ALLEGORIES IN THE TEACHING AND LEARNING OF MATHEMATICS

Reinert A. Rinvold

Hedmark University College, Norway

Andreas Lorange

NLA College of Teacher Education, Norway

This paper explores how the concept allegory from literature theory can be used in the teaching and learning of mathematics. A cognitive allegory theory is developed in analogy with the metaphor theory of Lakoff and Johnson (1980). The theory differs from the traditional view. For instance an allegory is also a cognitive mapping and not only a narrative. The paper draws upon data from a study of how teacher training students learn the concept of linear congruence equations. The students are given word problems which were translated to congruence equations and later used to solve other word problems.

INTRODUCTION

Researchers like Lakoff, Núñez, Sfard and Presmeg have elaborated the role of metaphors in mathematics and mathematics education, see for instance Lakoff and Núñez (2000), Sfard (1994) and Presmeg (1997). Allegory is another concept from literature theory which so far has been sparsely used in mathematics education. In this paper we suggest that the concept of allegory can be applied to this field. Our contribution is to develop allegory as a part of mathematics education theory, in a way similar to how metaphors have been used in the tradition initiated by Lakoff and Johnson (1980).

METAPHORS AND ALLEGORIES

Traditionally a metaphor is a figure of speech in which a phrase denoting one kind of object or idea is used in place of another. An example is “You are straight on target with your reply.” In this view of metaphors “straight on target” is a figure which means something else. The phrase can be translated to literal speech, for instance “precise and relevant”. In cognitive metaphor theory metaphors are not as in the old traditional view, seen as isolated phrases, but as systems structuring concepts and thought. Such systems map one domain into another such that the target domain inherits structure from the source domain. An example is “argument is war”, Lakoff and Johnson (1980, p. 4).

Warfare → Argument or discussion

Target is a concept from warfare. If an argument is compared to an arrow or a bullet, we can characterize the argument by describing how the arrow aims at the target. But, the metaphorical mapping can also express lots of other things. An example is “The teacher went into a defensive position when faced with critique”. ‘Defensive’ is also part of military jargon, just like ‘targets’ is. In the tradition initiated by Lakoff and Johnson it is stressed that metaphors create or modify abstract concepts. The metaphor “argument is war”, is modifying or giving a special interpretation of what argument is. In other cases metaphors create a complete new concept.

Allegories are similar to metaphors, but have the structure of narratives and are usually more extensive. The New Encyclopædia Britannica has this definition:

> ...allegories are forms of imaginative literature or spoken utterance constructed in such a way that their readers or listeners are encouraged to look for meanings hidden beneath the literal surface of the fiction. A story is told or perhaps enacted whose details when interpreted – are found to correspond to the details of some other system of relation (its hidden, allegorical sense) (Fadiman, 1986, p.110)

Like metaphors, an allegory maps one domain onto another one, but the source domain is a narrative. Different parts of the source narrative are mapped into different parts of the target domain. An example from the Bible is Galatians 4:24, in which the word ‘allegory’ appears in the King James Version of the Bible. Two covenants are compared to the first two sons of Abraham by a freewoman and a bondwoman. An allegory maps objects and persons of a narrative to a more abstract domain. Each woman is mapped to a covenant, and the story told by the Apostle Paul gives flesh and meaning to the rather abstract concepts of a new and an old covenant. Both this allegory of Paul and the parables of Jesus have clear didactical purposes. They are designed by a teacher. These kinds of allegories are the focus of this paper, but of course mathematical ideas and conceptions are the goal, not spiritual ones. The word ‘conception’ is used to avoid non-cognitive interpretations of the alternative word ‘concept’, see Sfard (1991, p. 3) and Rinvold (2007, p. 4). A conception is a cognitive network in which several allegories and metaphors can be nodes. It isn’t uncommon to think that concepts are primarily given by formal definitions. Such definitions are just an aspect of conceptions and not at all a complete description.

We restrict our attention to allegories which include a timeline. This means that the narratives move in time. All the parables of Jesus are like that and so are most narratives. In mathematics education many text problems have the form of narratives. Such problems will be called narrative text problems. In this paper ‘text problem’ will always mean ‘narrative text problem’. On the other hand, narrative is a wider concept than text problem. A narrative is neither necessarily a problem nor given by a text.

