STEPS TOWARDS A STRUCTURAL CONCEPTION OF THE NOTION OF VARIABLE

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If students acquire a new mathematical notion, according to Sfard (1991), they pass through different phases: an operational and a structural phase. At a grammar school in Bremen, Germany, students of age 12 to 14 first came into contact with the notion of variable using a simple programming language without a computer. As a part of the learning environment the students wrote imaginary dialogues in which they let two protagonists talk about different tasks. The imaginary dialogues of the students are analysed against the background of Sfard's theory of the dual nature of mathematical conception. In particular, the different steps towards a structural conception of the notion of variable in the context of the programming learning environment are elaborated.

INTRODUCTION

If we look at a mathematical notion, we can think about what it is in the mathematical world, how it is defined, which properties it has, and how it relates to other parts of mathematics or we can consider how a human being thinks about it and what kind of inner picture has been built. Anna Sfard (1991) distinguishes here between the word *notion* or *concept* on the one hand and *conception* on the other hand.

> The whole cluster of internal representations and associations evoked by the concept - the concept's counterpart in the internal, subjective "universe of human knowing" - will be referred to as a "conception". (Sfard, 1991, p. 3)

According to Sfard, a conception of a mathematical notion has two complementary sides, an operational and a structural one, in which a learner first passes through operational phases until a structural conception can be developed. She also points out that

> without the abstract objects all our mental activity would be more difficult. (Sfard, 1991, p. 28)

In this article the development of the conception of variable is considered. The underlying question of the presented analysis is: what are steps towards a structural conception of the notion of variable? To approach an answer the findings of a qualitative analysis of imaginary dialogues written by students of age 12 to 14 from one class will be presented.

THEORETICAL FRAMEWORK

The theory of reification

Sfard (1991) presents a theoretical framework for the acquisition of a mathematical notion. She distinguishes between an *operational* and a *structural conception* of the same mathematical notion. If a learner has acquired an operational conception, she or he will know how to operate with the notion, i.e. with algorithms, processes and actions. For a structural conception it is necessary to recognise the notion as a mathematical object. Sfard expects that the operational conception precedes the structural. In this process from operational to structural three steps occur: *interiorization*, a process with familiar objects, *condensation*, where the former processes become separate entities and *reification*:

> to see this new entity as an integrated, object-like whole. (Sfard, 1991, p. 18)

While a learner can come gradually from interiorization to condensation, Sfard speaks of a *leap* when it comes to reification:

> “Reification (...) is defined as an ontological shift – a sudden ability to see something familiar in a totally new light. Thus, whereas interiorization and condensation are gradual, quantitative rather than qualitative changes, reification is an instantaneous quantum leap: a process solidifies into object, into a static structure.” (Sfard, 1991, p. 19-20)

Sfard & Linchevski (1994) used the framework of the theory of reification to study the case of algebra. In particular, they focused on the transition from operational to structural regarding a variable as a fixed unknown on the one hand and in a functional context on the other hand. Sfard (1991) asks the question how to diagnose the stages towards a conceptual development and proposes:

> "It seems that we have no choice but to describe each phase in the formation of abstract objects in terms of such external characteristics as student's behaviour, attitudes and skills." (Sfard, 1991, p. 18)

Mathematical writing

Mathematical writing by students has been the issue of several studies, compare Borasi & Rose (1989), Clarke, Waywood & Stephens (1993), Gallin & Ruf (1998), and Shield & Galbraith (1998). Gallin & Ruf investigated the use of *journals* (in German: Reisetagebücher) in order to establish a written dialogue between the students and the teacher. While writing their journals the students can approach the regular mathematics in their singular way.

Imaginary dialogues are a different type of mathematical writing (Wille, 2008). In an imaginary dialogue the student lets two protagonists solve a mathematical task or talk about a mathematical question. Usually one protagonist understands the task better than the other. In this way the student can decide what particular themes she or he addresses. Unlike in journal writing, in an imaginary dialogue, one finds a lot of exploratory writing. On the other hand, in contrast to pure exploratory writing, like writing a letter to someone and explaining something, in imaginary dialogues the protagonists can develop a solution of a task and the protagonists can point at possible learning difficulties.

