

# THE USE OF ALGEBRAIC LANGUAGE IN MATHEMATICAL MODELLING AND PROVING IN THE PERSPECTIVE OF HABERMAS' THEORY OF RATIONALITY

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*In this paper we consider the use of algebraic language in modelling and proving. We will show how a specific model of rational behaviour derived from Habermas' elaboration allows to describe and interpret several kinds of students' difficulties and mistakes in a comprehensive way, provides the teacher with useful indications for the teaching of algebraic language and suggests further research developments.*

Key-words: Habermas, rationality, algebraic language, modelling, proving

## INTRODUCTION

Habermas' work has attracted the interest of many scholars in the domain of Sciences of Education (see the review of the translation into English of *Truth and Justification* by Tere Sorde Marti, 2004). We think that at least one of his constructs, that of "rational behaviour", is of specific interest for mathematics education, if we want to analyse complex mathematical activities (like conjecturing, proving, modelling) in a comprehensive way and to deal with them not only as school subjects and sets of tasks, but also as ways of experiencing mathematics as one of the components of western rationality. In a long term research perspective, we think that Habermas' construct is a promising analytic instrument in mathematics education if we want to connect the individual and the social by taking into account the epistemic requirements of "mathematical truth" in a given cultural context and the ways of discovering, ascertaining and communicating it by means of suitable linguistic tools. Indeed, according to Habermas' definition (see Habermas, 2003, Ch. 2), a rational behaviour in a discursive practice can be characterized according to three inter-related criteria of rationality: *epistemic* rationality (inherent in the conscious control of the validity of statements and inferences that link statements together within a shared system of knowledge, or theory); *teleological* rationality (inherent in the conscious choice and use of tools and strategies to achieve the goal of the activity); *communicative* rationality (inherent in the conscious choice and use of communication means within a given community, in order to achieve the aim of communication).

In our previous research we have dealt with an adaptation of Habermas' construct of rational behaviour in the case of conjecturing and proving (see Boero, 2006; Morselli, 2007; Morselli & Boero, 2009 - to appear). In this paper we focus our interest on the use of algebraic language in proving and modelling. Algebraic language will be intended in its ordinary meaning of that system of signs and transformation rules, which is taught in secondary school as a tool to generalize arithmetic properties, to develop analytic geometry and to model non-mathematical situations (in physics, economics, etc.). In particular, for what concerns modelling (according to Norman'

broad definition: see Norman, 1993, and Dapueto & Parenti, 1999, for a specific elaboration in the case of mathematics) algebraic language can play two kinds of roles: a tool for proving through modelling within mathematics (e.g. when proving theorems of elementary number theory) - *internal modelling*; or a tool for dealing with extra-mathematical situations (in particular to express relations between variables in physics or economy, and/or to solve applied mathematical problems) - *external modelling*.

Our interest for considering the use of algebraic language in the perspective of Habermas' definition of rational behaviour depends on the fact that our previous research (Boero, 2006; Morselli, 2007) suggests that some of the students' main difficulties in conjecturing and proving depend on specific aspects (already pointed out in literature) of the use of algebraic language, which make it a complex and demanding matter for students. In particular, we refer to: the need of checking the validity of algebraic formalizations and transformations; the correct and purposeful interpretation of algebraic expressions in a given context of use; the goal-oriented character of the choice of formalisms and of the direction of transformations; the restrictions that come from the needs of following taught communication rules, which may contradict private rules of use or interfere with them.

In this paper, we will try to show how framing the use of algebraic language in the perspective of Habermas' theory of rationality: first, provides the researcher with an efficient tool to describe and interpret in a comprehensive way some of the main difficulties met by students at any school level when using algebraic language; second, provides the teacher with some useful indications for the teaching of algebraic language; third, suggests new research developments, in particular those concerning the interplay between epistemic rationality and teleological rationality in the use of algebraic language, and those related to the role of verbal language as a crucial tool for a rational behaviour in the use of algebraic language, thus potentially adding new arguments to the elaboration presented in Boero, Douek & Ferrari (2008) and concerning the specific functions of verbal language in mathematical activities.

## **ADAPTATION OF HABERMAS' CONSTRUCT OF RATIONAL BEHAVIOUR TO THE CASE OF THE USE OF ALGEBRAIC LANGUAGE**

The aim of this section is to match Habermas' construct of rational behaviour to the specificity of the use of algebraic language in modelling and proving.

