-$b^2$$\leftarrow\rightarrow$(a+b)*(a-b) and it can successively be used to transform other expressions or part of them whose form match with it. Moreover, a specific command allows to represent every transformed expression on the algebraic line automatically. Through this command it is possible to verify quantitatively the preservation of the equivalence through the transformation, observing that all the transformed expressions belong to the same post-it when their variables are dragged along the line. These characteristics of the algebraic manipulator of Alnuset can have a great epistemic importance because they can be effectively exploited to support the comprehension of the algebraic manipulation in terms of formal proof of the equivalence between two algebraic forms. The second necessity is to support the integration of practice of quantitative and manipulative nature. In this manipulator three rules allow the user to import the root of a polynomial, the truth set of a proposition and the value assumed by a variable on the algebraic line from the Algebraic line component to be used in the algebraic transformation. For example the rule "Factorize" uses the root of polynomial found in the Algebraic Line to factorize it. The way in which this rule works, makes the factorization technique of Alnuset different from that of CAS. In CAS this technique is totally under the control of the system, and the result can appear rather obscure for not expert users. In Alnuset, the factorization can be applied on the polynomial at hand only if its roots have been previously determined on the algebraic line. In Alnuset the distribution of tasks between user and computer and the way they interact, can contribute to understand the link between the factorization of a polynomial and its roots. The third necessity is to offer more powerful rules of transformation when needed for the activity and when specific meaning of algebraic manipulation have been already constructed. Two specific rules, also present in the CAS are available in this manipulator. They determine the result of a numerical expression and the result of a computation with polynomials respectively. These rules of transformation contribute to increase the pragmatic value of the instrumented technique of algebraic transformation in Alnuset and they can be used to introduce to the use of CAS

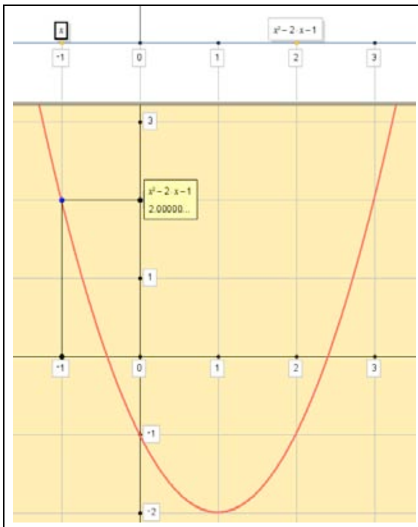
Moreover, the technique of algebraic transformation has been instrumented in this manipulator to provide non expert users with cognitive supports in the development of specific manipulative skills. A first support is the possibility to explore, through the mouse, the hierarchical structure that characterises the expression or the proposition to be manipulated. By dragging the mouse pointer over the elements of the expression or proposition at hand (operators, number, letters, brackets…), as feedback the system dynamically displays the meaningful part of the selected expression or proposition. In this way it is possible to explore all meaningful parts of an expression in the different levels of its hierarchical structure. Another feedback occurs when a part of expression has been selected. Through a pattern matching technique, the system, as feedback, activates only the rule of the interface that can be applied on the selected part of expression. This is a cognitive support that can be used to explore the connection among the transformational rules of the interface, the form on which it can be applied, and the effects provided by their applications.

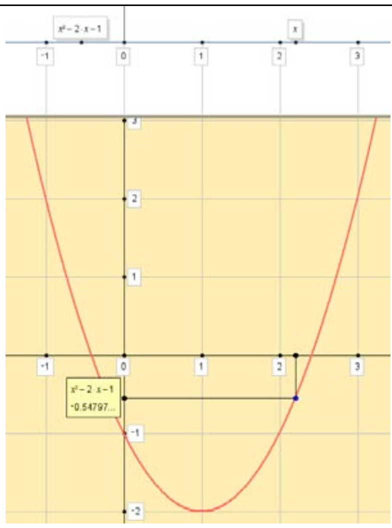
Functions component

The main idea characterizing the design of the Functions component is the possibility to connect a dynamic functional relationship between variable and expression on the algebraic line with the graphical representation of the function in the Cartesian plane. As a consequence, the interface of this component has been equipped with the Algebraic line and a Cartesian plane. This idea makes this component deeply different from other environment for the representation of function in the Cartesian plane. Through a specific command and the successive selection of the independent variable of the function, an expression represented on the Algebraic line is automatically represented as graphic in the Cartesian plane. Dragging the point corresponding to the variable on the algebraic line, two representative events occur:

- on the algebraic line, the expression containing the variable moves accordingly
- on the Cartesian plane, the point defined by the pair of values of the variable and of the expression moves on the graphic as shown in the following figure.

This instrumented technique supports the integrated development of a dynamic idea of function with a static idea of such a notion (Sfard 1991). The functional relationship between variable and expression is visualized dynamically on the algebraic line through drag of the variable point, and statically in the Cartesian plane through the curve. The movement of the point along the curve during the drag of the variable on the algebraic line supports the integration of these two ideas, showing that the curve reifies the infinite couples of values corresponding to the variable and to the expression on the line. This instrumented technique can be very useful to orient the interpretation of the graphics on the Cartesian plane and to develop important concepts of algebraic nature.





For example, it contributes to assign an algebraic meaning to the intersection of two curves (for the value of the variable that determines the intersection, the two expressions are contained in the same post-it on the algebraic line) or to the intersection of a curve with the x-axis (in this case the expression is contained in the post-it of 0).

Other examples are related to the construction of meaning for the sign of a function (position of the corresponding expression on the line with respect to 0), or to order among functions (positions of the expressions on the algebraic line)

CONCLUSIONS

In this paper we have presented the main ideas that oriented the realisation of Alnuset and the choices we made to instrument specific functions of algebraic activity that can be useful for the teaching and learning of algebra. We have shown that the quantitative, symbolic and functional techniques available in the three environments of Alnuset to operate with algebraic expressions and propositions have characteristics that are deeply different from the instrumented technique of CAS. The technique of Alnuset was designed having in mind two types of users, different from the target user considered by CAS designers. The former type of user is the student who is not an expert of the knowledge domain of algebra and uses the instrumented techniques of Alnuset to learn it carrying out the algebraic activity proposed by the teacher. The latter type of user is the teacher who has difficulties to develop algebraic competencies and knowledge in students and who uses the instrumented technique of Alnuset to acquaint them with objects, procedures, relations and phenomena of school algebra. The technique of Alnuset was designed to be easily controlled during the solution of algebraic tasks, to produce results that can be easily interpreted and to mediate the interaction and the discussion on the algebraic meaning involved in the activity. The techniques of Alnuset structure a new phenomenological space where algebraic objects, relations and phenomena are reified by means of representative events that fall under the visual, spatial and motor perception of students and teachers. This contributes to provide an appropriate balance between the pragmatic and epistemic values of the techniques made available by Alnuset. In the phenomenological space determined by the use of the instrumented technique of Alnuset algebra can become a matter of investigation as evidenced by Trgalova et al. (WG4) and Pedemonte (WG2) of CERME6.

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