CONCEPTION OF VARIANCE AND INVARIANCE AS A POSSIBLE PASSAGE FROM EARLY SCHOOL MATHEMATICS TO ALGEBRA

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Change and invariance appear at the very early stages of learning mathematics. In this theoretical paper, examples of topics and tasks from primary school mathematics with various kinds of interplay between variation and invariants are presented. Application of this approach might be a tool that helps to improve non-formal algebraic thinking of students. We present some examples of pre-service teachers' reasoning in terms of variances and invariance.

INTRODUCTION

For over fifty years, mathematics educators have studied ways of teaching algebra. Beyond viewing algebra as generalized arithmetic, various classifications for meaning of algebra, algebraic symbolism, procedures and skills have been proposed (Usiskin, 1988). In algebra, students have to manipulate letters of different natures such as unknown numbers (Tahta, 1972), parameters, and variables. Required skills include specific rules for manipulating expressions and an ability to construct and analyze patterns. These components form the basis for the structure of school algebra, which appears to students to be abstract and rather artificial. Through dealing with transformation of algebraic expressions, students can hardly recognize the core ideas of algebra, such as application of standard arithmetic procedures to unknown or unspecified numbers.

From the point of view of primary school teachers, algebra is comprised of letters, rules of operations with expressions, and formulas to solve equations. Moreover, the term *pre-algebra* in the school math curricula stands for some "advanced arithmetic" topics that are linked with future algebra, mostly chronologically but not conceptually.

Since 2005, the awareness of pre- and in-service teachers about algebra has been one of the "hot" issues of annual conferences on training primary school math teachers in Israel. In order to match the course *Algebraic Thinking* to the needs of pre-service primary school mathematics teachers, a systematic study on their vision of algebra has been initiated. Preliminary results of this research show that only a few of these students are aware of non-formal components of algebra (Sinitsky, Ilany, & Guberman, 2009).

What mathematical concept could help pre- and in-service teachers to construct relevant algebraic comprehension? School algebra is a combination of generalized arithmetic, calculations with letters, and properties of operations (Merzlyakov &

Shirshov, 1977). In general, it requires reasoning on connections and relations between objects, for example, finding similarities and dissimilarities between objects. The question "what changes and what does not change?" seems to be fruitful in a meta-cognitive discourse that concerns problem-solving activity (Mason, 2007; Mevarech & Kramarski, 2003). We propose to apply this question at the very early stages of mathematical learning as a possible tool to connect primary school mathematics with algebra.

WHY VARIANCE AND INVARIANCE?

The two notions of *variance* and *invariance* are strongly linked, since "invariance only makes sense and is only detectable when there is variation" (Mason, 2007). Mason claims that "invariance in the midst of change" is one of three pervasive mathematical themes. Watson and Mason (2005) have elaborated the theory of possible variation and permissible change for the needs of mathematical pedagogy. The use of the concept of variance and invariance with pre-service teachers can develop their algebraic thinking and provide them with tools to construct examples.

The issue of learning processes is related to the human ability to associate and to distinguish between different characteristics of the same object. Research (Stavy & Tirosh, 2000; Stavy, Tsamir, & Tirosh, 2002) shows that reasoning patterns "same A then same B" and "more A then more B" are prevalent among students, and direct analogy causes deep misconceptions in the learning of mathematics. Refining comprehension of various types of interconnections between change and invariance may be fruitful for improving cognitive schemes of students.

Starting from secondary school, students systematically face algebraic notation and formalism. The most significant feature of algebra for students is manipulating with letters. It seems to them (and to their teachers) as a switch from four arithmetic operations with numeric operands into *terra incognita* of some quantities that are both unknown and tend to change.

Although the abilities to deal with varying objects, to explain, and to formulate are the very essence of secondary school algebra, students are expected to grapple with these based on their experience in primary school. In the framework of systematic construction of formal algebraic concepts, pre-algebra is responsible for the development of pre-abstract apprehensions of algebra (Linchevski, 1995).

In this paper, we bring up some issues from primary school mathematics and observe these problems in terms of change and invariance. We refer, at a non-formal level, to the main components of school algebra mentioned by Linchevski, i.e. using variables and algebraic transformations, generalization, structuring, and equations.

We proposed related mathematical activities for pre-service primary school mathematics teachers, and discuss some relevant classroom findings in the last paragraph and in the appendix.

VARIANCE AND INVARIANCE INTERPLAY IN PRIMARY SCHOOL

Word problems and algorithms of school algebra often have an origin, or an analogy, in primary school mathematics. Despite the concrete numerical form of arithmetic problems, they usually enable some algebraic generalizations into patterns for several number sets with suitable restrictions. For example, the property of being divisible by 9 is invariant in relation to any change in order of digits. Analysis of mathematical problems of primary school from the point of view of algebraic concepts may be fruitful for students as a step to constructing their algebraic thinking.