Not all narrative text problems are allegories. This is only the case if a narrative text problem is going to represent or create something else, which usually is more abstract. Consider the following text problem: “John was hiking in the mountains. The first day he walked 20 km and the next day 25 km. What is the total distance he walked these two days?” This problem isn’t likely to represent something outside itself. Most students will solve the problem, forget it, and go on to the next one. The following narrative is different:

> Peter has an urn containing balls. On each ball it's written a prime number. The urn may contain more than one ball with the same number. Peter asks Andrew to draw as many balls as he wants. Then Andrew is asked to find the product of the numbers on the drawn balls. When Andrew has told what the product is, Peter starts calculating. After a while he says: "I know which balls you have drawn". How is this possible? What would happen if composite numbers had been written on the balls?

With possible guidance from a teacher, this story can help the students to understand unique factorization in prime numbers and the role of such numbers. Drawing of a ball represents a factor. The information that Peter is able to tell which balls Andrew has drawn, corresponds to uniqueness of prime factorization. The fact that he isn't able to tell the order the balls were drawn, represents the commutative law. The narrative is used to create understanding of an abstract property of numbers. The problem isn't just a problem among many, but may have a lasting effect.

Even if the teacher tells a story intended to be an allegory, the learners don't always understand it in this way. Relating to a constructivist epistemology, it is the learners themselves who develop allegories. An allegory may be idiosyncratic and has elements of individual variation.

METHOD

This paper uses data from a study of how teacher training students learn the mathematical concept of linear congruence equations. The study was conducted in March 2008 by the authors on our own students. The data consist of participant observation of a teaching and learning session and three videotaped and partially transcribed interviews. One of the researchers interviews the teacher of the lesson, who at the same time is the other researcher. Both researchers together interview groups of either two or three students. As part of the sessions, students work together with a text problem. The researchers then ask questions helping the students to describe their reasoning process. The student interviews were conducted two days after the lesson, and the teacher interview the day after that. The transcriptions, descriptions and interpretations of the teacher interview were read by the interviewee, discussed with the researcher and then adjusted. Later, in the process of writing the paper, the teacher sometimes remembered thoughts and events from the lesson which can't directly be read from the data. Such thoughts aren't presented as data, but have without doubt influenced interpretations and directions of the paper.

CONGRUENCE CALCULUS

The lesson is based on several text problems given to the students. The first problems concern week days. Exercise 1 asked them to calculate the weekday of 31st March, given that 1st January was a Tuesday. The teacher gave comments and discussed solutions in between. The students themselves made tables resembling calendars.

0	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30				

After their work the teacher showed them the table above on a blackboard. Then he introduced the signs '≡' and 'mod' for congruent numbers. From 1st January to 31st January is 30 days. He pointed to the numerals 2 and 30 in the table and connected them with a red line. Then the teacher said that 2 and 30 are in the same column and wrote $30 \equiv 2 \pmod 7$. Mathematically this means that 30 and 2 have the same remainder upon division by seven. In other words, the difference between 30 and 2 is an integer multiple of 7. Practically, the meaning is that 30 days from now and 2 days from now differs with a number of integer weeks. The identity was followed by $29 \equiv 1 \pmod 7$ and $31 \equiv 3 \pmod 7$ since 2008 is a leap year and March has 31 days. Finally the teacher wrote

$$30 + 29 + 31 \equiv 2 + 1 + 3 \equiv 6 \pmod 7.$$

The move of six days forwards from a Tuesday gives a Monday, so that is the weekday of 31st March.

NARRATIVE TEXT PROBLEMS

After three other text problems having to do with calculation of week days, the students were given what we call the Duckburg problem:

> A ship arrives at the harbour of Duckburg today, which is a Monday. Then the ship arrives at the harbour every third day. Some days later the ship arrives at Duckburg harbour on a Wednesday (two weekdays later). How many arrivals later can this be?

