SURFACE SIGNS OF REASONING

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Abstract

In this paper, we explore forms of verbal expression undergraduate mathematics students employ while working in pairs on geometric tasks in a computer environment, focusing in particular on the connectives (notably 'because') they use as well as the modal expressions in their talk as they discuss ideas with their partner. We use this data to bring together C. S. Peirce's idea of abduction, the linguistic notion of hedging and Toulmin's argumentation scheme, and argue that in trying to identify abductions, the presence of hedges (of which Toulmin's 'modal qualifiers' are an instance) or a particular use of 'because' may provide some evidence.

> It is a commonplace of philosophical logic that there are, or appear to be, divergences in meaning between, on the one hand, at least some of what I shall call the formal devices—$\sim$, $\wedge$, $\vee$, $\supset$, $(\forall x)$, $(\exists x)$ (ιx) (when these are given a standard two-value interpretation)—and, on the other, what are taken to be their analogues or counterparts in natural language—such expressions such as *not, and, or, if, all, some* (or *at least one*), *the*. (Grice, 1975/1989, p. 22)

In this paper, we wish to explore some of the natural language markers (in English) that are employed in students' *spoken* mathematical reasoning. One motivation for doing so is a realisation of how different, on occasion, even experienced mathematical undergraduates *speak* when working on problems in pairs, from the conventional way formal mathematics is supposed to be *written* (e.g. Morgan, 1998). A second was the difficulty we had at times in identifying the nature of the reasoning from the speech of the participants. A third arose from our growing interest in the notion of *abduction*, which has been receiving attention in the past few years within mathematics education (e.g. Mason, 1995; Pedemonte, 2007; Reid, 2003; Rivera, 2008; Sinclair, Lee and Strickland, under review), as well as possible connections to the linguistic notion of hedging (see, e.g., Rowland, 1995) and Toulmin's argumentation scheme (see, e.g., Inglis et al., 2007).

In mathematical discourse, there are significant differences between speech and writing. We are not claiming that there are disjoint vocabularies, but there are some words that are usually only spoken (including a few that require invented spellings for transcription e.g. 'cuz', 'gonna', 'gotta') and some that are much more comonly written (hence, therefore, consequently). The formal written mathematical register is quite tightly specified in terms of particular conjunctions to be used in proofs, particularly at the beginnings of sentences to mark the relation between the preceding and subsequent comments (e.g., 'let', 'hence', 'therefore', 'if', 'since', 'conversely'). This is another level of difference beyond that to which Grice is drawing attention.

However, one linguistic challenge arises from the fact that mathematical purposes are not the only functions that these words encode. The language of ‘if …, then …’, for instance, so common in written mathematics, is also the language of threats. Many of the conventional connectives in other circumstances carry a space, time or sequencing connotation (e.g. then, since, when, hence) – for more on mathematics and time, see Pimm (2006). In conversation, the *then* of ‘if …, then …’ is often elided, and there are occasions when even the *if* marker can be absent.

In this paper we wish to go further than Paul Grice in differentiating logical operators from what he terms ‘natural language’, by distinguishing spoken from written natural language. Unlike Grice, however, we will offer attested speech data for consideration rather than invented data. In the opening chapter to his book *Text and Corpus Analysis*, linguist Michael Stubbs (1996) criticises the dominant tradition since Chomsky (and including Grice) for basing extensive theoretical arguments on no real language data. Nevertheless, Stubbs (see below) supports Grice’s specific claim about the non-congruence between logical connectives and English words and goes further, paying close attention to the role of modality in verbal communication.

This paper draws on data collected within a larger study of mathematical reasoning in undergraduate students. The data consist of twenty videotaped episodes (ranging from ten to twenty minutes in length) in which pairs of students are working at computers, using *The Geometer's Sketchpad* (Jackiw, 1991) to solve geometric tasks. These tasks include, among many others, using *Sketchpad* to construct a parabola, to identify the particular transformation that relates two given shapes, to solve the Apollonius problem and to figure out the fractal dimension of given curves.

