ROLE OF AN ARTEFACT OF DYNAMIC ALGEBRA IN THE CONCEPTUALISATION OF THE ALGEBRAIC EQUALITY

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In this contribution, we explore the impact of Alnuset, an artefact of dynamic algebra, on the conceptualisation of algebraic equality. Many research works report about obstacles to conceptualise this notion due to interference of the previous arithmetic knowledge. New meanings need to be assigned to the equal sign and to letters used in algebraic expressions. Based on the hypothesis that Alnuset can be effectively used to mediate the conceptual development necessary to master the algebraic equality notion, two experiments have been designed and implemented in Italy and in France. They are reported in the second part of this paper.

***Keywords**: Alnuset, semiotic mediation, conceptualisation of algebraic equality*

INTRODUCTION

The research reported in this paper is carried out in the framework of the ReMath project (http://remath.cti.gr) addressing the issue of using technologies in mathematics classes "*taking a 'learning through representing' approach and focusing on the didactical functionality of digital media*". The work is "*based on evidence from experience involving a cyclical process of a) developing six state-of-the-art dynamic digital artefacts* [DDA] *for representing mathematics […], b) developing scenarios for the use of these artefacts for educational added value, and c) carrying out empirical research involving cross-experimentation in realistic educational contexts*". This paper presents the research concerning Alnuset, one of the 6 DDA developed within the project. First, some theoretical considerations related to the notion of algebraic equality, at stake in this paper, are presented. Next, our research hypotheses are discussed and Alnuset is briefly presented. Finally, two experiments involving this artefact are described and the main results are discussed.

THE NOTION OF ALGEBRAIC EQUALITY

Important conceptual developments are needed to pass from numerical expressions and arithmetic propositions to literal expressions and elementary algebra propositions. As a matter of fact, in arithmetic only numbers and symbols of operations are used and the control of what expressions and propositions denote can be realized through some simple computations. In elementary algebra, instead, letters are used to denote numbers in indeterminate way and new conceptualisations are necessary to maintain an operative, semantic and structural control on what expressions and propositions denote (Drouhard 1995; Arzarello et al. 2002). The necessity of this conceptual development emerges clearly with the construction of the notion of algebraic equality. On the morphological plan, equality is a writing composed by two expressions or by an expression and a number connected by the “=” sign. On the semantic plan, equality denotes a truth value (true/false) related to the statement of a comparison. When the expression(s) composing the equality is (are) strictly numerical, it is easy verifying its truth value through some simple calculations (e.g., 2*3+2=8 is true while 2*3+2=9 is false). Experiences with numerical equality contribute to structure a sense of computational result for the “=” sign. This sense can be an obstacle in the conceptualisation of algebraic equality as relation between two terms, as highlighted by several researches (Kieran 1989, Filloy et al. 2000). When the expression(s) composing the equality is (are) literal the equality can present different senses because the value assumed by the letter can condition differently its truth value. In these cases the “=” sign should suggest to verify numerical conditions of the variable for which its two terms are equal. There are cases where the two terms could never be equal whatever the value of the letter is, as in 2(x+3)=4x-2(x-1). In other cases to interpret equality on the semantic plane, it is necessary to distinguish if it has to be considered as equation or as identity. The “=” sign assigns to the equality the sense of equation when its two members are equal only for specific values of the letter. For example, the equality 2x-5=x-1 is true only for x=4 and it is false for all other values. Instead, the “=” sign gives to the equality the sense of identity when its two members are equal whatever the numerical value of the letter is, as in 2x+1=x+(x+1). In order to master algebraic equality, a conceptual development of notions of equation, identity, truth value, truth set and equivalent equation is necessary. Moreover, to express the way in which a letter can condition the truth value of an equality, it is necessary to develop a capability to use universal and existential quantifiers, even though in implicit way.

RESEARCH HYPOTHESIS

Traditionally, conceptual construction of algebraic equality is pursued through solving equations using techniques of symbolic manipulation. Empirical evidence and results of research have highlighted that in many cases this approach does not favour a construction of an appropriate sense either for the notion of algebraic equality or for that of solution of equation. In more recent years, a functional approach to algebra has been introduced within the didactical practice allowing to articulate algebraic and graphical registers of representations (Duval 1993). Even in this approach difficulties emerge. These regard the interpretation of a graph. For example, for the solution of equations of the type ax+b=cx+d, the intersection of the two lines in the graph has to be interpreted as indicator of the fact that the equation has a solution. Moreover this solution has to be read on the x-axis in correspondence of the intersection point of the lines. As Yerushalmy and Chazan (2002) observed, this approach is not devoid of obstacles: students can interpret the graph as comparing two functions (y=ax+b and y=cx+d) or as a solution set of a system of two equations in two unknowns, instead of an equation in a single variable. Our research hypothesis is that Alnuset, an artefact of dynamic algebra recently developed, can be effectively used to mediate conceptual development necessary to master the notion of algebraic equality. Further in the paper we discuss this hypothesis referring to some results of two experimentations.

