STRUCTURE OF ALGEBRAIC COMPETENCIES

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This paper reports a research study that aims at understanding interrelationships between algebraic abilities. Theoretical considerations drawn from the literature suggest various interconnections. To gain empirical evidence a test was developed and the findings analyzed by fitting different statistical models.

INTRODUCTION

Ideally, algebra lessons lead students to develop a profound understanding of algebraic concepts and the ability to see algebra as a central and connected branch of mathematics and the ability to apply algebra to a wide range of topics. If this happens, then students can be said to have a high algebraic competency. Even with this aim in mind, it is not clear how to design algebra courses. There are many approaches to the teaching of algebra (see e.g. Bednarz et al. 1996) and they obviously differ in the algebraic concepts that are given priority. The field of algebraic concepts is very broad, e.g. mastering the concept of an equation is a long process in which various aspects of the equation concepts are learned and they all interact with other algebraic concepts. To help in planning the algebraic learning process, it would thus be useful to gain more insight into the inner structure and dependencies of these algebraic concepts.

Such insight can be expected from empirical studies of various designs. Interpretative studies are valuable and some have been performed, especially as they allow to link theory and observations. However, they usually focus on a small number of students and it often remains unclear, how representative they are. Quantitative studies, on the other hand, often lack a deeper connection to theories.

The quantitative study reported here tries to apply advanced statistical models on a test that was developed to reflect certain theoretical assertions about the learning of algebra. In this paper, only results from a single use of the test are reported but this study is part of a larger research project that will collect longitudinal data as well.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

Algebra deals with a lot of objects, including numbers, variables, expressions, functions and relations, and each of these can play many different roles. School algebra thus is composed of many ingredients. Several theories have been developed that give some structure to this large field and we will mention some of them that were used implicitly in our study.

Variables play the central role in our investigation because they are a link between most of the other objects mentioned. Variables are used in many different ways in algebra. Küchemann (1979) gave six ways of using variables. From the perspective of integrating these modes of variable usage into a scheme we found that these modes, although useful in explaining students results, are bit unhandy.,When looking at algebra problems from textbooks we found that his “Letter ignored” is not of great importance and test items regarding it seem always a bit artificial. Moreover, it may be subsumed to the aspect of a variable as generalized number. The use of a variable as a reference to a non-arithmetical object “Letter as object” is (which restricts itself to standard school algebra) an important misconception that is viable only in a very limited subset of algebra. As a misconception it should not be included into the structure of abilities that are to be mastered by the students. Malle (1993) gave a short list of three aspects which proved a bit coarse when classifying textbook problems and test items. A synthesis of these approaches that works well for the classification of the role of variables in different problems turned out to be very similar to the one found by Drijvers (2003) in his empirical study, see below. It is worth to make explicit the operations that are linked with the different roles of variables. This shall emphasize the fact that the role of a variable is not only determined by the algebraic context but also by the subject working with it, e.g. the x in $2x+1=4$ may be viewed as an unknown which is to be determined or as a placeholder were one can insert numbers or expressions.

- Placeholder P (operation: substitute (not only numbers but general expressions))
- Unknown U (operation: determine)
- General number G (undetermined; operation: expressing relations)
    - Ga: General number used in analyzing expressions
    - Gm: General number used for modeling (describing)
- Variable as changing quantity V (operation: change the value)
    - Vi: independent variable (operation: change at will)
    - Vd: dependent variable (operation: observe change)
    - Vr: variable in a relation without predetermination what variables is changed independently as in Ohm’s law $U=RI$.
- Variable as a symbolic element of the symbolic algebraic calculus: C (i.e. operation: use as structure-less object in symbolic manipulations)

Different researchers have advocated the point of view that mathematical objects are constructed from operations (Sfard 1991, Dubinsky 1991, Gray & Tall 1994). While the theories of these authors differ in detail, the broad picture seems similar and naturally explains e.g. the creation of symbolic expressions as encapsulated calculation sequences. It is not as clear to which processes the concept of a variable is linked. Therefore, we associated the above mentioned processes to each aspect of variables. Obviously, different operations lead to different objects, but nevertheless, mathematicians look at variable as a single concept which can be used under different aspects. It is therefore interesting to note that the operation of substitution has tight relations to all the other operations except those operations associated with the last aspect of the above list. We therefore formulate the hollowing hypothesis:

Substitution is a central operation in algebra and the competence to use it properly is at the heart of algebra in the sense that it makes other operations easy as well, with the exception of the symbolic calculus aspect. Put in more technical language, this states that the ability to use substitutions should be a good indicator variable for performance in other algebraic tasks.