LEARNING ENVIRONMENT

The learning environment is designed for first experiences with the notion of variable. The students do not start with a single variable as a fixed unknown. Instead, they get to know a simple programming language which is executed by the students without a computer but with a little wooden robot on a sheet of paper with a coordinate grid. The programming language has similarities to LOGO (Papert, 1980). Here, as a "memory" each robot needs matchboxes on which letters for the names of variables like "a" and "b" are written. These matchboxes serve as *preset reifications* of the notion of variable, which the students fill by hand instead of assigning a number to a symbolic variable. For example to move three steps forward, the program will look like this

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a ← 3
forward(a)
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While executing the first line it must be assured that exactly three matches are in the matchbox named "a". In the second line, the robot will be moved into the direction it faces. The matchboxes must be used in order to move a robot, since the direct command "forward(3)" is not part of the programming language. Next to these commands there is also the command "turnaround()", which lets the robot turn by 180°. Furthermore there are a right and a left turn, commands to place the robot on a certain intersection point on the coordinate grid and different command loops. That way students can write and execute programs in order to move their robot on the grid while assigning variable by filling matchboxes with matches.

In the learning environment the programming of the robot can be combined with writing imaginary dialogues. One of the first tasks can be the following: The students get a sheet of paper with "a ← " and "b ← " on top and "turnaround()" in the middle. On another sheet of paper eight paper commands "forward(a)" and eight paper commands "forward(b)" can be cut out. The students get the following exercise with the name "cut out and explore":

> On the next sheet of paper you see a program that is not finished yet. You can use commands out of a construction kit and put them above and below the command "turnaround()". 1. Cut out as many commands as you need and write a program with them. 2. Execute your program with the matchboxes and the robot. 3. Try to write such a program that the robot comes back to his starting point. 4. For which values a and b does your program function? Are there different possible values? 5. Write your favourite program and name many values with which it works.

Right after this lesson the students get the following homework (*dialogue A*):

> Two students talk about the last task “cut out and explore”. One of the students can do it easily, the other has more difficulties. Write a dialogue in which the two students talk about the task. Write at least one page.

In the next task a simple program is presented, where over the turnaround command there are two commands “forward(a)” and under it one command “forward(b)”. There is also a table given for a and b with values (1,2), (2,4), (3,6) and (4,7). A beginning of a dialogue is also part of the task where two students talk about whether the numbers in the table should be switched. One protagonist draws also the following picture:

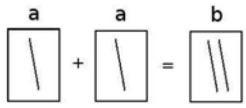


Figure 1

The students are asked to work with the program first, decide, if the table is correct and finish the dialogue (*dialogue B*). After further tasks with the robot a third imaginary dialogue task (*dialogue C*) is given. The students get the following picture:

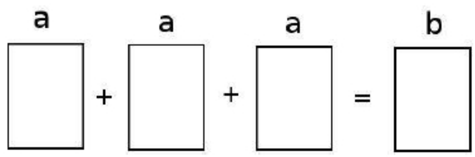
Programm

a ← ____
b ← ____
vorwärts(a)
vorwärts(a)
vorwärts(a)
umdrehen()
vorwärts(b)

Tabelle

a	b
1	3
2	6
3	9
4	12

Schachteldiagramm



Gleichung

$3 \cdot a = b$

Fig

ure 2

Now the students are asked to think of an interesting program of a similar form, find the proper presentations like in Figure 2 and write an imaginary dialogue about it.

METHOD

The study was carried out in a class of a grammar school (Gymnasium) in Bremen, Germany, in 2008 with the above mentioned learning environment. The students wrote three different dialogues A, B and C. Dialogue A was written after the second lesson, dialogue B after three more days and dialogue C after about three weeks. The imaginary dialogues A and B were given as homework, dialogue C was written in the classroom. Since not all students did their homework or some let the protagonists talk about only non-mathematical tasks, for the analysis 16 A-dialogues, 15 B-dialogues and 22 C-dialogues could be used. For the qualitative analysis of the imaginary dialogues the framework of Sfard's theory of reification was used. The analysis was carried out in three steps:

13. examination by *four criteria*: recognised structures, occurring aspects of the notion of variable, phase in which the student is (i.e. interiorization, condensation, mixed form/indistinct, or reification), mentioned preset reification
14. creation of a *mind map* of the seen structures for each dialogue A, B, and C
15. creation of *tables* that includes the information of the mind maps and the phases

In order to examine by the four criteria, most dialogues were first transcribed and than interpreted in detail. The students' development was classified according to the phases according to these criteria:

- *interiorization*: the student can handle the program: processing the program, filling matchboxes with matches, etc.
- *condensation*: the student deals with variables as with objects but does not see them as objects, the input and output is more important than the process itself
- *mixed form/indistinct*: it cannot be decided if the student already reificated the notion of variable, variables are used in a tight relation to preset reifications
- *reification*: variables are seen as independent objects

FINDINGS

All imaginary dialogues mentioned here were written in German and translated by the author.