A consideration of variation, change, and invariance may help to provide a non-formal algebraic vision of arithmetic issues. Every mathematical situation provides a variety of variance–invariance links. Moreover, a suitable set of variations and related invariants that describe a task may provide a way to solve it. We illustrate the appearance and application of the "change and invariance" concept in a number of topics from primary school mathematics.

Quantities and numbers

The most fundamental example of invariant is human ability to count (Invariant, n. d.). It starts with the transition from objects to quantities and develops through numerous activities of counting objects of different nature. At this stage, quantity is invariant of physical properties of specific objects. Children also learn to count a given set of objects in different ways, and discover that the result is invariant of various (correct) counting procedures.

Thus, a basic conception of equality of quantities arises: the equality represents the fact that the same quantity is obtained or described in two different ways. There is also the possibility of inverting the problem: which changes are allowed within a given quantity? This question seems concerns a misconception of equality. Linchevski and Herscovics (1996) have connected cognitive difficulties in the transition from arithmetic to algebra to dual procedural-structural algebraic thinking. A well-known example of such difficulties is the comprehension of the expression $34+7=$ as a command to carry out an action (Gray & Tall, 1991). Accordingly, in the equation $8+4=\Delta+5$ the unknown is interpreted by students as the result of adding $8+4$. In contrast, the idea of equality as an idiom of invariance invites possible changes.

An appropriate didactical scheme for primary school students is to focus on problems of decomposition of given number into a sum of two addends. Typical questions require producing additional presentations based on a given one as demonstrated in this activity:

- $8=3+5$ How can you split the same number 8 into another sum of two addends?
- How does a change in the first addend influence the second one?

- How does the change of addends of two “adjacent” decompositions vary? (At a higher level this leads to a conclusion on invariance of parity for differences of addends for several decompositions of the same number)
- For a given odd (or even) number, what can you say about the parity of addends in each decomposition?

This activity invites students to discover the role of invariant quantities in a game of changing in.

In discussions with pre-service teachers, the same questions were followed by further generalizations. For instance, the last question on parity leads to a conclusion on the invariance of parity of algebraic sums of numbers, with arbitrary distribution of +/- signs, through an analogy to the arithmetic expression. A choice of signs +/- does not influence the parity of the expression $a_1 \pm a_2 \pm \ldots \pm a_k$ (for integers $a_1, a_2, \ldots a_k$).At an advanced level, the same mathematical situation leads to combinatorial tasks, such as:

- In how many ways can we split a given natural number into the sum of equal addends?
- Can you arrange any presentation of an arbitrary multiple of three as a sum of consecutive addends by first splitting it into a sum of equal addends?
- In how many ways can we split a given natural number into sum of consecutive addends?

In the appendix, we present examples of pre-service primary school math teachers' response to some of these questions.

With this cluster of problems, we explored the concept of permissible changes within a given invariant in a variety of mathematical questions and levels.

Comparison of quantities in terms of change and invariance

In addition to invariance, the very basic process of counting deals with *variation* of quantity. Adding each new object to a given set of objects generates a new quantity that is greater than the given one. These examples are taken from the Curricula for Primary School in Israel (Curriculum, 2006): the sum 5+1 is greater than 5, and the sum 67+2 is less by1 than the sum 67+3.

From the point of view of invariance and change, students try “to find the same” in a pair of arithmetic expressions. The same operand plays a role of a parameter, i.e. arbitrary but the same number. The only cause for different values of given expressions is the difference in second operands. Therefore, to compare two quantities one looks at them in a structural manner: namely, noting the similarity and the difference between them. For example, comparing the results of other arithmetic operations when one of the operands is the same for both expressions:

- Which one of the differences is greater: 856 – 47 or 856 – 44?
- What is the difference between the two products: 84 X 123 and 83 X 123?

- Shirli arranged dolls in nine rows with the same number in every row. She added two dolls to each row. By how many dolls did the total number of dolls increase?

In school algebra, the presence of an unknown quantity typically turns the simple problem of comparing two similar expressions into a difficult one for students. For example, the comparing the pair *a-7* and *a+7* as opposed to the pair *7-a* and *7+a*.

Further, in order to compare more "remote" arithmetic expressions, one can try to interpret them as a different change of the same connecting expression. When pre-service teachers discussed how to compare two differences, i.e. 1234-528 and1243-516, they constructed intermediate expressions, 1234-516 or 1243-528. In a similar way, they proposed using the product 83×123 for comparison of products 84×123 and 83×124. This method of comparison is also an algebraic one: two expressions *a*b* and *c*d* are interpreted as changes of the same basic structure *a*d* or *c*b*.