Checking the validity of this hypothesis is one of our research questions. The next question is much more open: To what extent do these aspects of ‚variable' depend on each other?

METHODOLOGY

There exist many tests for algebraic achievement but most items test syntactic term rewriting or formal equation solving capabilities. Far fewer test items exist that assess algebraic understanding and algebraic concepts developed by the students. A notable exception is Küchemann's work in the late 70s and early 80s. For this study we developed a new test that is somewhat in the spirit of Küchemann and uses many of his items, but most items were developed to reflect the various aspects of variables described above. In addition, there were test items on the relation between equations and functions.

The study was conducted at the beginning of grade 11 (age approximately 16 years) of a German high school (Gymnasium).There were 141 students from six classrooms in the study. Unlike most other German schools this particular high school starts at grade 11 and thus collects students who were recently at a large number of different schools. Although this sample is not representative of German students, it can be expected to span the breadth of the population better than samples from classes that had the homogenizing effect of a common school culture. However, the mean achievement level is supposed to be below that of an average grade 11 high school.

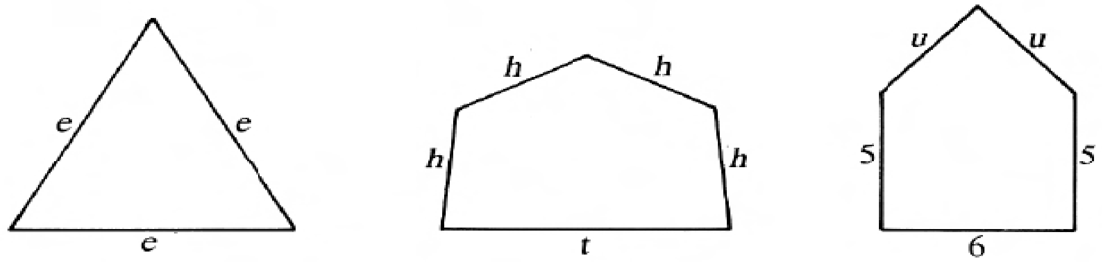
The test was compiled for this study but most of the items had been used in our research group before. The test consists of 43 items, two of which are multiple choice items, while the others ask for a free form response. The answers were rated on a point scale as the following example of a rating rule indicated:

> Item 2a (from Küchemann 1979): Give a short answer and explanation: What is greater? n+2 or 2n?
> 0 Points= no response; false response without argumentation
> 1 Points= example; some explanation; wrong answer with detailed explanation
> 2 Points= example with explanation; detailed explanation without case distinction
> 3 Points= almost correct with case distinction
> 4 Points= completely correct

Some examples of the test items are shown below; their association to aspects of variable' are shown in square brackets:

Item 4: (based on Küchemann 1979) Let r be the number of rolls and c the number of croissants bought at a bakery. A roll costs 30ct, a croissant is 70ct.
a) What is the meaning of $30r+70c$? [G]
b) How many parts have been bought all together? [G]

Item 6a,b,c (from Küchemann 1979): Work out the circumference of the following figures:



[G]

Item 9: a) Assume that the equation $a=b+3$ always holds. What happens to a if b is increased by 2? [V] (from Küchemann 1979)

c) Assume that the equation $a=2b+3$ always holds. What happens to b if a is increased by 2? [V]

Item 13: It is known that $x=6$ is a solution of $(x+1)^3 + x = 349$. How then can one get a solution of $(5x+1)^3 + 5x = 349$? [G] (from Küchemann 1979)

Item 14: Simplify the following expressions a) $(a-3)^2 - a^2$ b) $(x-x^3)\cdot(x+x^3)$ c) $\sqrt{36+4a^2}$ d) $\frac{1}{n}-\frac{1}{n+1}$ [C]