SPOKEN MARKERS OF REASONING

> A third case of the interaction of pragmatic and syntactic matters is provided by the so-called logical connectors (e.g. *and, but, or, if, because*). Their uses in everyday English are not reducible to their logical functions in the propositional calculus, but have to do with speakers justifying their confidence in the truth of assertions, or justifying other speech acts. (Stubbs, 1996, p. 224)

Any modal utterance contains both propositional information and the speaker’s attitude towards the information. Echoing Grice, Stubbs uses modality to distinguish between different functions of connectives. He claims *because* is representative in having two distinguishable uses, which he terms *logical* and *pragmatic*: the first has the structure of ‘effect plus cause’, the second ‘assertion plus justification’. Stubbs notes that the pragmatic use of *because* is often signalled by the addition of epistemic *must* (‘he must have been drunk because he fell down the steps’). In addition, He provides a number of syntactic criteria to help distinguish the two uses. He claims these points are also true for the pragmatic use of *if, or, but,* and *and.*

An example of the logical use comes from Birkhoff and Mac Lane (1941/1956) “Because of the correspondence between matrices and linear transformation, we need supply the proof only for one case" (p. 227). Similarly, in Spivak (1967), we find: "Because this sequence varies so erratically near zero, our primitive mathematical instincts might suggest that $\lim_{n\to\infty} f_n(x)$ does not always exist" (p. 414).

There is no scope within this paper for a detailed corpus analysis of connectives in our data, though we wish to remark on the prevalence of 'so' and 'which means' as markers of deductive utterances. From our data, we find very few logical uses of because.

A: Well, **because** those two don't, for sure, lie in the circle, so if we rotate it around that point, it's not gonna be exact.

In A's statement above, the cause is signalled by '**so if**.' Far more often, the uses of because are pragmatic, as in the following two examples.

D: No, **because** the rotation point is gonna be over here.

E: Yeah, the original one **because** then O_1 will convert to a line and through … never mind. That didn't work. We did it wrong.

In both these and other similar instances, what we find is students hypothesising or positing justifications for claims they are making. This connects in an interesting manner to the theme we turn to in the next section, namely abduction as a form of inferring, which is proving challenging to us to identify confidently. This brief look at 'because' suggests that one place to look for abductions is in pragmatic uses of the connective 'because'.

TWO SHORT EPISODES OF STUDENT REASONING

Here are two episodes of student mathematical problem solving where we found the form of reasoning less clearly identifiable, less likely to be deductive, and replete with modal utterances. We provide a brief contextualisation of each episode in this section, and then offer two tentative analyses—one using Peircean abduction and the other Toulmin's model of argumentation— of each episode in the following section.

Example 1

Two students (Lucie and Brad) are trying to solve the problem of geometrically constructing a parabola in *Sketchpad* given a focus point *P* and a directrix line *j*. The students have already constructed the envelope of the parabola by tracing the perpendicular bisector of *PB* where *B* is a point on *j* that can be dragged back and forth along the line. The students begin looking for ways to construct a point that depends on B so as they move B along *j* it will trace out the parabola.

At first, they place a point on the segment *PB* right where the segment first touches the envelope edge. When Lucie drags B, they both realise that this point does not always lie on the curve, so they delete this point. In turn 1 below, Brad notices that if the solution point is placed on *PB*, then it could never reach the upper parts of the parabola (given that *PB* is a segment). This seems to give rise to an anomaly for Brad

– that the point will have to be able to travel high up the sides of the parabola. Indeed, his expression is emphatic and strong-voiced and the modal verb ‘can’t’ is also strong: “We can’t have …”. Indeed, he tries to convince Lucie of what he’s noticing: “see that point”. In turn 3, Brad makes a deductive inference, first using the word ‘so’ and then “which means” to indicate the implication that the point *cannot* be on *PB*.

1	Brad:	We can’t have [..] [1] Well, like, [….] like, see that point has to be able to get up here, right? (He points to *j* with his pen and then points to the top left of the curve with his pen and then his finger.)
2	Lucie:	Uhuh.
3	Brad:	So, which means it can’t touch the line.

Lucie then proposes that this point lies on a line passing through *P* perpendicular to *j*.

4	Lucie:	Yep [….] So then [….....] Let’s say […………] (Constructs the line through *P* perpendicular to *j*, as in Figure 1.) Maybe that’s the line […] ‘cause um [..] the distance from like [..] here to here would be the same as that one? (Points to distance between the envelope of the curve on the left and her new line.) But I don’t know if that’s right. (Points to her new line and the curve on the right.)
5	Brad:	So what line did you just create?
6	Lucie:	The perpendicular line to the bottom through P. But I don’t think it’s right.

