FROM AREA TO NUMBER THEORY: A CASE STUDY

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In this paper we examine the way two 10th graders cope with a tiling problem that involves elementary concepts of number theory (more specifically linear Diophantine equations) in the geometrical context of a rectangle's area. The students' problem solving process is considered from two perspectives: the interplay between different approaches relevant to the conceptual backdrop of the task and the range of executive control skills showed by the students. Finally the issue of the setting of modeling problem solving situations into number theory tasks is also commented.

INTRODUCTION

Modeling problem solving situations into generalization tasks related to number theory is useful for learning mathematics and includes two stages: modeling and solving the number theory tasks that emerge. On the one hand, solving generalization tasks dealing with number theory serves as a tool for developing patterns, as a vehicle towards appreciation of structure, as a gateway to algebra, as a rich domain for investigating and conjecturing at any level of experience (Zazkis, 2007). However despite of their significance number theory related concepts are not sufficiently featured in mathematics education. Consequently many issues related to the structure of natural numbers and the relationships among numbers are not well grasped by learners (Sinclair, Zazkis & Liljedahl, 2004). On the other hand according to Mamona-Downs and Papadopoulos (2006) when students have an accumulated experience on problem solving they can affect changes in approach and are able to take advantage of overt structural features appearing within the task environment. Moreover they can show a deeper understanding of the nature of mathematical generalizations. In their work which lasted 3 years they followed some students from the 5th grade up to their 7th with emphasis on problem solving techniques relevant to area. Three years later we follow two of these students who currently attend the 10th grade (15 years old) during their effort to cope with a non-standard task concerning problem solving activity relevant to elementary number theory concepts. The case is interesting since it displays executive control skills related to the way the students proceed when they have to work on a new domain and to the handling and establishment of a 'model' that could lead to the generalization. This is why we try to explore in this paper the interplay of the students among different approaches during their problem solving path towards generalization and at the same time to refer to the actions of the students concerning decision making and executive control. In the next section we present the task and describe the students' background. After that in the next two sections we present the problem solving approaches followed by our students (Katerina for the first, Nikos for the second). These are followed by a discussion section trying to shed light on these two axes (i.e., the interplay and the control issues) and finally the conclusions.

DESCRIPTION OF THE STUDY AND STUDENTS' BACKGROUND

Katerina and Nikos were 10^{th} graders and they had participated in an earlier study conducted by Mamona-Downs and Papadopoulos (2006) aiming to explore and enhance the students' comprehension of the concept of area with an emphasis on problem solving techniques for the estimation of the area of irregular shapes. Their participation in this resulted in the creation of a "tool-bag" of available techniques as well as in an accumulated experience on the usage of these techniques. The conceptual framework now mainly lies in number theory. However in the official curricula (for 10^{th} graders in Greece) the only reference to number theory concepts is a tiny one commenting the divisibility rules for the numbers 2, 3, 5, 9, 10.

This is the problem we posed to the students:

> Which of the rectangles below could be covered completely using an integral number of tiles each of dimensions 5cm by 7cm but without breaking any tile?
>
> Rectangle A: dimensions 30cm by 42cm
>
> Rectangle B: dimensions 30cm by 40cm
>
> Rectangle C: dimensions 23cm by 35cm
>
> Rectangle D: dimensions 26cm by 35cm.
>
> For each rectangle that could be covered according to the above condition show how the tiles would be placed inside the rectangle.
>
> Now, we want to cover a rectangle with an integer number of (rectangular) tiles. Each tile is of dimensions 5cm by 7cm. What could be the possible dimensions of the rectangle?

The mathematical problem is: define a set of necessary and sufficient conditions on a, b so that there exists a rectangle of dimensions a by b, that can be covered completely with tiles of dimensions 5 by 7. Look at the side of length a: if there are s tiles that touch it with the side of length 5 and k tiles that touch it with the side of length 7, then $a= 5s+7k$. The same reasoning applied to b gives $b=5s'+7k'$, where s, k, s', k', are non negative integers. Now if c denotes the total number of tiles used then the area ab of the rectangle should be 35c. Therefore 35 divides ab. Thus, there are three cases: i) 35 divides a, ii) 35 divides b, or iii) non of the previous, but since 35 divides ab, 7 must divides a and 5 divides b (or vice versa). Consequently, a and b should satisfy one of the following necessary conditions: i) $a = 35 m$, $b=5s'+7k'$, ii) $b=35n$, $a= 5s+7k$ ii) $a=7q$, $b=5t$ (or vice versa). It easy then to be shown, that these conditions are also sufficient. Thus, even though the context of the task seems to be geometrical with its relevance to area, however a crucial aspect in solving the task is the usage of a Diophantine linear equation $ax+by=c$ where the unknowns x and y are allowed to take only natural numbers as solutions. The task consists of two parts. In the first part four rectangles have been carefully selected to help the solver when finishing the first part to be able to reach the generalization asked in the second part.