### **Epistemic rationality**

It consists in:

- *modelling requirements*, concerning coherency between the algebraic model and the modelled situation: control of the correctness of algebraic formalizations (be they *internal* to mathematics - like in the case of the algebraic treatment of arithmetic or geometrical problems; or *external* - like in the case of the algebraic modelling of physical situations) and interpretation of algebraic expressions;

- *systemic requirements* in the use of algebraic language and methods. In particular, these requirements concern the manipulation rules (syntactic rules of transformation) of the system of signs usually called algebraic language, as well as the correct application of methods to solve equations and inequalities.

### **Teleological rationality**

It consists in the conscious choice and finalization of algebraic formalizations, transformations and interpretations that are useful to the aims of the activity. It includes also the correct, conscious management of the writer-interpreter dynamics (Boero, 2001): the author may write an algebraic expression under an intention and, after, interpret it in a different goal-oriented way, by discovering new possibilities in the written expression.

### **Communicative rationality**

In the case of algebraic language we need to consider not only the communication with others (explanation of the solving processes, justification of the performed choices, etc.) but also the communication with oneself (in order to activate the writer-interpreter dynamics). Communicative rationality requires the user to follow not only community norms concerning standard notations, but also criteria for easy reading and manipulation of algebraic expressions.

### **Some comments**

The previous requirements define a model of “rational behaviour” in the use of algebraic language in modelling and proving.

We are aware of the existence of several analytical tools to deal with the teaching and learning of algebraic language. In the case of most of them, the researcher adopts a specific point of view, performs in-depth analyses according to it, but usually does not take into consideration the connections between the different aspects of the use of algebraic language and suggests only partial indications for its teaching. In our opinion, Arcavi's work on Symbol sense (Arcavi, 1994; 2005) offers the most comprehensive perspective for the use of algebraic language. With different wordings, it includes concerns for teleological rationality and some aspects of epistemic rationality. Comparing our approach with Arcavi's elaboration, we may say that we add the communicative dimension of rationality. We will see how it will allow us to account for: the possible tension between private rules of communication in the intra-personal dialogue, and standard rules; and the interplay between verbal language and algebraic language. Moreover we will see how our distinctions between the epistemic dimension and the teleological dimension, and between the modelling and the systemic requirements of epistemic rationality allow to deal with the tensions and the difficulties that can derive from their coordination.

In order to justify a new analytic tool in Mathematics Education it is necessary to show how it can be useful in describing and interpreting students' behaviour, and/or in orienting and supporting teachers' educational choices, and/or in suggesting new

research developments. The aim of the following Sections is to provide evidences for all the three mentioned aspects of the use of the adapted Habermas' model.

## DESCRIPTION AND INTERPRETATION OF STUDENTS' BEHAVIORS

The following examples are derived from a wide corpus of students' individual written productions and transcripts of *a posteriori* interviews, collected for other research purposes in the last fifteen years by the Genoa research team in Mathematics Education. In particular, we will consider four categories of students:

- (a) 9<sup>th</sup> grade students who are approaching the use of algebraic language in proving;
- (b) 11<sup>th</sup> grade students who are learning to use the algebraic language in modelling;
- (c) students who are attending university courses to become primary school teachers;
- (d) students who are attending the third year of the university course in Mathematics.

A common feature for all the considered cases is that the individual tasks require not only the solution, but also the explanation of the strategies followed to solve the problem. Each individual task was followed by *a posteriori* interviews. However, while in the cases (c) and (d) the explanation of the strategies is inherent in the didactical contract already established with the teacher for the whole course, in the cases (a) and (b) such explanation is only an occasional request.

### EXAMPLE 1: 9<sup>th</sup> grade class

The class (22 students) was following the traditional teaching of algebraic language in Italy: transformation of progressively more complex algebraic expressions aimed at « simplification ». In order to prepare students to the task proposed by the researcher, two examples of “proof with letters” had been presented by the teacher; one of them included the algebraic representation of even and odd numbers.

THE TASK: “Prove with letters that the sum of two consecutive odd numbers is divisible by 4”.

Here we report some recurrent solutions (in parentheses the number of students who performed such a solution; note that “dispari” means “odd” in Italian)

- E1 (4 students):  $d+d=2d$

In this case, we can observe how the *systemic requirements of epistemic rationality* are satisfied (algebraic transformation works well), while the *modelling requirements* fail to be satisfied (the same letter is used for two different numbers).