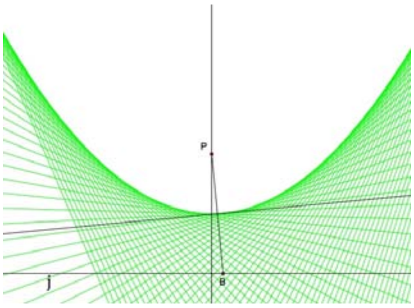

Figure 1: The envelope of a parabola with focus P and directrix *j*

Brad seems to think that Lucie’s line “couldn’t be” the right one, but acknowledges her statement about equidistance. At this point, the instructor intervenes and redirects the students’ attention to the more pertinent equidistance relationship (to point P and line *j*). The students eventually figure out how to construct the point on the parabola as the intersection between the perpendicular bisector of *PB* and the line perpendicular to *j*, passing through B.

Example 2

Two students (Gloria and Peter) are trying to figure out which isometry maps a given shape on the computer screen onto another and then to construct the specific transformation. The students have studied the composition of reflections (and found that the composition of two reflections gives a rotation, unless the two lines of reflection are parallel). In turn 1, Gloria has already identified two corresponding segments of the shape and asks “can we continue these two lines?”

1	Gloria:	Rotation right? [..] Which is two reflections but I don't know how to do that. (Points to the right edge of top figure and top edge of bottom one – see Figure 2 below.) OK, can we continue these two lines?
2	Peter:	Probably two reflections.
3	Gloria:	Can we, yeah, or a rotation, same difference.
4	Peter:	[inaudible]. (Gloria draws a straight line extending the right-hand vertical edge of the top figure.)
5	Gloria:	Can we make this a straight line and find out what this angle is, and then rotate it that much? [……….] Um […..] That'd work, wouldn't it?

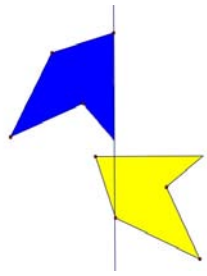

Figure 2: Line extending one side of the top shape

In turn 4, Gloria extends the line and then, in turn 5, infers that the intersection of the line and the horizontal side of the lower shape will form an angle that corresponds to the angle of rotation necessary between the two shapes.

INTERPRETING THE EPISODES

In each episode, we see mathematical reasoning that plays an important role in the problem-solving process of the pairs, but that does not fall easily into the two most commonly-discussed categories of inductive and deductive reasoning. We thus begin by interpreting the two episodes described above in terms of Peircean abduction. We then interpret the same episodes using Toulmin's (1958) structure of argumentation.

Focus on Peirce's different types of inferences

> Deduction proves that something *must* be; Induction shows that something *actually is* operative; Abduction merely suggests that something *may be*. (Peirce, 1931/1960, 5.171)

Peirce's description of the three forms of inference, as quoted above, marks a shift in interpretation away from the logical *form* of a given inference (how it might be characterised through syllogistic propositions) toward its *use*, by the inquirer, in the process of inquiry. While researchers such as Reid (2003) and Cifarelli (2000) claim to have identified student abductions based on these logical forms, Mason (2005) cautions, "The tricky part about abduction is locating at the same time the appropriate rule and the conjectured case" (p. 5). In many cases, neither of these propositions will be uttered out loud in spoken conversation – they must be inferred from context.

While logical forms are sometimes easy to identify in written language (especially in mathematics texts), they can be much harder to identify in speech, which is frequently less planned and more emergent in real time, especially in the context of pairs jointly co-constructing the talk. While some students will state that something "must be" (or 'has to be' or 'gotta be') true, others may choose to express their certainty through other means, both verbal and non-verbal. Peirce's emphasis on the *uses* of deduction, induction and abduction invites attention to the intentions of the inquirer, but these intentions, about what must be, what actually is, and what may be, can't always be clearly identified either. Thus, one challenge facing researchers is how to work with the surface elements of language in order to make interpretations about the type of inference demonstrated in particular conversational exchanges. The short list given by Grice in our opening quotation, which includes clear, propositional terms of inference, is completely insufficient when looking at real people reasoning in conversational pairs about mathematics.

Considering episode 1, we can see Brad's inference that the point cannot lie on PB as a deduction, since he states what *must* be the case. Here, the logical form is quite easy to identify, as are the linguistic features. By contrast, Lucie's proposal that the point lies on the perpendicular to *j* through P can be seen as an abduction, since it indicates what *may be* true, as exemplified by her own words "Maybe that's the line" and her later hedged statement of hesitation "But I don't know if that's right." Lucie's inference satisfies two additional characteristics of abduction: (1) it involves the generation of a new idea (the line she constructs did not exist before, and stands as a genuinely new and plausible solution); and (2) it is not logically derivable from true statements (and, indeed, the line she proposes is not the right one). Further, the use of "'cause" is a pragmatic one, in Stubbs's sense as described above.