Item 16: Given the examples $7\cdot9=8^2-1$ and $11\cdot13=12^2-1$, formulate a general rule and justify it. [G]

Item 17: A function is defined by: $f(x)=x^3-2$. Determine

a) $f(2)=$ b) $f(y)=$ c) $f(x+1)=$ d) $x\cdot f(x)=$

[P]

Item 19: What must be substitute for x in the expression $2(x^2-1)$ to obtain the desired result? [P]

Desired Result	Substitute x=….
6	
-2	
$2((a+1)^2-1)$	
$2(b^2+2b)$	

The test items were classified by the aspects of variables they involve and by the relevance of the abilities to handle functions (Fun), relations (Rel), syntactical expression manipulation (Syn), working with unknowns (Unk), handling substitutions (Sub) and translating between algebra and geometry (Geo). Of course, this classification is build upon assumptions about typical solution strategies.

Besides more traditional statistical methods, this study uses structural equational modeling as a tool to model dependencies. While this technique is frequently used in many empirical sciences, it seems that its use in the mathematics education community not as widespread and I know of no application of this technique to gain insight into concepts of algebra. However, I believe that this statistical tool is appropriate here, because it allows us to work with hidden variables that cannot be observed directly (e.g. the person's understanding of a variable as a general number) and to model relations among latent and observed variables.

RESULTS AND INTERPRETATION

The test contained several items developed and used by Küchemann 30 years ago. Despite the passage of time, our results were very similar, thereby underpinning the validity of his study. The order of empirical difficulty of the items turned out to be precisely the same as that found by Küchemann. Also the percentage of students that solved the items were remarkable close (despite the fact that we tested 16 year old students while Küchemann tested 14 year olds), with one interesting difference regarding the 'letter as object' aspect. We found Item 6a was solved by 74% while Küchemann found 94% (for 6b and 6c we found 74%, 58%, Küchemann found 68%, 64%). These numbers become interesting when combining with the result that item 4a was solved only by 14% and 4b only by 7%. Most students that failed on 4a showed a clear object interpretation reading $30r+70c$ to mean 30 rolls and 70 croissants. However, many more students were able to solve 6a and 6b, which are described by Küchemann as items that can be solved successfully using 'letter as object'. Using a variable as reference to an object should be differentiated into two aspects: The misconception that a variable can stand as shorthand for any object, and the conception that a variable stands for some measureable quantity, such as the length of a segment. This latter interpretation is at the heart of an approach to algebra by Davydov, Dougherty and others (see Gerhard 2008) that is suitable also for younger children. Interestingly, the sum of points of 6a and 6b show a correlation with the total test score of $r=0.62$ indicating that the ability to solve these items show much more than a misconception.

Next we gather some results from analyzing cumulative variables as described above. Together, these variables accounted for approximately 70% of all test items. According to the Kolmogorov-Smirnov-Test they can be considered to be normally distributed. Then a multivariate regression of the total score to these Variables was performed. The standardized beta-weights (with standard errors) were:

Variable	**Standard. Beta(SE)**
Syn (syntactic manipulation)	0.15(0.03)
Geo (geometry)	0.28(0.03)
Sub (substitution)	0.26(0.04)
Gen (working with general numbers)	0.22(0.04)
Fun (functions)	0.07(0.04)
Rel (relations)	0.38 (0.04)

The interpretation of these numbers must of course take into account that they reflect to some extend the composition of the test. There were eight items that were taken together to form the Rel variable, but only four that formed the Fun variable. Yet this can’t explain the dramatic difference in beta weights. We conclude that understanding of algebraic relations is an important component of algebraic competency. It is also interesting that the Geo variable that consists of only five items is that important. One may draw the conclusion that expressing relations among quantities is at the heart of algebra. It is therefore justified to exercise this extensively in introductory algebra lessons.