According to our observations, all students made a table with the weekdays from Monday to Sunday in the first row. There was some variation in the content of the tables, but in some way all students marked the days when the ship arrived. They all discovered the first solution of the problem, and some even found a formula for the number of arrivals when the ship arrives on a Wednesday. The student work was followed up by the teacher in a plenary session. As support for the introduction of congruence equations, he made a protocol for the arrivals of the ship by writing the identities

$$3 \cdot 1 \equiv 3 \pmod 7,\ 3 \cdot 2 \equiv 6 \pmod 7,\ 3 \cdot 3 \equiv 2 \pmod 7,\ 3 \cdot 4 \equiv 5 \pmod 7,\ \ldots$$

He simultaneously said things like "three times four is in the same column as five", referring to the table. Then the teacher related the text problem to the mathematical formulation “which multiplies of three are in the same column as two when divided by seven”. Finally the congruence equation $3x \equiv 2 \pmod 7$ was presented as a translation of the Duckburg text problem. The lesson continued with the demonstration of algebraic techniques for solving the equation. These techniques are part of the motivation for the translation, but we don’t discuss the solving process in the paper.

The Duckburg narrative is built upon the culturally shared concepts of days, weeks and calendars and the well-known phenomenon of ships regularly arriving at harbour cities. The name Duckburg, which is the domicile of the Disney figure Donald Duck, is used to make it clear that we are talking about a fantasy world in which details can be changed. Duckburg is a name which is easy to remember and with positive associations for most students. Also, this cartoon city is placed close to the sea, Grøsfjeld (2007), so arrivals of ships are relevant.

Later in the lesson the students were given the running track problem:

> An athlete runs intervals of 300 m on a 400 meters running track. She starts at the starting line, runs 300 meters and stops. She continues this way. After a while she stops 100 meters after the starting line. How many 300 meter intervals has she run?

The students at first worked with the task themselves using a table. A drawing of the track was introduced afterwards by the teacher in the plenary. He used the drawing to simulate the intervals of the runner. This problem also corresponds to a linear congruence equation, but the situation is sufficiently different from the Duckburg problem to supplement it.

FROM NARRATIVE TO PROTOTYPE

Lakoff and Johnson (1980) claim that usually we place things and phenomena in categories by comparing with a typical or prototypical member. A prototypical bird has wings, is able to fly, lay eggs and has a beak. A picture of a blue jay is used by some dictionaries when defining birds. The blue jay is a candidate for a prototypical bird in countries where this bird is well known. We will use interview data to argue that the Duckburg problem has the potential to be a prototypical text problem for linear congruence equations. The argument is based on the way the Duckburg problem is used by the group of three students to solve the following text problem which also corresponds to a linear congruence equation:

> Oda is sick and has to take a tablet every fifth hour, both day and night, in order to get well. She takes the first tablet at five in the morning. A friend calls her when her watch has just passed one o’clock. Her watch is analogue, that is, has rotating hands. How many tablets has Oda taken? There are several correct answers.

The students work for about twenty minutes with the problem and are then interviewed. In the interview one of the students was passive and seemed to participate only to a restricted degree. The active ones were Kari and Lise. A reason why they used so much time is that the problem is structurally more different from the Duckburg problem than we intended. In particular, Lise mentioned several times in the interview that she was confused because the problem was unclear. In fact, one of the researchers had forgotten to specify that Oda had just taken a tablet when the friend called. However, the students demonstrated understanding of the problem and were able to solve it with the extra constraint when asked to. Some statements by Kari support the claim that the Duckburg problem and some of its structure were used in the solution process. One example appears when Kari and Lise had written the congruence equation $5x \equiv 8 \pmod{12}$ on their sheets. When asked why the right hand side is 8, Kari said:

Kari: I remember when we worked on the problem with the ships which arrived at the harbour, we started with a Monday. Then we were going to find Wednesday, which was two days later, so we would have two there.

The student refers to the Duckburg problem which corresponds to the equation $3x \equiv 2 \pmod{7}$. The numeral 2 on the right hand side corresponds to 2 days later. In analogy, one o'clock is 8 hours later than five. We think this is the reason why the students wrote 8 on the right hand side of the equation $5x \equiv 8 \pmod{12}$. This is supported by another statement from the interview:

Kari: Then we draw a table with 12 columns. We started with the hour she took the first tablet, which was at five o'clock. (...) Then we counted every fifth hour...