We might also attempt to interpret Lucie's abduction in the following logical form, where the case is the only thing Lucie knows to be true, and the result has been hypothesised as a plausible situation in light of the novel rule.

case: The (solution) point has to go up

rule: If it's on that line, it would go up

result: The point is on that line

In contrast with the linguistic interpretation offered above, the logical form fails to capture the interlocutor's degree of conviction when she hedges her proposal both with 'maybe' and "I don't think that's right'. Additionally, there is a close link between this formulation of abduction and Stubbs's pragmatic category of connective use, as noted above in relation to "'cause". Curiously, Stubbs's term 'pragmatic' seems to evoke Peirce's work on pragmatism.

We turn now to episode 2, where Gloria and Peter are trying to identify the isometry relating two shapes. In turn 1, Gloria asks, after pointing to the two lines in question, "can we continue these two lines?" She has not explicitly stated that she is trying to identify the angle of rotation (or the angle between the two lines of reflection), but this becomes explicit in turn 5, where she asks (again): "Can we make this a straight line and find out what this angle is?" We see this as an abductive inference, since it follows the use of what *may be* true, as evidenced by her questioning tone of voice, her use of the hedge tag phrase "can we" and the final, doubtful, tagged utterance "That would work, wouldn't it?"

We find further evidence of this as an abductive inference by the fact that it introduces a new idea (the technique of extending lines had not been previously used in class), which, in this case, turns out to be fruitful. Once again, we could offer an interpretation based on the 'underlying' logical form of the inference, but the preceding analysis seems to offer an identification consistent with Peirce's conceptualisation of abduction in its pragmatic function.

Focus on Toulmin's forms of argumentation

In work on forms of argumentation and informal logic, Toulmin's (1958) scheme has had its place. But, as Inglis et al. (2007) clearly point out, it is a reduced form of Toulmin's scheme that has been commonly used in mathematics education, one which leaves out two of the six components: the rebuttal and, of greater relevance for us here, modal qualifiers. Inglis *et al.* worked with the production of individual oral arguments of graduate students in mathematics, exploring a range of mathematical conjectures. We were struck in their paper by the fact that modal qualifiers are precisely hedges, those statements of propositional attitude concerning the degree of conviction the speaker is willing to express. This made us wonder about the connection between overt hedging and abduction, which suggest that the student was to some extent aware of the making of an abduction that consequently required a more tentative assertion.

Inglis *et al* (2007) give a visual summary to illustrate Toulmin's model of argumentation (Figure 3). The argument would read: based on the data (D) given, the warrant (W) – which is supported by the backing (B) – justifies the connection between D and the conclusion (C), unless the rebuttal (R) refutes it. The modal qualifier (Q) qualifies the certainty of the conclusion by expressing degrees of confidence.

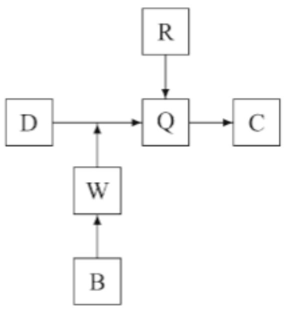

Figure 3: Toulmin's model of argumentation

We now run the first episode above through Toulmin's model to obtain Figure 4. The data include the point P, the directrix *j*, the point B on *j*, as well as the segment *PB*. Lucie's conclusion, that the point lies on the line perpendicular to *j* and passing through P is qualified by her hedged utterances “Maybe” and “But I don’t know if that’s right”. We see her statement regarding the equidistance of the line to each side of the parabola functioning as the warrant, even though it is offered *after* the argument – following some hesitation and speculate that it is the presence of her partner that makes her verbalise this at all. The backing includes the fact that the point must be on some line (instead of a line segment like PB), but one that should somehow involve both P and *j* (the givens in the situation). The rebuttal is not evident in her argument and may not exist at all.

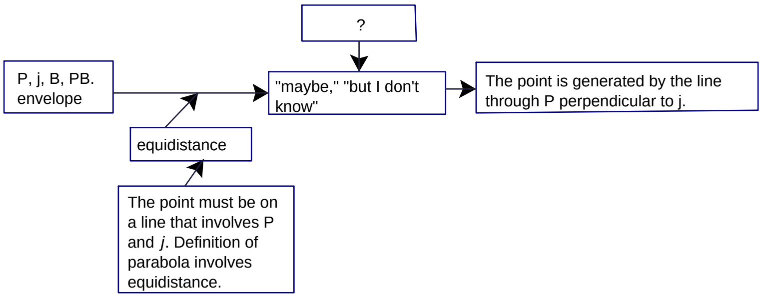

Figure 4: Lucie’s argument expressed using Toulmin’s scheme

Turning now to the second episode, we can also run Toulmin’s scheme on Gloria’s argument (in Figure 5).