- E2 (8 students):  $d+d+2=2d+2$

In this case, both the *systemic and the modelling requirements of epistemic rationality* are satisfied, but the requirements inherent in *teleological rationality* are not satisfied: students do not realize that the chosen representation does not allow to move towards the goal to achieve (because the letter  $d$  does not represent in a transparent way the fact that  $d$  is an odd number) and do not change it.

- E3 (5 students):  $d=2n+1+dc=2n+1+2n+1+2=4n+4$  (or similar sequences)

We can infer from the context (and also from some a-posteriori comments by the students) that "dc" means "**dispari consecutivi**" (consecutive odd numbers).

In this case *epistemic rationality* fails in the first and in the second equality, but *teleological rationality* works well: the flow of thought is intentionally aimed at the solution of the problem; algebraic transformations are used as a calculation device to produce the conclusion (divisibility by 4).

### **EXAMPLE 2: University entrance, primary school teachers' preparation**

The following task had been preceded by the same task of the Example 1, performed under the guide of the teacher. 58 students performed the activity.

THE TASK: Prove in general that the product of two consecutive even numbers is divisible by 8

Very frequently (about 55% of cases) students performed a long chain of transformations, with no outcome, like in the following example:

- E4:  $2n(2n+2)=4n^2+4n=4(n^2+n)=4n(n+1)=4n^2+4n=n(4n+4)$

In this case, we see how both requirements of *epistemic rationality* are satisfied: *modelling requirements* (concerning the algebraic modelling of odd numbers and even numbers); and *systemic requirements* (correct algebraic transformations). The difficulty is inherent in the lack of an interpretation of formulas led by the goal to achieve, thus in *teleological rationality*. The student gets lost, even if the interpretation of the fourth expression would have provided the divisibility of  $n(n+1)$  by 2 because one of the two consecutive numbers  $n$  and  $n+1$  must be even. We can also observe how (in spite of the didactic contract) in general no substantial verbal comment precedes or follows the sequence of transformations (sometimes we find only a few words: "I use formulas"; "I see nothing").

In the following case, both *modelling* and *systemic requirements* are not satisfied: the same letter is used for two consecutive even numbers (note that "**pari**" means "even" in Italian) and the algebraic transformation is affected by a mistake.

- E5:  $p*p=2p^2$ , divisible by 8 because  $p$  is divisible by 2 and thus  $p^2$  is divisible by 4.

The student seems to work under the pressure of the aim to achieve: having foreseen that the multiplication by 2 may be a tool to solve the problem, she tries to justify it by considering the juxtaposition of two copies of  $p$  that generates "2". Indeed in the interview the student said that she had made the reasoning " $p$  is divisible by 2 and thus  $p^2$  is divisible by 4" before completing the expression. In this case we can see how *teleological rationality* prevailed on *epistemic rationality* and hindered it.

We have also found cases like the following one:

- E6:  $p*(p+2)=p^2+2p=8k$  because  $p^2+2p=8$  if  $p=2$



Also in this case, from the *a posteriori* interview we infer that probably the lacks in *epistemic rationality* depend on the dominance of *teleological rationality* without sufficient epistemic control:

I have seen that in the case  $p=2$  things worked well, so I have thought that putting a multiple  $8k$  of  $8$  in the general formula would have arranged the situation.

### EXAMPLE 3: The bomb problem

**TASK:** A helicopter is standing upon a target. A bomb is left to fall. Twenty seconds after, the sound of the explosion reaches the helicopter. What is the relative height of the helicopter over the ground?

The problem was proposed to groups of third year mathematics students in seven consecutive years, and to two groups of 11<sup>th</sup> grade students (high school, scientific - oriented curriculum). According to the school levels, some reminds were provided (or not) about the fact that the falling of the bomb happens according to the laws of the uniformly accelerated motion, while the sound moves at the constant speed of 340 m/s. However no formula was suggested.

The problem is a typical applied mathematical problem, whose solution needs an *external modelling* process. In terms of *teleological rationality*, the goal to achieve should result in the choice of an appropriate algebraic model of the situation, in solving the second degree equation derived from the algebraic model, and in choosing the good solution (the positive one).