The students made the same type of table as in the Duckburg problem. Five was the first column in the tablet case, as Monday was the first in the Duckburg problem. They counted how many hours after the first tablet she takes the next and would have got the same result if the first one was taken at for instance two o'clock. Another argument is this mentioning of the running track case:

Kari: Recognizing the running track task. Then 0 and 12 were the same. It was the starting line. Do we have to start with 0 then? But, now 0 is at 5 o'clock. If she starts at 5 there, then…

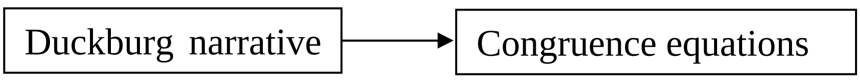
Without doubt, Kari now uses five o'clock as the zero point. The students however, didn't notice a minor difference between the questions in the problems. In the Duckburg problem the question is how many days *after the first* arrival the ship arrives at Wednesday. In the tablet case we asked how many tablets she has taken, *including the first* one. If Oda had taken the first tablet at hospital, and we had asked how many tablets Oda has taken at home, their equation had been correct. Then $5 \cdot 1 \equiv 5$ would have meant that she took the first tablet at home 5 hours after the one at hospital. The identity $5 \cdot 4 \equiv 20 \equiv 8$ would have meant that she took the fourth tablet at home 20 hours after the one at hospital. In some sense the wrong equation is more convincing than $5x \equiv 1 \pmod{12}$, which has $x \equiv 5$ as solution. In the latter case the students could just have put in the numbers 5, 1 and 12 given in the problem, without any understanding.

The students' use of the Duckburg problem and its structure is an argument that the Duckburg problem is on its way to becoming a prototype for a category of narratives. A more thorough study would have been necessary in order to claim with strength that some text problem has been established as a prototype. A possible weakness in our study is that only one student orally indicates this kind of reasoning. However, the students wrote the equation $5x \equiv 8 \pmod{12}$ collaboratively and Kari said that "we worked with the problem". This may indicate that at least Lise also shared her ideas.

ALLEGORIES AND GENERALIZATION

The transformation of a narrative text problem to a prototype for a class of such problems is an important step in giving a problem lasting value in mathematical thinking. This is one aspect of making the special case represent something general. In the Duckburg problem we can change the involved numbers without changing the structure of the narrative. Clearly, there is nothing special in "every third day" or "two weekdays later". The general is represented by the special case. The related "principle of generalization" is investigated in Rinvold (2007). To change the numbers of weekdays from seven to something else is also possible, but needs more imagination because weeks with seven days are so deeply established in our culture.

We think that the narrative of Duckburg has the potential of becoming an allegory for linear congruence equations with one unknown. When the narrative is turned into a prototype, each part of the narrative represents a part of a generalized narrative. For instance "arrives every third day" represents "a tablet every fifth hour" and "runs an interval of 300 meter" in the two other example problems. But, the parts of the Duckburg narrative also represent parts of a formal linear congruence equation. These representations can be made clearer with the help of mappings. In the latter case the source domain is the Duckburg problem and the target domain is the class of congruence equations.



"Every third day" is mapped onto 3x, "two days later" onto 2 and the number of weekdays is mapped onto 7 in the equation $3x \equiv 2 \pmod{7}$.

In the lesson the students were given some context free congruence equations and told how to translate these into Duckburg problems. When given the equation $2x \equiv 3 \pmod{8}$, one of the groups introduced a new weekday and drew a table. They quickly realized that the problem had no solution. With eight columns, steps of two weekdays can't lead to the same place as a change of three weekdays. The students said that it was a cheating exercise since there was no solution. In the beginning of the interview the students were asked about their experience of the lesson.

Kari: When we used the practical situations as starting points, we could in the end see a congruence equation, and then the numbers gave meaning. We could know what 4x really represents. When I recalled the ships, it gave meaning.

This could refer to the equation $4x \equiv 1 \pmod 7$ which was one of the translation problems from the lesson. The formal congruence equation in the beginning seems to give little meaning to the students. The ships were part of the Duckburg problem and are used by the student to refer to that problem. We infer that translation of context free congruence equations into variants of the Duckburg problem was a main source of the meaning which emerged.