Computational algorithms and techniques

In school algebra, most procedures cause changes in algebraic expressions yet preserve equality or inequality. This issue is not new for students. Almost every process of computation includes some transformation of a given arithmetic expression to another one. The transformation is valid provided it keeps invariant the value of the expression. In fact, both the rules of arithmetic operations and standard computational algorithms preserve the invariants:

- To calculate the sum 123+456, one groups similar units of addends, 123+456=(100+400)+(20+50)+(3+6) – this is a direct analogy of gathering similar terms in algebraic expressions.
- The difference 123-49 can be replaced by a new expression that retains the value of the given one: 123–49=124–50.

Fraction reduction and expansion are additional examples in elementary school of variation that preserves value.

The ability to find a suitable variation of a given expression that preserves its value is a useful starting point for oral calculations. A necessary condition to apply is the invariance of the value under the change of form of the calculated expression.

We have studied the strategies pre-service primary school math teachers apply to calculate sums of arithmetic progressions (Sinitsky & Ilany, 2008). Only 5% of the students succeeded in recalling a suitable formula and applying it correctly. After taking part in series of assignments concerning interplay of change and invariance, the students were given similar tasks. They tried to calculate sums by reducing them to

known series in various ways
(2+3+...+26→1+2+...+25; 3+6+9+...+60=[2+4+...+40]+[1+2+...+20]).

Number properties and range of generalization

When students manipulate algebraic expressions, the application of natural intuitive reasoning schemes “same A then same B” or “more A then more B” leads them to false reasoning: “$x^2 = y^2$ *implies* $x = y$”, “ $-x > 2$, *therefore* $x > -2$”. In terms of change and invariance, this is a problem of connection between different invariants.

There are numerous examples of correct ways of reasoning when letters A and B stand for the property of numbers. Examples of correct propositions concerning squares of natural numbers: “If the unit digits of two numbers are the same their squares have the same unit digit”; “The squares of numbers with the same parity are also of the same parity”`; “As natural numbers increase so do their squares”.

Such a convenient tie between invariants and changes invites a wide generalizing. Accordingly, questions that lead to counter examples and determination of range of possible changes or invariants are crucial: “Does changing the order of a sum change the result?”; “Does equal square/rectangle/parallelogram area imply the same perimeter?”; “Does multiplying a number by 2 increase the number of its divisors?”

Generalizing regularities and solving problems without algebraic formalism

An equation composed to solve a word problem algebraically expresses an invariance of some (typically unknown) value. For example, in problems that concerns motion, the same distance that two vehicles cover in different manners is the invariant of the two processes involved. Hence, the ability to identify invariance through some changes is useful for solving mathematical problems.

At primary school level, the search for invariance is an effective tool to discover regularities in numerical tables and in tables of arithmetic operations. For example, in the hundred table (see appendix, example 2) numbers increase constantly, but the change between adjacent cells in any row or column is invariant of the cell position. Similarly, the difference of products of diagonals of any 2×2 square is an invariant of the choice of square.

The next stage of proving those propositions typically involves some algebraic manipulation. Detecting a proper invariant for the problem can help avoiding formal algebra and provide a transparent proof with a generic example (Mason & Pimm, 1984). This type of reasoning is presented in the appendix.

Coming back to word problems and relevant equations, we illustrate another aspect of interaction between variation and invariance in pre-algebra mathematics. This interplay may provide non-algebraic solutions for some word problems. For example:

> John bought two kinds of items: pencils that cost 30 cents each and pens that cost 50 cents each. He paid 6.20 euro for 16 items. How many pencils and how many pens did John buy?

We restate here a well-known arithmetic solution of the problem with an emphasis on variation and invariance. We start with the possibility that John bought 16 pencils at a cost of 4.80 euro. Now we need *to vary* the cost, keeping *invariant* the number of items. The answer to the question "How many pencils do we need to exchange for pens to increase the total price by 1.40?" provides the solution of the problem. In this approach, the total number of items is an invariant of the process. An alternative method of solution starts from any combination of items that provides the desirable cost (for example, 10 pens and 4 pencils). The next step is to vary the number of items keeping the total cost invariant.

A taxonomy of change and invariants

Due to many characteristics of each object or process, every variance results in several changes and introduces invariants as well. Alternatively, preserving some invariant permits variances of other properties. Thus, there are many possibilities of interrelation between change and invariance. The same sort of connection can occur in various mathematical problems and topics.

From the above and other examples, we have derived a suggested taxonomy for change, variance, and invariance:

- An invariant is given *a priori*, and the focus is on possible changes and related invariants.
- To understand the action of *prescribed change*, we look for imposed variations and for given invariants.
- To solve a problem, it is necessary to *find some key invariant* of all the procedures involved.
- To treat a mathematical situation, we *introduce a suitable variation* or a sequence of variations.