SHORT DESCRIPTION OF ALNUSET

Alnuset is constituted of three components, Algebraic Line, Symbolic Manipulator and Functions, strictly integrated with each other. They enable quantitative, symbolic and functional techniques to operate with algebraic expressions and propositions.

The main characteristic of Algebraic Line component is the representation of an algebraic variable as a mobile point on the numerical line, which can be dragged with the mouse along the line. This feature has transformed the number line into an algebraic line where it is possible to operate with algebraic expressions and propositions through techniques of quantitative and dynamic nature. These techniques focus on numerical quantities indicated by an expression when its variable is dragged along the line or on numerical quantities that make true a proposition. These techniques make a dynamic algebra possible. The main characteristic of Symbolic Manipulator component is the possibility to transform algebraic expressions and propositions through a set of particular commands. These commands correspond to basic properties of operations, properties of equality and inequality, logic operations among propositions, operations among sets. Another characteristic is the possibility to create a new transformation rule once it has been proved. These characteristics support the development of skills regarding the algebraic transformation and they contribute to assign a meaning of proof to it. The main characteristic of Functions component is the possibility to operatively integrate Algebraic Line with Cartesian Plane, where graphs of expressions can be represented automatically. Moreover, dragging the point corresponding to the variable on the algebraic line makes the expression containing the variable move accordingly on the line. On Cartesian Plane, the point defined by the couple of values of the variable and of the expression moves on the graph. These characteristics support two integrated conceptions about the notion of function: a dynamic conception developed on Algebraic Line and a static one associated to the graph on Cartesian Plane. For a more detailed description of Alnuset, we refer to the work of Chiappini and Pedemonte presented in this edition of CERME within the working group 7.

EXPERIMENTATIONS

As we mentioned above, the development of DDAs was followed by a design of learning scenarios involving these tools and the implementation of these scenarios "*in realistic contexts*". ReMath partners decided that each DDA would be experimented not only by the designer team, but also by an other team that did not participate to the DDA development. Such "cross-experimentation" of the DDA was intended to highlight the impact of theoretical frameworks and of contextual issues on the design of both DDA and learning scenarios. Indeed, each team was free to set up educational goals taking account of institutional constraints and to choose theoretical approaches to frame the scenario design process. Thus, the experiments involving a given DDA were not meant to be compared, but rather to validate design choices related both to the DDA and the learning scenarios.

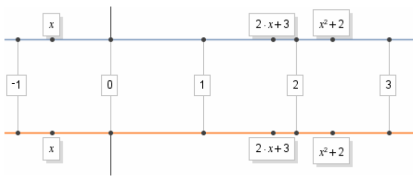
Italian experimentation

The experimentation activity reported below, lasting 1h40, has involved a class of 15-16 year-old students (Grade 10) attending a Classic Lyceum. The students worked in pairs using Alnuset. Previously, they had carried out 6 activities with Alnuset centred on notions concerning algebraic expressions. The whole teaching experiment lasted about 20 hours. The activity considered in this paper is centred on solving a 2nd degree equation. In the previous school year, students had learnt to solve 1st degree equations through symbolic manipulation. In this activity notions of conditioned equality, solution of an equation, equivalent equations, truth value of an equality and truth set of an equation are addressed. The didactical goal is the conceptual development of these notions while the research goal is the study of Alnuset mediation in this conceptual development. The activity comprises several tasks. The first task aims at allowing students to explicit their own conception of the algebraic equality notion.

Task: *Consider the following two polynomials:* x^2+2; $2x+3$. *Explain what it means putting the equal sign between them, or, in other words, how you interpret the following writing* $x^2+2=2x+3$.

Many students attribute to the "=" sign the meaning of computation result. Nevertheless they were already faced with 1st degree equations. A typical students' answer is: "*To put the equal sign between two polynomial expressions means that these expressions have the same result*". For many students inserting the equal sign between two expressions suggests the idea that the computation result of the two terms has to be equal when a value is assigned to the letter.