The problem solving session lasted one hour, without any intervention from the researchers, and the students were asked to vocalize their thoughts while performing the task (for thinking aloud protocol and protocol analysis, see Schoenfeld, 1985). Protocol analysis gathered in non-intervention problem-solving session is considered especially appropriate for documenting the presence or absence of executive control decisions in problem solving and demonstrating the consequences of those executive decisions (Schoenfeld, 1985). The students' effort was tape-recorded, transcribed, and translated from Greek into English for the purpose of the paper.

THE FIRST PROBLEM SOLVING APPROACH - KATERINA

Katerina's first criterion for deciding whether the four rectangles can be covered completely by the tile was based on whether the dimensions of the four rectangles were multiples of the dimensions of the tile. This is why her answer was positive only for the rectangle A (since 30=5*6 and 42=7*6) and negative for the remaining three ones. She used the quotient of their areas (E1/E2, E1 the area of rectangle A and E2 the area of the tile) as a way to determine the number of the tiles required for the covering and not as a criterion to decide whether the tiling is possible). She tried then (according to the task) to show how the tiles will be placed inside the rectangle. The visual aspect of this action made the student to realize her mistake and to re-examine the four rectangles:

K.1.23. The tiles could be placed in any orientation in the interior of the big rectangle.

K.1.24. It is not necessary to be placed all of them in a similar orientation.

After that she verified that the rectangle A could be covered according to the task's statement. For the rectangle B she worked with an interplay between an arithmetical and geometrical-visual approach and she realized that the case of tiles with different orientation could mean that she could work with an 'equation' since she was not able to proceed geometrically. Now, it is the first time a linear combination is involved:

K.1.37. It could be ….. 5x+7y=30

K.1.38. It must be a rectangle with length of 30cm and this has to be expressed with tiles of length 5cm and 7cm.

She was not able to express her thought using proper mathematical terms. Her intention was to say that this equation did not have integer solutions (the case for an unknown to be equal with zero is excluded). So she decided to use terms such as 'round numbers' to show that it is needed for x and y to be integer numbers:

K.1.42. However this case is not possible… (*the above mentioned equation)*

K.1.43. We could not expect to have 'round' numbers for x and y.

For the rectangle C she decided to rely on the question whether the length of the side of the rectangle could be written as a linear combination of the dimensions of the tile. The lack of relevant knowledge on this domain provoked a certain technique for overcoming this difficulty. She worked with successive multiples of 7 plus the remainder (expressed in multiples of 5). She followed the same line of thought for the rectangle D. The criterion of the linear combination was already established and by the technique of the successive multiples she founded that:

K.1.67. For the side of 26cm it is necessary to have 3 tiles of length 7cm and 1 tile of 5cm.

Immediately she turned to the visualization in order to verify that indeed this can be done, working independently on each dimension of the rectangle D (Fig. 1, left).

For the second part of the task she started with two steps that according to her opinion could help her:

K.1.74. I will use drawings because it seems to me easier in that way

K.1.76. How could I use the findings of the first part of the task?

She rejects the condition of E1 being an integer multiple of E2 as the unique criterion since:

K.1.87. …it might be necessary for a tile (or some tiles) to be split.

Her model for finding the possible dimensions of any rectangle that could be covered by tiling using an area unit (tile) with dimensions 5 by 7 includes two cases exploiting her previous findings of the first part of the task.

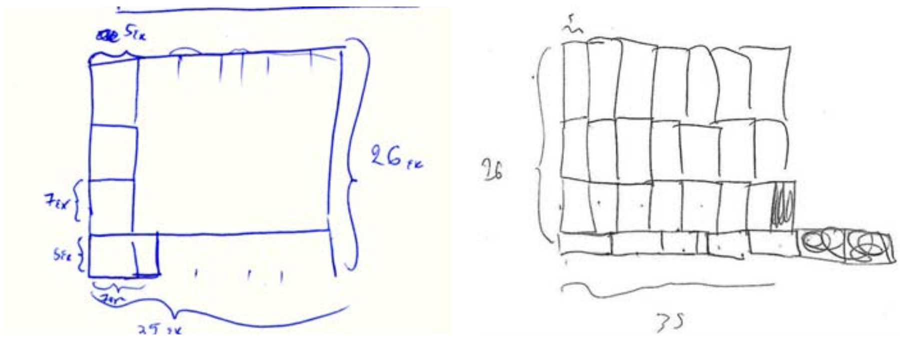

Fig.1 Katerina's (left) and Nikos's (right) visual approach on rectangle D

So, in the first case:

K.1.92. If all the tiles are oriented uniformly then the asked dimensions of the rectangle could be multiples of 5 or 7.