Then an analysis of covariance gave first insight into interdependences. The interesting findings were: There is almost no correlation between the syntactic manipulation (Syn) and Geo (r=0.09), Sub (r=0.09), Gen (r=0.10), Fun (r=0.13), Rel (r=0.02). The scale Syn consists of item 14 (which has two more sub-items than shown) on the simplification of expressions and of two items on solving linear equations. The result means that syntactic manipulation and conceptual understanding are two different dimensions. The assumption implicit in some teachers position on teaching algebra that learning the symbolic algorithms will lead to insight seems thus to be false. To further support this point we give the following two-way table:

Number of students		Score on syntactical items	
		Above average	Below average
Score on other items	Above average	37	27
	Below average	36	41

The χ^2-test gives p=0.19 on that, compatible with the assumption of independence (which is certainly not correct, but there is only a very weak relationship.)

This almost-independence result was stronger than expected and future studies should investigate this again. An interesting observation is that the connection is somewhat stronger for higher achieving students.

On the other hand the highest correlation (r=0.63) is between Rel and Subs. Subs also correlates with Geo (r=0.44), Gen (r=0.54) and Fun (r=0.54). All of these correlations are highly significant (p<0.01). This supports the hypothesis about the fundamental role of substitution given above.

Next, we report some results from the path model study. Although this interpretation was not intended by Drijvers (2003) we made up a structural equational model (more specific, a path diagram) from his diagram given below (Fig. 1). The model fit was acceptable according to Hair’s (Hair et al. 1998) recommendations with CMIN/df=1.96<2.0 and Parsimony-Adjusted Measure PCFI=0.56 . We found that the concept of placeholder loads most on the changing quantity (our role V of a variable; path weight and standard error: 1.14(0.48)), then on Unkown (U, weight 0.37(0.12)) and negligible on the generalizing aspect (G, weight 0.08(0.04)). The other arrows carry small weights as well. While the first two results are plausible, the question arises what influences the important aspect of a variable as a generalized number if not the placeholder aspect.

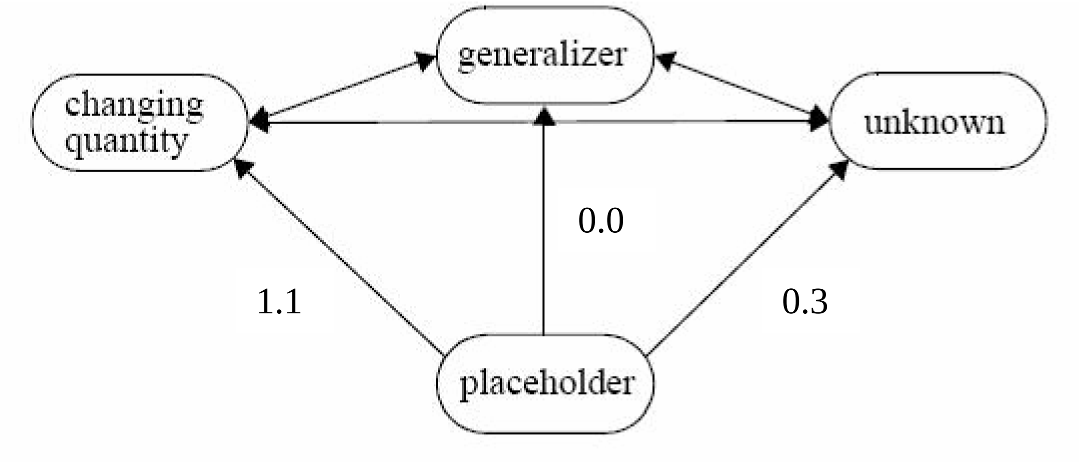


Fig. 1

The following model (Fig. 2) includes all of our five variable aspects. The latent variables are named by the short cuts of the variable aspects defined in the theory section. This model provides almost good model fit CMIN/df=1.53, PCFI=0.67. Nevertheless many of the estimates for regression weights are rather small and we will refine and modify the model shortly to get better results. Nevertheless this model shows some interesting results. First the arrows that relate the calculus aspect C with other aspects carry small weights. This feature is common to all models we tried and reflects the fact mentioned above, that syntactic manipulation is almost independent from the rest of the test. Another interesting fact is that there is a substantial (and significant) weight for the arrow from G to V. This is naturally interpreted as the implication that a general number can be viewed as standing for changing numbers. On the other hand, students learning algebra may first master the aspect of changing quantities and only later develop the general concept of a variable that stands for a general number without reference to a particular number. Therefore we omit this arrow in later models.