In the following task students were asked to represent the two expressions on the algebraic line of Alnuset to verify their answers. Dragging the mobile point x along the line (and observing that the points corresponding to the two expressions move accordingly), all students noted that there are only two values of x for which the points of the two expressions are close to each other, almost coincident. Through this exploration students experienced that equality of two expressions is conditioned by numerical values of the variable, which is crucial to develop the conditioned equality notion. In previous activities with Alnuset, students experienced that every point of the algebraic line is associated to a post-it that contains all expressions constructed by the user denoting that point. In order to verify equality of two expressions, the students tried to find values of x for which the two expressions belong to the same post-it. Since these irrational values had to be constructed on the line, the students could not verify this directly: "*we don't*



understand what is the number…it will be 2 point something…even if we use zoom in we don't understand …". The technique mediated by Alnuset to find these irrational numbers requires transforming the equation into its canonical form (x^2-2x-1=0), representing its associated polynomial on the line and using a specific command to find roots of this polynomial. Our hypothesis was that this technique could favour a conceptual development of notions of equivalent equations and of truth value of an equation. The transformation was realized in the Symbolic Manipulator and was guided by the following task:

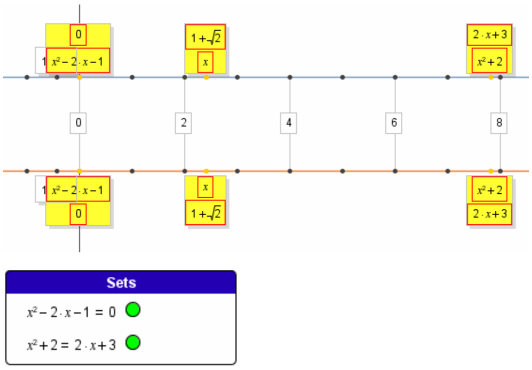
Task: *Select the equation and use the rule A=B ⇔ A-B=0 to transform it. Translate the result produced by this rule into natural language.*

This task focuses on the rule A=B ⇔ A-B=0 of the manipulator through which it is possible to transform the equality preserving the equivalence. We report two students' answers: "*If two terms are equal, then their difference is zero*"; "*it means that if two expressions are equal, subtracting them the result will be zero*". The conditional form of these sentences reflects a construction of an idea for the notion of conditioned equality used to justify the result produced by the rule. This does not mean that the students have understood the equivalence between the two equations in terms of preservation of the same truth set. Such understanding is the aim of the whole activity and its achievement requires several conceptual developments. First of all, students have to understand that the values of x for which x^2+2 is equal to 2x+3 are the same for which x^2-2x-1 is equal to 0.

The following task was assigned to favour exploring such quantitative relations:

Task: *Make a hypothesis about the relationship among the three polynomials x^2+2; 2x+3; x^2-2x-1 imagining what you could observe if you represented them on the algebraic line and if you dragged x. Use algebraic line to verify your hypothesis.*

A posteriori, we realized that the formulation of this task was misleading since it oriented the students to search for a relation among the three polynomials rather then between couples of terms of the two equations. Some students dragged the variable to explore if there were values of x for which the three polynomials could denote the same value on the line. They verified that such a value does not exist. Even if this exploration was not expected, it proved an important reference to overcome the following misconception, quite common in the students, concerning the equivalence of equations: two equations are equivalent if all their terms are equal for some values of the variable. A new formulation of the task by the experimenters allowed students to focus on couples of terms of the two equations. Exploiting the drag of the variable x they understood that, in order to find values of x for which x^2+2 is equal to 2x+3, it is sufficient to find values of x for which x^2-2x-1 is equal to 0. Subsequently they used the command E=0 to find the irrational roots of the polynomial x^2-2x-1 and to automatically represent them on the line (the student drags x to approximate the polynomial to 0 and the system automatically produces the exact value of the root). Through this experience an idea of equivalent equation begin to emerge. This idea will be consolidated through the exploitation of a new dynamic feedback offered by the system. We note that in the algebraic line environment expressions are represented on the line while equalities are represented in a specific window named “sets” and they are associated to a marker (a little dot) whose colour is managed automatically by the system. The marker is green if, for the current value of the variable on the line, the equality is true and, conversely, it is red if the equality is false. Dragging the variable allowed students to explore the truth of equalities and to construct a meaning for this notion, as shown in the following dialogue.



Student: If I drag x on $1+\sqrt{2}$ and on $1-\sqrt{2}$, the expressions of the first equation belong to the same post-it, namely x^2-2x-1 and 0 are coincident for these values of x.

For the same values of x even x^2+2 and 2x+3 belong to a same post-it.