K.1.93. I will make a draw

K.1.94. It is a shape whose length is multiple of 7 and its width multiple of 5.

The second case resulted mainly as a consequence of the rectangle D and two conditions must be satisfied: one side must be multiple of the Least Common Multiple of the dimensions of the tile and the second dimension linear combination of them.

K.1.101. Length must be common multiple of 7 and 5 whereas width must be sum of tiles that are oriented some of them horizontally and some vertically.

She tried then to refine her model asking for a rule that governs the common multiples of 5 and 7 (i.e., of 35). For the number 5 she knew the divisibility rule (the last digit must be 0 or 5). However she could not give any rule for the 7 or the 35. Finally she concluded with a recapitulation of her model trying to describe in a more formal way the second case of the model:

K.1.110. The rectangle in the second case should have one of its dimensions common multiple of both 5 and 7 and the other one sum of multiples of 5 and 7 at the same time.

THE SECOND PROBLEM SOLVING APPROACH - NIKOS

Nikos's first step was to interpret the statement of the problem in terms of conditions for the correct tiling: a) there is a rectangular region that has to be covered and b) the tile is a structural element of the task:

N.1.5. It means that each rectangle must be covered and for the measurement I must use an integer number of tiles

N.1.6. So we could consider this rectangle of 5 by 7 as a measurement unit

In his work and for each one of the four rectangles we can distinguish a concrete line of thought. For the rectangle A, his criterion was (as in Katerina's case) the proportionality of the sides, i.e. whether the dimensions of the rectangle were multiples of the dimensions of the tile. We have to mention here that his way of reading the task was non-linear in the sense that he did not follow the instructions of the task in the given order. Thus, he did not initially give answers for all the rectangles but after deciding for each rectangle, he proceeded to the specification of the way the tiles could be placed in the rectangle. In case there was not proportionality among the lengths of the sides of the rectangle and the tile -as it happened in the rectangle B- he used the criterion of E1/E2 as a way to ensure a negative answer. This quotient was not an integer number and this meant that there could not be coverage according to the task's statement. As he explained:

N.1.20. Because the ratio of their areas is not an integer

Now, in the rectangle C, the E1/E2 was an integer but the dimensions were not proportional. It is interesting the fact that his decision about E1/E2 is justified by the fact that E2(=35cm^2) is a factor of E1(=23*35), a relationship often overlooked even by pre-service elementary school teachers (Zazkis & Campbell, 1996). In their study and in an analogous quotient, teachers first calculated the product and then divided. At that point, Nikos asked for the linear combination that satisfies one of the dimensions since the second is multiple of 5:

N.1.24. When the area is 23 by 35, then obviously this product is divided by 35 which is the area of the unit (*tile*)

N.1.27. The point is **the way** the tiles must be placed

N.1.29. We could have 3*7+2, 2*7+9

N.1.34. 5+5+5+8, 4*5+3,….

N.1.35. For the 23 cm I can't make any **combination** of 5s and 7s.

In the rectangle D, he applied directly the rule of the linear combination that could satisfy the side of 26cm since the other one (35cm) was multiple of 5 (Fig.1, right). Trying to describe how the tiling will take place he worked initially independently on each side. However the way the tiles will be placed in one dimension affects the way the tiles will be placed in the second. This made him to turn towards a consideration of both dimensions at the same time. Despite this method could be considered adequate for him to give an answer for each rectangle, he preferred to re-check all the given rectangles, to verify his answers before making his final decision.

For the second part of the task he started with an impressive conjecture:

N.1.83. Obviously, if we want to cover a rectangle with this specific unit of dimensions 5 by 7, then the rectangle's sides must be the sum of multiples of 5 and 7 at the same time.

N.1.84. The case of 0*5 and 0*7 must be included in this.