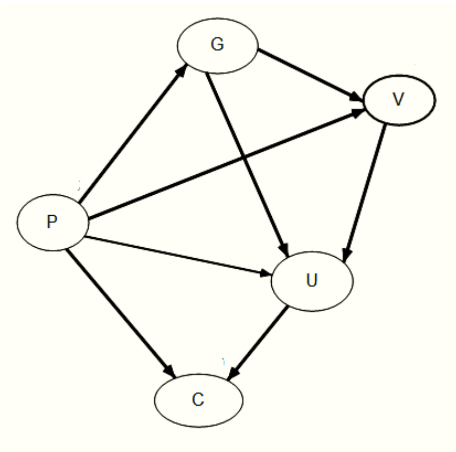


Fig. 2

The above path-model can be refined by splitting the aspect of general number as indicated in the theory section into the aspect of using the general number for analyzing or for modeling. Furthermore, we will omit the syntactic aspect of a variable as an element of algebraic calculus, because it is essentially independent from the rest. With these decisions made we tried out many linear structural equational models but concluded that the following one is the best choice. Some other models provide a slightly better model fit, but this model (Fig. 3) has two important properties: It is plausible from the theoretical point of view and can therefore be easily interpreted. Its advantage from the statistical point of view is that most of its path coefficients are either significant or close to significant. The model fit is adequate with CMIN/df=1.92 and PCFI=0.55. The estimates for regression weights (with standard errors in parentheses) are:

Place holder P	→	Unknown UK	0.15 (0.07)
Place holder P	→	General Number Ga	0.031 (0.024)
Place holder P	→	General Number Gm	-0.003 (0.341) ≈0
Place holder P	→	Variable V	-0.76(0.42)
Unknown UK	→	General Number Ga	0.30 (0.16)
Unknown UK	→	General Number Gm	-0.64(2.8) ≈0
Unknown UK	→	Variable V	6.6 (2.0)
General Number Ga	→	General Number Gm	2.5 (1.9)
Variable V	→	General Number Gm	0.077 (0.44) ≈0

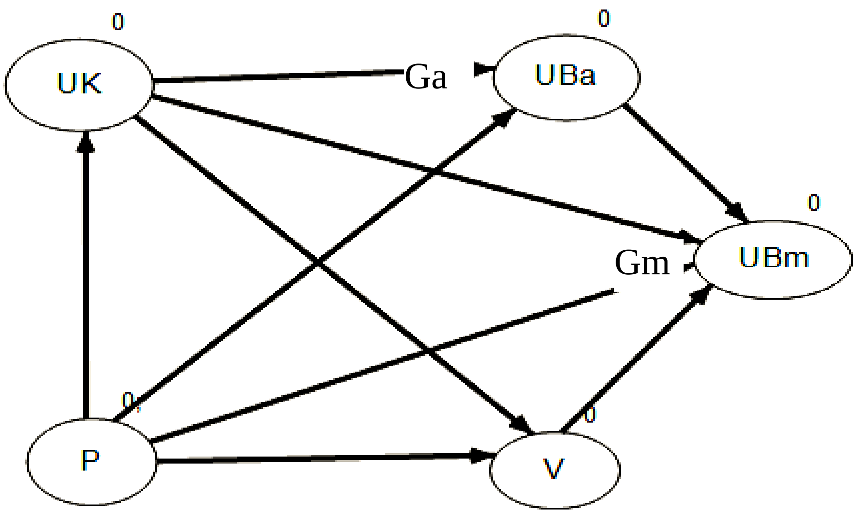


Fig. 3

Compared to the above model based on Drijvers diagram it may seem strange that the arrow P→V has a negative weight. This result does not claim that there is a negative correlation between these abilities but only that the direct influence is negative taking into account the large influence from the arrows P→UK and UK→V which both have positive weights. In fact, when omitting the UK→V arrow from the model, the arrow P→V gets massively positive (1.6). The negative weight in our model is therefore plausible: Learning to handle variables as placeholders may pave the way to seeing

variable as unknowns and this in turn helps develop the full concept; however students who can only deal with placeholders are unlikely to see variables as quantities that can change because a placeholder once filled with a number is constant.
The path weight for Ga→Gm was 2.5(1.9). When reversing the arrow it became negligible. This can be interpreted to mean that learning to analyze situations with variables is a prerequisite to modeling situations that are initially free of algebraic symbolization. On the other hand the aspect V is not helpful for algebraic modeling. This may give a hint that at the level of modeling situations by algebraic equations one is working at a rather high level where individual values of variables and their change is not considered. We hypothesize that the aspect of change is not important in forming the model but in its validation. But this conclusion can't be drawn from the data of this study.