The first difficulty students meet is inherent in the time coordination of the two movements: it is necessary to enter somewhere in the model the information that the whole time for the bomb to reach the ground and for the sound of the explosion to reach the helicopter is 20 seconds. The second difficulty is inherent in the space coordination of the two movements: the space covered by the falling bomb is the same covered by the sound when it moves from the ground to the helicopter.

Let us consider some students' behaviours.

Most students are able to write the two formulas:

- E7:  $s=0,5 gt^2$ ,  $s=340 t$

They are standard formulas learnt in Italian high school in grades 10<sup>th</sup> or 11<sup>th</sup>, in physics courses. About 25% of the high school students and 20% of the university students stick to those formulas without moving further. From their comments we infer that in some cases the use of the same letters for space and time in the two algebraic expressions generates a conflict that they are not able to overcome. We can see how general expressions that are correct for each of the two movements (if considered separately) result in a bad model for the whole phenomenon. *Teleological rationality* should have driven formalization under the control of *epistemic rationality*; such control should have put into evidence the lack of the *modelling requirements* of *epistemic rationality*, thus suggesting a change in the formalization.

In the reality for those students such an interplay between *epistemic rationality* and *teleological rationality* did not work.

In other cases (about 10% of both samples) the coordination of the two times was lacking, and the idea of coordinating the spaces (together with the formalization of both movements with the same letters) brought to the equation:

- E8:  $0,5gt^2 = 340t$

with two solutions  $t=0$ ,  $t=68$  that some students were unable to interpret and use (because 68 is out of the range given by the text of the problem). But other students found the height of the helicopter by multiplying  $340 \times 68$ ; the fact that the result is out of the reach of a helicopter did not provoke any critical reaction or re-thinking, probably because it is normal that school problems are unrealistic!

One part of the students who introduced the third equation  $t_b + t_s = 20$  added it to the first two equations without changing the name of the variable ( $t$ ).

Less than 60% of students of both samples wrote a good model for the whole phenomenon:

$$t_b + t_s = 20$$

$$h = 0,5gt_b^2 = 340t_s$$

and moved to a second degree equation by substituting  $t_s = 20 - t_b$  or  $t_b = 20 - t_s$  in the equation:  $0,5gt_b^2 = 340t_s$

Many mistakes occurred during the solution of the equation (mainly due to the management of big numbers). Once two solutions were got (one positive and the other negative), in most cases the choice of the positive solution was declared but not motivated. *A posteriori* comments reveal that the fact that a negative solution is unacceptable (given that the other solution is positive!) was assumed as an evidence, without any physical motivation.

In terms of *epistemic rationality*, three kinds of difficulties arose; they were inherent: first, in the control that the chosen algebraic model was a good model for the physical situation; second, in the control of the solving process of an equation with unusual complexity of calculations (big numbers); third (once the valid equation - a second degree equation - was written and solved), in the motivation of the chosen solution.

In terms of *communicative rationality*, we can observe how (in spite of the request of explaining the steps of reasoning) very few students of both samples were able to justify the crucial steps of the solving process. How is it possible to interpret this kind of difficulty? In some cases the steps were derived from a gradual adaptation of the equations to the need of getting a “realistic” solution. In other cases the equations were written as if the idea of coordination of the spaces and times of the phenomenon was supported by an intuition, but no wording followed. *A posteriori* interviews revealed that most students who had been unable to justify their choices were sure about their method only afterwards, when checking the positive solution and finding

that it was “realistic”, thus putting into evidence a lack in *teleological rationality* (lack of consciousness about the performed modelling choices). However a number of solutions was quite realistic, even if got through a bad system. Many authors of the correct solution were not able to explain (during the comparison of solving processes) why the other solutions were mistaken. This suggests that lacks in *communicative rationality* (as concerns verbal justification of the validity of the performed modelisation) can reveal lacks in *teleological rationality* (motivation of choices with reference to the aim to achieve) and even in *epistemic rationality* (control of the validity of the steps of reasoning). This conclusion can be reinforced if we consider the fact that almost all students who were able to provide a verbal justification for their modelisation were also able to explain why the other solutions were not acceptable (even if results were realistic).

## DISCUSSION

As remarked in the second section, the usefulness of a new analytical tool in mathematics education must be proved through the *actual* and the *potential research advances* and the *educational implications* that it allows to get.