Student 1: When x is $1-\sqrt{2}$ the two expressions are equal and these [dots] are green. So, since the solution of this equation is $1-\sqrt{2}$ then also for the other equation is the same.

Student 2: and for the other value [$1+\sqrt{2}$] it is true the same

Student 1: yes, for these values the two equations are true

To support the conceptual development necessary to master the notion of truth set of an equation, two other operative and representative possibilities of the algebraic line were exploited: a graphic editor to construct the truth set of an equality and a new feedback of the system to validate it. The graphic editor allows to operate on the line to define a numerical set that the system automatically translates into the formal set language associating it to a coloured marker. We note that the green/red colour of the marker means that the current variable value on the line is/is not an element of the set. As expected, students used this feedback to validate the defined numerical set as truth set of the equation, verifying the green colour accordance between equation marker and set marker during the drag of the variable on the line: “*for the values* $1+\sqrt{2}$ *and* $1-\sqrt{2}$ *the two equations* $x^2+2=2x+3$ *and* $x^2-2x-1=0$ *have the same truth set. In our opinion, the two expressions from one side and the other side of the = sign belong to the same post-it when x assumes the values of their solutions*”.

French experimentation

Let us remind that the French team that experimented activities described in this section was not involved in the development of Alnuset. Therefore, a preliminary step before designing a learning scenario with Alnuset consisted in an analysis of the tool from the usability and acceptability point of view (Tricot et al. 2003). This analysis brought to light main functionalities supposed to enhance learning of functions and equations, notions at the core of the Grade 10 math curriculum: dynamic representation of the relationship between a variable and an expression involving this variable and possibility to articulate different registers of representation of algebraic expressions (Krotoff 2008). In addition, praxeological analysis (Chevallard 1992) of the above mentioned mathematical objects allowed identifying types of tasks and comparing techniques available in Alnuset with institutional techniques identified in the Grade 10 textbook. This analysis shows that while institutional techniques are based on algebraic transformations on algebraic expressions, Alnuset techniques rely on visual observations of expressions (their position on the algebraic line, colour feedback…), and (almost) no algebraic treatment is needed when applying these techniques (Krotoff 2008). Thus, Alnuset seemed to be an appropriate tool to help students develop conceptual understanding of notions of function and equation, without adding difficulties linked to algebraic treatment that many students do not master well enough.

Although the French experiment was designed independently from the Italian one presented above, both experiments shared some didactic goals, in particular conceptual understanding of notions related to the notion of equation: meaning of a letter as variable or as unknown and of the "=" sign, understanding of what a solution of an equation means. Therefore, below we present only activities and results related to these common concerns. Our research goal was both to investigate to what extent the new representation of algebraic expressions provided by Alnuset contributes to the conceptual understanding of the notions at stake, and to study instrumental geneses (Rabardel, 1995) in students when interacting with Alnuset.

The experiment took place in a Grade 10 class with 34 students (15-16 years old), during two sessions lasting 3 hours altogether, held in a computer lab where students worked in pairs on a computer. Their work was framed by worksheets describing tasks and asking questions the students had to answer. Written productions are one kind of gathered data. Moreover, a few student pairs' verbal exchanges were audio recorded and this data provided us with the possibility to carry out case studies, namely as regards studying instrumental genesis in students. Results reported below draw mostly on these case studies.

The first task involving equations was finding solutions of $f(x)=4$, with $f(x)=x^2$, after having studied the function f with Alnuset. The task was intentionally quite simple: the students could either solve the equation algebraically and verify the result with Alnuset, or solve the equation with the tool by dragging x along the algebraic line and looking for values for which x^2 coincides with 4. Both strategies appeared to almost the same extent. However, students who used the exploration strategy to find solutions with Alnuset succeeded better than those who used the tool just to verify the results found by solving the equation algebraically, since these often provided only one, positive, solution. Alnuset turned out to be an efficient tool helping students to overcome their conception $x^2=k^2 \Leftrightarrow x=k$.

The next task, solving the equation $x^2=3x+4$, was proposed to prompt students to use Alnuset technique of dragging x on the line and searching for values for which the equality is true. Indeed, the students did not know yet algebraic techniques for solving such 2nd degree equation. Using the Alnuset technique requires to make sense of the "=" sign as meaning that the two expressions have the same value for some value of x, and thus also to distinguish between a letter standing for a variable and for an unknown. The students were first asked to determine whether 1, –1 and 2 are solutions of the equation. This question was intended to reveal students' conceptions of the notion of solution of an equation. Almost all students succeeded the activity. However, the following dialogue between two students reveals the student's S1 conception of a solution linked to the arithmetic sense of the "=" sign:

S1: You have to find 1. No, 3x+4 must be equal to 1, the solution.