Is it possible to assign students a single latent variable "algebraic competence"? To test this we fitted two simple models to the data. One model with only one latent variable "algebraic competence" and one model with latent variables "Univariate" and "Multivariate". The model with two latent variables has a model fit of CMIN/df=1.78, while the model with a single latent variable has a model fit of CMIN/df=2.99. This substantial difference may be seen as support for the hypothesis that algebraic competency is a higher dimensional construct, because here we have a higher dimensional modeling that fits the data better. Nevertheless, the test as a whole fits the assumptions of the one-dimensional Rasch model. Hence we conclude, that structural equational models can reveal detailed results.

CONCLUSION AND OUTLOOK

The findings of this study lead to two different kinds of conclusions. The first kind concerns the results from analysis of covariance and fitting the structural models. They indicate that the activities of describing general geometric situations algebraically are good indicators for overall performance. Similarly, substitution is a fundamental operation in algebra that shapes the meaning of algebraic constructs.

The second kind of conclusion concerns the level of algebraic competency reached in grade 11 and this is more specific to the situation in Germany (although the study does not claim to be representative for all German schools). While some areas (in particular, solving linear equations and using binomial formulas) show acceptable results, other parts of algebraic thinking, especially those that serve as a backbone in introductory calculus courses, reveal a serious lack of competence. Either a solution has to be found to cure the algebra decease or one should consider curricular changes in grades 11 and later that eliminate the need for those kinds of algebraic thinking; however, this would mean dropping calculus from the curriculum.

The future work of this research project is aimed at improving the situation. In collaboration with schools we aim to use this test as diagnostic instrument to help us assign tasks that will improve the construction of algebraic meaning. This includes the use of new algebraic technology (Oldenburg 2007) and the use of experiments (Ludwig & Oldenburg 2006).

REFERENCES

Bednarz, N., Kieran, C. & Lee, L. (Eds.) (1996). *Approaches to Algebra*. Dordrecht, Kluwer.

Dougherty, B. (2007). Measure up for understanding: reflect and discuss. *Teach. Child. Math*. 13, No. 9, 452-456.

Drijvers, P. (2003*). Learning Algebra in a Computer Algebra Environment.* Utrecht.

Dubinsky, E. (1991). *Reflective Abstraction in Advanced Mathematical Thinking.* In: David Tall (Ed.). *Advanced Mathematical Thinking.* Dordrecht, p. 95-123.

Gerhard, S. (2008). Problemsolving without numbers - an early approach to algebra, CERME Preprint.

Gray, E., Tall, D. (1994). Duality, Ambiguity and Flexibility: A Proceptual View of Simple Arithmetic. In: *The Journal for Research in Mathematics Education* 26(2) 115-141

Hair, J. F. et al. (1998). Multivariate Data Analysis (5. edition), p. 577-667, Upper Saddle River, New Jersey: Prentice Hall.

Küchemann, D. Algebra. In: K. Hart (Ed.) (1981): *Children's Understanding of Mathematics: 11 - 16*. John Murray Publishing .

Küchemann, D. (1979). Children's Understanding of Numerical Variables, *Mathematics in School*, 7, pp. 23-26.

Ludwig, M. & Oldenburg, R. (Eds.) (2007). Experimentieren. *mathematik lehren* Heft 141.

Malle, G. (1993): *Didaktische Probleme der elementaren Algebra*. Wiesbaden, Vieweg.

Oldenburg, R. (2007). Algebraic Modeling using a dynamically linked geometry and computer algebra system. CERME proceedings 2007, 1480-1488.

Sfard, A. (1991). On the dual nature of mathematical conceptions. In: *Educational Studied in Mathematics* 22 (1991) 1-36.