However he still considers the two dimensions separately. Trying to figure out what would be the general case for the asked dimensions of the rectangle he created some arithmetical examples, fulfilling the need for linear combination for each dimension, without considering the fact that there is an interrelationship among the two dimensions since the area of the rectangle must be a multiple of 35:

N.1.102. We could say that a=5x+7y (where 'a' is one of the rectangle's dimensions)

N.1.103. and similarly b=5z+7w

N.1.104. The product of these dimensions a and b will be the area

N.1.105. I can choose for a and b any sum of multiples. For example, a=5+14=19, b=15+28=43. So, the area is 19*43

N.1.106. However in that case I have for the area a number that is not divided by 35.

N.1.107. So, 35 must divide the product a*b which is the area of the rectangle.

N.1.112. Thus, a=5x+7y, b=5z+7w and the quotient ab/35 must be an integer.

Trying to establish a model that would describe all the possible cases he was also influenced by the four rectangles of the first part of the task. He decided that his model would include two types of rectangles:

N.1.141. The first type concerns rectangles with one side multiple of 5 and the other multiple of 7. So, a=5x and b=7y, which is a=5x+0*7 and similarly b=0*5+7y.

N.1.142. Consequently the area of such a rectangle divided by 35 gives an integer number as quotient.

N.1.154. And it is in accordance with the general form I conjectured earlier

For the second type he decided that:

N.1.159. One of the rectangle's side will be a sum of multiples of 5 and 7 at the same time

N.1.160. whereas the second side will be a multiple of 35

N.1.171. that is a=5x+7y and b=35z

N.1.172. I think that these latter conditions form the most general form for the dimensions of any rectangle able to be covered with rectangular tiles 5 by 7.

After that, Nikos applied this most general form for each of the four rectangles examined in the first part to check the validity of this form. Furthermore he made clear that the first type of rectangles could be incorporated in the second:

N.1.188. …to incorporate the first type which essentially is a special case in the second type which is more general..

Finally Nikos proceeded to a refinement of his model determining the circumstances that do not allow a rectangle to be covered according to the task giving a certain counterexample:

N.1.213. The second side must be always multiple of 35 and it can be constructed using either 5s or 7s.

N.1.218. This is the only solution because 35 is the Least Common Multiple of 5 and 7

N.1.219. This means that it is not possible to have a rectangle for which both its dimensions are linear combinations of 5s and 7s.

N.1.220. When I say that a is a linear combination of 5s and 7s, I mean that a=5x+7y but not a multiple of 5 or 7.

DISCUSSION

In relevance to our research questions we could make some comments on our fieldwork.

1. Interplay among differing modes of thinking

During their attempts to solve the problem the students worked in tandem with two pairs of modes. The first pair included the arithmetical mode and visualization. Both students started arithmetically even though the context of the task was relevant to area that is geometrical. Katerina from the very beginning used the visual aspect as a tool. She started arithmetically but when she was unable to proceed with numbers she preferred to make drawings that would help her (K.1.74). In the same spirit some times she moved from the visual context to algebra. At some point she clarified that the tiles could be posed not necessarily with the same orientation. However she was not able to proceed geometrically and she preferred to turn to algebra asking for an equation (K.1.37). Nikos did not choose to work with this pair of modes. He mainly worked arithmetically and he turned to the visual aspect only to show the way the tiles could be placed in the interior of the four rectangles in the first part of the task. The second pair of modes has to do with the way students dealt with the dimensions of each rectangle. Working with the first mode dimensions were considered by the students separately as two unconnected objects (arithmetical mode). Thus, they made calculations (they summed, multiplied, divided) to determine the way the tiles should be placed in one dimension. In the second mode the dimensions were interrelated (geometrical mode, relevant to area). The fact is that the way the tiles will be placed in the first dimension influences the way the tiles will be placed in the second dimension. Working independently in two dimensions does not guarantee that the total area of the rectangle will be integer multiple of 35 which is the tile’s area. Both students made successive movements between these two modes. Their initial approach was to work separately for each dimension and only then they made the connection about the interrelation of the two dimensions. For example in Nikos's work (N.1.102-N.1.112) it is clear that his working on the two dimensions separately resulted in a rectangle that could not be covered with integer number of tiles since its area was not multiple of 35.

As a result of this interplay emerges -for Nikos in particular- the issue of putting forward a set of conditions (N.1.112) that are evidently realized as being *necessary* and later an equivalent set of conditions (N.1.172) that are seen as *sufficient* (because the covering of the relevant rectangles can be explicitly constructed).