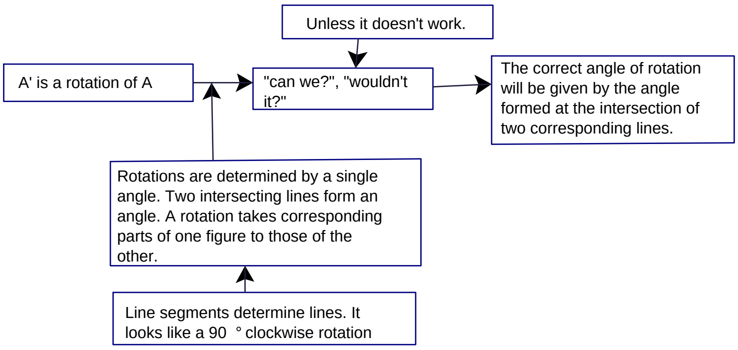

Figure 5: Gloria’s argument expressed using Toulmin’s scheme

This time the modal qualification is not expressed through specific words, such as ‘maybe’ or ‘probably’, but instead in the intonation of Gloria’s statement, which is made in question form: “Can we […]?”. In this episode, we also find no evidence of a rebuttal, though presumably Gloria had an immediate and pragmatic rebuttal in mind, which was to actually see whether the angle of rotation created by intersecting the line and side segment would work to rotate the pre-image to its image. Filling in the scheme, Gloria’s conclusion is that the angle of rotation between the two shapes is the angle created by the intersection of two corresponding sides (one extended).

CONCLUSION

The above analyses show that it is possible to interpret the two excerpts of paired student reasoning in conversation using either Peirce’s idea of abduction or

Toulmin's model of argumentation. Both are challenging to use as interpretational frameworks, and this is so for several reasons. First, both Peirce and Toulmin tended to work with made-up examples to illustrate their inferences or arguments; and, as we have seen, real speech is much messier – some phrases are omitted, others are communicated non-linguistically, and so on. Second, and especially for abduction, we have already noted that the most important component of the abductive inference – the stating of the general rule – must often be inferred from context. However, even in Toulmin's case, what counts as data, warrant, and backing is not always obvious, and certainly not objectively knowable. Third, neither Peirce nor Toulmin has conversational reasoning in mind when articulating their theories. In some senses, Toulmin's emphasis on argument is *post hoc*, given that the interaction between two students (in our own data) frequently involved negotiation of meanings, and subsequent attempts to explain and/or convince.

The analyses we conducted reveal interesting similarities and differences. Most remarkable of the former related to the importance attached to the degree of confidence held by the reasoner. Toulmin includes modal qualifiers in his model in order to account for the variety of certainty that one might have about a claim. Pierce's abductions are seen as hypothetical *may be*'s. Their attention to uncertainty might seem strange in the context of mathematics, where one frequently seeks precisely the opposite. Yet both Peirce and Toulmin seem to care about how the reasoner can make advances in inquiry, and take it as given that many advances will be tentative. A particular resonance such a perspective has in mathematics education can be found in the work of Rowland, who has studied the notion of hedging in the mathematics classroom. We suggest that this notion could be used productively to help identify and analyse and interpret student reasoning in terms of Toulmin or Peirce. Lastly, the pragmatic use of 'because' also appeared as a surface marker in one of the two episodes that may help identify abductions in some cases.

Toulmin is concerned with trying to identify the structure and form of an existing argument, whereas Peirce is more concerned with examining the process of scientific discovery. Peirce draws attention to the way in which problem solving may require abductive 'leaps of faith', where one is reasoning ahead of more explicit or acknowledged deductive or inductive means. This seems to us an important awareness in educators involved in supporting and eliciting mathematical problem solving. Toulmin's analysis of an argument acknowledges the qualification involved in any emergent complex argument, and serves to draw attention to argument structures and resources that may not have been apparent in the more 'logical' literature analyzing the form and nature of mathematical arguments.

By juxtaposing the results of each analysis of the same two mathematical episodes, as well as identifying hedging as one surface linguistic phenomenon common to both, we have attempted to highlight how one might ground each theoretical account in the specifics of moment-to-moment conversation, as well as thereby drawing attention to commonalities across the two accounts that have not been made before.

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