Within this classification, the two latter cases seem to be more complicated since they involve *construction* of relevant objects or procedures. On the other hand, a specific kind of relation between variation and invariance is connected more with the method of solving the problem than with the problem itself. Thus, various solutions of the same problem may bring into play different kinds of interaction of change and invariance or even a combination of those interactions.

PEDAGOGICAL ASPECTS OF THE APPROACH

We require that primary school mathematics teachers be competent to recognize relevant kinds of variations and invariants in various issues and problems of elementary mathematics. We need to start introducing this concept in teachers' education to ensure that they can construct an additional didactical tool for mathematical discourse in a classroom.

To test the influence of discourse in terms of interplay between variance and invariance on algebraic thinking of students, we designed an experimental study. The research involved future and current teachers of mathematics at elementary school. We tried to learn if, and to what extent, discourse on variance and invariance influenced beliefs and knowledge on the ability of further application of non-formal algebraic reasoning. In addition to checking the validity of our conjectures, we would like to improve the awareness of school educators about the use of variation and invariance at primary school level.

So far, pre-service teachers have participated in the study through problem solving activities in the framework of their courses in pedagogical colleges. Throughout these activities, they have discussed the ideas of variance and invariance with specific mathematical issues. We have found that future teachers have begun to construct examples for teaching in elementary school that invite algebraic thinking and argumentation in terms of change, comparison and invariants (Sinitsky & Ilany, 2008).

To promote this concept, we designed additional mathematical assignments. Each task includes a cluster of math problems on different issues at various levels of difficulty *united by the same relation of variance and invariance*. The starting point is part of the school curriculum, should be familiar to every pre-service teacher, and is a basis for further generalizations and analogies. The style of all the assignments is that of open problems in order to stimulate various approaches and strategies.

CONCLUSION

In this paper, we discussed applications of conception of changes and invariants in primary school mathematics. We looked at numerical problems from a point of view that is general and in many cases algebraic. The same types of connection can be detected in different mathematical issues. The ability to recognize variation and invariants may be an effective tool in constructing non-formal algebraic thinking of students. However, as a necessary stage, it requires the awareness of teachers on the subject. Some preliminary evidence on pre-service teachers' activities seems encouraging and invites further wide-scale research.

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APPENDIX: IT LOOKS LIKE ALGEBRA

Two samples of reasoning involving variance and invariance interplay are presented.

1. Representations of natural number as a sum of consequent addends – fragment of transcript of discussion with pre-service primary school mathematics teachers

Students wrote down all the pairs with the given product, 30, and constructed sample sums of equal addends.

Student A: *“I start with equal addends. Now, for 30=10+10+10, I keep the total sum but vary the addends:* (she moves a finger from the first term to the third one and has marked it with an arrow) *30=10+10+10. We get 30=9+10+11, and it is possible to do this for each of these sums of equal addends! For example, I can derive from this sum* (she points 30=6+6+6+6+6) *another sum of consequent addends: 30=4+5+6+7+8 and... No, it does not work with 30=15+15: we need the sum to be invariant but also keep a middle term, and there is no middle addend here. Ah, I can try to split each one of 15s, but it changes the number of addends...”*

Students also obtained representation of 30 as a sum of four consequent addends: 30=6+7+8+9, and tried to derive sum of consequent addends from the sum of fifteen equal ones.

Student B: *“But we need negative numbers. Aha, after the cancellation we get exactly the same sum! It means that for every presentation of natural number as a sum of consequent natural numbers we can make more sums if we use integer numbers that will be cancelled after that, for example, 12=3+4+5 and also 12=(-2)+(-1)+0+1+2+3+4+5, because (-2)+(-1)+0+1+2=0”*

2. Divisibility of differences of two-digit numbers with “inverted” digits – sample proof

Conjecture: The difference of two two-digit numbers, where the second number has the same digits as the first one but in inverted order, is a multiple of 9.

How can we introduce the justification of this proposition without algebraic formalism in the framework of discussion with the students?

Let us check, what is the same in each pair of these numbers? They have the same digits, therefore also the same sum of digits. Now, let us mark an arbitrary pair of these numbers in a hundred table, for instance, 62 and 26. Their difference is just a distance between cells. Can we construct the route from 26 to 62 that keeps **invariant** the sum of digits? The route passes through 35, 44 and 53 before reaching 62. Each step increases the number by 9 (see “decomposition” of one of the steps in the table), therefore the total difference is a multiple of 9. Moreover, the difference between inverted two-digit numbers equals the number of such steps multiplied by 9.

1	2	3	4	5	6	...
11	12	13	14	15	16	...
21	22	23	24	25	26	...
31	32	33	34	35	36	...
41	42	43	44	45	46	...
51	52	53	54	55	56	...
61	62	63	64	65	66	...
...	...	...	...	...	...	...