S2: No, you have to put x on 1 and the… what do you call it [*pointing at 3x+4*]… Because x^2 should be equal to… the thing, equation and this isn't the case (Fig. 2a).

S1: But it's the result this [*pointing at 1*].

Indeed, it seems that S1 considers a solution of an equation to be the "result" or the value of the expressions: if 1 is a solution of $x^2=3x+4$, then (x^2=) 3x+4=1. This conception emerged also when the students checked for -1. The student S2 grasped the targeted technique: "*On the other hand, -1 is the solution since f(-1) equals this equals this equals this*" (Fig. 2b), and explains it to S1: "*To find the solutions, you drag x until x^2 and 3x+4 overlap*".

Figure 2. (a) 1 is not a solution since x^2 and 3x+4 do not overlap when x is on 1; (b) –1 is the solution.

The students were then asked to find other solutions of the equation if there are any. This task was much more difficult for the students. Only half of the pairs succeeded it. The main obstacle was the fact that when x=4 (the other solution), the expressions x^2 and 3x+4 went out of the screen. The students did not spontaneously resort to using "tracking" functionality allowing to keep visualising the expressions taking bigger values, which the students had used previously. Teacher's intervention was necessary to remind the availability of this functionality, which helped the students to successfully finish the task. Such observations point to the issue of instrumental genesis in students, which can be a rather long-term process, especially in the case of innovative functionalities such as “tracking” or “E=0” command as we will see in the following example.

Next, the students were asked to find solutions of the equation $x^2=x+3$. This equation has irrational roots, therefore the technique based on dragging x and making the expressions overlap is not efficient anymore. The aim was to introduce the E=0 command allowing to find irrational roots of the expression x^2-x-3 and thus bring the idea of equivalent equations A=B and A-B=0. Most students used first the strategy relying on dragging x on the line and either provided approximate values of solutions (e.g., 2,3 and –1,3) or framed the solutions by integers (e.g., $-2<x<0$ and $2<x<4$). Teacher intervention was necessary to clarify that exact solutions were to be found and suggest using the E=0 command. Students encountered two main difficulties with using this command. The first difficulty was making a link between the expression E(x) they needed to find to be able to solve the given equation of the type A=B (the question intended to guide them was “What equation of the type E(x)=0 allows solving the given equation? Explain.”). The teacher had to state more precisely that Alnuset only provides a tool for solving equations with the right side equal to 0, and that it is then necessary to transform the given equation in a way to have 0 on the right side. Such intervention helped most students to find an adequate expression and use the E=0 command. The other difficulty was linked to the use of the E=0 command. In fact, to solve an equation with Alnuset, one has to use this command as many times as the equation has solutions. Although the students were aware that the equation has two solutions (most of them provided two approximate values at the beginning of the task), they did not think of using the command twice in order to find both solutions, and thus provided only a single solution. This difficulty is linked to the development of a scheme of using the E=0 command, which supposes to anticipate the number of solutions of a given equation and to be aware of the fact that applying the command gives a single solution at a time. This is quite unusual comparing to traditional algebraic techniques.

CONCLUSION

These two experimentations enable a first evaluation of the mediation offered by Alnuset. In both experiments Alnuset was exploited both as a tool to verify already developed conjectures and as a tool to explore algebraic phenomena in order to arise and validate new conjectures. It allows designing learning scenarios with characteristics that are deeply different, according to given contexts (institutional, cultural, social…) and educational goals to be pursued. The two experimentations lasted differently and this allowed to evidence that: (i) the instrumental genesis of the Alnuset instrumental techniques may be quite short for some of them (e.g., using drag mode for determining equivalence of two expressions) and longer for others (e.g., using E=0 command to solve polynomial equations and interpreting associated feedback); (ii) the instrumented techniques can be controlled by mathematical justifications and previous knowledge, correct or not. On the other hand, the French experiment showed that when the previous mathematical knowledge is rather fragile and the students are not very confident with it, resorting to the tool can help them carry out successfully the tasks they would not succeed without using the tool; (iii) the instrumented techniques produce representative dynamic events that can be easily related to algebraic notions and meaning involved in the activity.

Both experiments evidenced the importance of teacher’s role in supporting the development of students’ instrumental genesis at the beginning of the activity with Alnuset. Moreover, the role of the teacher remains very important during the whole activity to orient discussions and considerations about instrumental issues that have to be intertwined with algebraic knowledge involved in the activity.

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