Mini-statistics

We can observe a shift of the students of this class from interiorization to reification as Sfard predicted. It must be mentioned that the tasks for the dialogues A, B and C were similar, but different. Thus, there is the possibility that the observed shift also depends on the different tasks. In the following table, the number of students in a certain phase of a certain dialogue is denoted:

	i	c	m	r	Total
A	11	2	2	1	16
B	5	5	3	2	15
C	4	5	6	7	22

Table 1: number of students in a certain phase

Structures recognised by the students

The structures that were recognised by the students are shown in the tables of the Figures 3 and 4. The tables should be read like a tree from left to right where each row is a branch. It is also listed which phase is assigned to the specific imaginary dialogue, in which the student recognised the structure. The letters i, c, m and r stand for the phases interiorization, condensation, mixed form/indistinct and reification. There are several crosses, if several students see the same structure. Some of the structures that can be seen as examples of *preliminary steps of reification* are discussed below. In the following, for example "Figure 3, structures in A, 7" refers to the seen structure in A written in row 7 which is here "segmentation of the distance – in segments a and b".

structures in A

		i	c	m	r
comparison of the number of steps		x			
independence of the notation	name of variable is free to choose	x			
segementation of the commands	search for a segementation	x			
	same amount of comands with a and b			x	
	with an example				x
segmentation of the distance	in steps without a and b	x			
	in segments a and b		x		
preform of abstraction	forward(a) and box content synonymical			x	
preform of substitution	preform of "a is 1"			x	
preform of reification	abstraction of matchboxes		x		
	paper commands "a's" and "b's"			x	
	relationship between the number of "a's" and "b's"			x	
	a and b as steps			x	
abstraction	a as a value in different contexts				x

structures in B

		i	c	m	r
correlation of the values in the table		x			
pre-understanding of equations	preform of symmetry in equations	x			
	correlation between program and substitution in equation	x			
	preform of the understanding of equations		x		
structure of explanation	abstract, concrete, example				x
correlation between term and distance			x		x
correlation between a and b	comparison of undefined step sizes		x		
	with the notion "a" and "b"			x	
	b is the double of a			xx	
abstraction	variables as independent objects				x

Figure 3: structures in A and B

structures in C

			i	c	m	r
1	commutativity of the commands				x	
2	different substitutions possible					x
3	structure in table	multiple addition		x		
4		as the times table			x	
5		proportion between a- and b-values			x	
6		correlation between values in the table			x	
7	variable as common name for something	“>a<” as name for a mathbox, column and field			x	
8		generic term for values, commands and distances			x	
9	correlation between a and b	as values				x
10		b is the double of a			x	xxx
11		it is “the same” with other values				x
12		how many of “a” yield “b”				x
13		concrete assignment unimportant			x	x
14	letters a and b	instead of a number of intersection points			x	
15		as a fixed unknown			x	
16		stand for a number of steps				x
17		stand for numbers in the language use		x		
18		stand for numbers		xx		x
19	step size as a scale					xx
20	preform of reification	“a)” and “b)” instead of concrete numbers in the language use		x		
21		notation “a=7” for values of the table			x	
22		a and b as generic term for values in the table			x	
23	understanding of equations	symmetry			x	
24		cancellation				x
25		equation as a summary of examples				x
26	expansion of the number of variables					x
27	abstraction	variables as independent objects				xxxx
28		terms as independent objects				x

Figure 4: structures in C

Independence of the notion

In the imaginary dialogue of a student (Figure 3, structures in A, 2) we can read that for him the name of the matchbox is free to choose. One of his protagonists explains:

> “You put arbitrarily many matches of the 16 and label the matchbox with a letter, let me say an example: “N”. You position the robot on the sea bottom and now you must give commands to the robot: for example: forward (for example N). Hence, he goes forward as much as you have put matches into the matchbox.”