2. Executive control and decision making issues

The students realized many actions that indicate interesting executive control and decision making skills. Katerina rejected her initial approach which was based only on the criterion of proportionality among the rectangle's and the tile's dimensions. This was because her turn to visualization made her to realize that it was not necessary for the tiles to be placed in a uniform orientation. This turn seemed to be in practice an important act of control. The task's statement did not give any direction concerning the way the tiles could be placed inside the rectangle. It was up to her to interpret correctly the statement. Later when she tried to solve the Diophantine equation she applied the technique of the successive multiples. According to this technique if one has to solve the equation ax+by=c starts with positive multiples of a and then examines whether c minus ax is multiple of b or vice versa (i.e., one starts with multiples of b). This is an act of control since the solving of the equation was dealing with the task's limitation to use an integer number of tiles without breaking any of them. When she decided to deal with the second part of the task her first thought was to use her previous results (K.1.76). Moreover, an important act of control was the 'model' she proposed for estimating the possible dimensions of any rectangle that could be covered with an integer number of tiles according to the statement of the task (K.1.92, K.1.110). She exploited her previous findings (the four rectangles of the first part), and progressively she established this 'model' checking step by step its accordance with these rectangles as also with examples generated by herself. The choice of examples is especially important since not every example facilitates a successful generalization. Nikos also made an analogous proposition of a 'model'. He was also based on the four rectangles of the first part of the task. The steps followed by his line of thought reveal presence of control: First look if there is proportionality among the dimensions. See also whether E1/E2 is not an integer. This means that your answer has to be negative. It is not necessary always to make the long division E1/E2. Instead, see whether E2 is factor of the E1(N.1.24). Now if sides are not proportional and E1/E2 is an integer, then construct the Diophantine equation and apply a strategy to find integer solutions. He also used to check always the consistency of his generalization model against particular examples and this is important. The continuous checking of their steps that both students showed is especially significant as an act of control since students checking is not usually part of the algebraic thinking of the students when they make generalizations (Lee and Wheeler, 1987). A capable problem solver recognizes a correct approach and insists on it. This evaluation of a specific approach could also be considered as an act of control. Nikos recognized the applicability of the linear combination and he used it to check the plausibility of his answers always according to task conditions (N.1.154). This often turn to the tasks' statement was a common pattern for both students. However, perhaps the most important act of control of both students was their effort to refine their model regardless of whether they succeeded. Katerina tried without success to achieve a condition for the second side to be common multiple of 5 and 7. Nikos however did manage to refine his 'model' determining whether it is impossible for a rectangle to be covered according to the task's requirements (N.1. 219). Such an asking for a counterexample actually is an important act of control.

CONCLUSIONS

According to Douady and Parzysz (1998) when a problem allows the solver to move between different modes during the problem solving process then an interplay between these different modes is caused. They claim that the effort of the solver to reach the solution results to the relations of these modes as well as to the usage of some tools that belong to each of them. Additionally "...this interaction provides new questions, conjectures, solving strategies, by appealing to tools or techniques whose relevance was not predictable under the initial formulation..." (p. 176). Both of our students were able to apply this interplay among two pairs of modes. In the first pair (arithmetical-visual) this interplay was used as a way that allowed overcoming difficulties about how to proceed or for verifying or checking the validity of an argument. In the second pair of modes the one mode (arithmetical, working on one dimension) was indicative of a surface understanding of the structural elements of the task. However it seemed that finally the students did show a deeper understanding of these elements through the other mode considering both dimensions at the same time (geometric, interrelated dimensions).

'Executive control' and 'decision making' constitute in general the issue of control in problem solving. Executive control is concerned with the solver's evaluation of the status of his/her current working vis-à-vis the solver's aims (Schoenfeld, 1985). In general, this requires mature deliberation in projecting the potential of the present line of thought, married with an anticipation how this might fit in with the system suggested from the task. In our study and despite their age, these 15-years old students showed considerable control skills in relation to the task's requirements on the one hand and the specification of the 'model' they proposed for solving the task on the other. The existed experience enabled students becoming capable to make generalizations.

Concluding we could refer to some final remarks that emphasize the significance of our results. It is common thesis that the task design is a crucial parameter for teaching and learning algebra at every level. So, in reference to our work, we could claim that the setting of modelling problem solving situations into number theory tasks allows students to:

5. transfer knowledge from one domain to another during their successful interplay among different modes of thinking (algebraic thinking and geometrical one).

6. construct and propose a 'model' that possibly describes the situation and facilitates the generalization

7. generate examples that check the consistency of their model, and

8. generate counterexamples that result to the refinement of the proposed 'model'.

Obviously it would be an exaggeration for these conclusions to be generalized since we dealt with two students and this study could be better considered as a case study. However these finding were encouraging enough to call for a design of a future research on these aspects of problem solving.

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