ALLEGORIES AND THE SOLVING OF TEXT PROBLEMS

When solving text problems allegories can be intermediate stages between the given narrative and a mathematical model.

Text problem → Allegory → Mathematical model

The idea of prototypes means that new text problems given to the students won't be directly mapped to a mathematical model, but first to a prototype like the Duckburg problem.

Narrative → Prototypical narrative → Mathematical model

A prototypical narrative in the learning of mathematics is a mathematized narrative. The given text problem or narrative also has to be mathematized to some degree in order to be mapped onto the prototype.

A crucial question is which qualities these mappings have for the students. Certainly, their versions of the mappings can differ from the intentions of the teacher. At best, the mappings reflect the mathematical structures effectively, but the mappings may also be based on superficial aspects of the text problems. Clement (1982) identified a syntactic and a semantic way of thinking when students tried to solve word problems for equations. The syntactic variant consists of a word by word translation of the text problem to algebraic language. Another kind of syntactic translation is based on possibly superficial similarities between the text problem and other text problems known to give a specific mathematical model. When working with the tablet problem, the student Kari made the following utterance:

Kari: We thought that it was 5x because it was every fifth hour she had to take the tablet and that was because the ship arrived every fifth day.

We see that the phrase "every fifth" appears in both problems. This may be interpreted as a sign that the student compared the appearance of words in the two problems. However, the ship in the Duckburg problem arrives every third day, not every fifth. If fact, some of the students, certainly including Kari, during the lesson also solved a variation of the Duckburg problem corresponding to the congruence equation 5x ≡ 3 (mod 7). This at least indicates that she compared with the appropriate version. Another argument that the translation has a semantic flavour is that in the lesson Kari, together with a group of students, generalized the Duckburg problem. They investigated what happens when the interval between arrivals or the number of weekdays ahead were changed.

Part of our theoretical thinking is that allegories are one of the sources for semantic meaning. When one text problem has been transformed to an allegory, the comparison with other text problems will no longer be just syntactical. Clement's semantic way of thinking means a mapping from a narrative or text problem to a mathematized version of the problem, and then a mapping to the congruence equation.

Narrative → Mathematized narrative → Mathematical model

This is similar to the mappings of Parzysz (1999):

Real situation → Pseudo-contrete model → Mathematical model

In the case of the Duckburg problem, the emphasis on the table, the columns and the introduction of mathematical signs means that the teacher intended to support the development of a mathematical structuring of the narrative. In the lesson the teacher explicitly sets up a mapping from the pseudo-concrete model to the congruence equation. For instance "the numbers which are multiples of three" was translated to '3*x*'. The text problem is still a real situation for the student, but mathematical language has been introduced in order to change the students' interpretation of the situation, making the translation to formal mathematics precise and smooth.

The term "real situation" is not as clear as commonsense language may suggest. We interpret 'real' as "real for the student", as in RME, the Dutch approach to mathematics education (see van den Heuvel-Panhuizen, 2003). One point is that 'real' doesn't have to mean practical or related to everyday life. The student Kari used the phrase 'practical situation' referring to the Duckburg problem, but imagined situations such as weeks with eight days can also be 'real'. A situation isn't something objective, but an experienced or imagined phenomenon. A narrative may create a situation in the mind of the student, but the process of mathematization also has a role in shaping the situation for the student. The degree of mathematization and semantic interpretation decides the quality of the mappings.

QUESTIONS FOR RESEARCH

This paper introduces the idea of cognitive allegories in mathematics education and supports this by discussions based on one limited empirical study. Obviously there is a need for more studies to establish that the concept of allegories is a fruitful one for the use of narratives and text problems for conceptual learning in mathematics. It is necessary to have more thorough studies to establish that students transform introduced narratives into prototypes and allegories and how they do this. Other mathematical concepts and other potential allegories have to be studied. We also need to develop criteria for the design of such narratives. Another interesting task is to study how several allegories can be used for the same concept. We think that a single allegory usually isn't enough to develop a rich intuition. In our study the running track problem is a candidate for a complementing allegory, but only very limited evidence for this can be inferred from the data.

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