The students writes “forward(for example N)” which shows that he points out that he could have chosen another name for the matchbox. If we transfer this to variables, we can call it an aspect of the independence of the name of variable. This aspect has its relevance, if we think about students who might know for example the binomial formulas with a and b, but have difficulties, when different variable names are used.

Name of variable as a generic term for multiple objects

Variables can simultaneously represent multiple values and can be abstracted from multiple real objects, like distances or the quantity of something. Hence, a preliminary step for this abstraction is to use different objects synonymously or to use a variable as a generic term for multiple objects. We can see the use of different objects synonymously in a dialogue by a student (Figure 3, structures in A, 8) who first wrote:

> “because (a) and (b) are most probable of different size.”

After this she inserted the words “forward” from above, such that the sentence looks like this:

> “because forward(a) and forward(b) are most probable of different size.”

We do not know, if she means by “(a)” the box content or a value of an abstract a, but we might consider that she uses the command “forward(a)” and whatever she thinks of as “(a)” synonymously.

The next step is to use a variable as a generic term for multiple objects as in the following dialogue (Figure 4, structures in C, 7). Here, the protagonists are named “S” and “D”.

S: Well, the table has two columns. A+b. As the two matchboxes. In >a< are two matches, and in b 8. In column >a< 2 are added in each row. In column >b< it is the same.

D: Like a times table? Where in each row it increases by 2 or 8 respectively?

S: Yes! Precisely. Now to the matchbox diagram. The field >a< stands for the number >2<. The field >b< stands for >8<. That way the diagram is eventually: 2+2+2+2=8.

When the student mentions her notation “>a<” the first time it means a matchbox. After this it is a column and the end a field which can be substituted. We can also observe that the student does not use the letter a without relating it to an object. It does not appear in a complete abstract manner.

A different student (Figure 4, structures in C, 8) uses variables as a generic term for commands,

> “We have the commands A, B, & turnaround.”

values,

"But how do I know, what is the value of A & B?"

and distances:

"If you go the distance a() + b(), then it makes no difference, if you go back a() + b() or b() + a()."

Talking about a and b as talking about objects

A student talks in his dialogue (Figure 3, structures in A, 11) about a and b as if they were objects. Possibly he thinks about the paper commands while talking about them.

"If a is equal to 1 and b is equal to 2: First you must (you can) go with all a's forward and with the half of the b's backward and you are again on the same point."

Since he says "with the half of the b's", the "b's" are some kind of objects to him.

Correlation of different variables

Several students discuss the correlation between different variables (compare Figure 3, structures in B, 7-9 and Figure 4, structures in C, 9-13). One example is where the student recognises that b must be the double of a (Figure 3, structures in B, 9):

"If the robot moves two half steps (a) and he must go back steps which are bigger, then b must have the double, thus an entire step."

A different student formulates the correlation by fitting a number of a into b (Figure 4, structures in C, 12):

S2: Well, if a and b stand for the number of steps and you can turnaround only once, then you must find out how many of a yield b.

S1: Thus, if a is 1 and b 4 then one must find out how often a fits in b.

S2: Exactly!

What are a and b?

Some students discussed the topic of what the letters a and b are. Most often they used the words "stands for" instead of "is". We find passages, all in dialogue C, saying for example that a or b stand for a number of steps (compare the preceding example), or for numbers (Figure 4, structures in C, 18):

2: Exactly and for the equation you must do this in a multiplication exercise.

1: Without numbers?

2: The letters stand for numbers, for example out of the table.

1: But there are multiple numbers. Which ones do I take?

2: That is easy. You can take every number you like. Just make sure that a has the double value.

SUMMARY

The analysis of the imaginary dialogues written by the students indicates the process from the phase interiorization, passing condensation to reification, as predicted by Sfard (1991). In the tables we see all structures that were recognised by the students. Among those structures we can also identify several preliminary steps toward a structural conception of the notion of variable: the independence of the notion, using the name of variable as a generic term for multiple objects, talking about variables as about objects, recognising correlations between different variables, and actually discussing what a letter stands for. Whether these preliminary steps eventually lead to a complete reification or not, we cannot predict. But we can observe that several students in dialogue A are tight to the preset reification of the notion of variable in form of the matchboxes or paper commands, while reading the dialogues B and C, the preset reifications disappear in many writings and the language use becomes more and more regular.

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