### Research advances

In the frame of our adaptation of Habermas' construct, the distinction between *epistemic rationality* and *teleological rationality* allows to describe, analyse and interpret some difficulties (already pointed out in Arcavi's work), which depend on the students' prevailing concern for rote algebraic transformations performed according to *systemic requirements* of *epistemic rationality* against the needs inherent in *teleological rationality* (see E4). Moreover, the distinction in our model between *modelling requirements* and *systemic requirements* of *epistemic rationality* offers the opportunity of studying the interplay between the *modelling requirements* and the requirements of *teleological rationality* (see E7); we have also seen that formalization and/or interpretations may be correct but not goal-oriented (like in E2 and E4), or incorrect but goal-oriented (like in E5, E6 and E8).

Together with the other dimensions of rationality, *communicative rationality* allows to describe and interpret possible conflicts between the private and the standard rules of use of algebraic language, and the ways student try to integrate them in a goal-oriented activity (see E3).

At present, we are engaged in establishing how the requirements of the three components of rationality intervene in the phases of production and interpretation of algebraic expressions.

Further research work should be addressed to establish what mechanisms (meta-cognitive and meta-mathematical reflections based on the use of verbal language? See Morselli, 2007) can ensure the control of *epistemic rationality* and the intentional, full development of *teleological rationality* in a well integrated way. With reference to this problem, taking into account *communicative rationality* (in its



intra-personal dimension, possibly revealed through suitable explanation tasks and/or interviews) suggests a research development concerning the role of verbal language (in its mathematical register: see Boero, Douek & Ferrari, 2008, p.265) in the complex relationships between *epistemic*, *teleological* and *communicative rationality*. In particular, previous analyses (see E3, E4 and Example 3) suggest not only that the request (related to *communicative rationality*) of justifying the performed choices can reveal important lacks in *teleological rationality*, but also that the development of a kind of personal “verbal space of actions” can be relevant for a successful development of the activity (even if algebraic written traces are not satisfactory from the *systemic-epistemic rationality* point of view, like in the case E3). The respective role of the space of verbal actions and of the space of algebraic manipulations should be investigated on the *teleological rationality* axe. Here Duval's elaboration about the productive interplay between different registers in mathematical activities might be borrowed to better understand and frame what students do (see Duval, 1995). Also the results by Mac Gregor & Price (1999) could help highlighting the relations, as emerged from our data, between the production of verbal justifications and the effective use of algebraic language to achieve the goal of the activity.

### **Educational implications**

We think that the analyses performed in the previous section can provide teachers as well as teachers' educators with a set of indications on how to perform educational choices and classroom actions to teach algebraic language as an important tool for modelling and proving. Some of those indications are not new in mathematics education; we think that the novelty brought by Habermas' perspective consists in the coherent and systematic character of the whole set of indications.

First of all, the performed analyses suggest to balance (at the students' eyes, according to the didactical contract in the classroom) the relative importance (in relationship with the goal to achieve) of:

- production and interpretation of algebraic expressions, vs algebraic transformations;
- flexible, goal-oriented direction of algebraic transformations, vs rote algebraic transformations aimed at “simplification” of algebraic expressions.

These indications are in contrast with the present situation in Italy and in many other countries: teachers' classroom work is mainly focused on algebraic transformations aimed at “simplification” of algebraic expressions, and most simplifications are performed by elimination of parentheses, thus suggesting a mono-directional way of performing algebraic transformations. At the students' eyes, the importance of the formalization and interpretation processes is highly underestimated. The fact that algebraic expressions are given as objects to “simplify” (and not as objects to build, to transform according to the aim to achieve, and to interpret during and after the transformation process in order to understand if the chosen path is effective and correct or not) has bad consequences on students' *epistemic rationality* and *teleological rationality*. As we have seen, many mistakes occur in the phase of

formalization (against the *modelling requirements*), and even when the produced expressions are correct, frequently students are not able to use intentionally them to achieve the goal of the activity (against the *teleological rationality requirements*).

A promising indication coming from our analyses concerns the need of a constant meta-mathematical reflection (performed through the use of verbal language) on the nature of the actions to perform and on the solving process during its evolution. At present, the only reflective activity in school concerns checking the correct application of the rules of syntactic transformation of algebraic expressions (thus only one component of rational behaviour - namely, the *systemic requirements of epistemic rationality* - is partly engaged).

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