EQUALITY RELATION AND STRUCTURAL PROPERTIES [1]

Marchini Carlo [a], Cockburn Anne [b], Parslow-Williams Paul [b], Vighi Paola [a]

[a] Mathematics Department University of Parma - ITALY ; [b] School of Education and Lifelong Learning University of East Anglia, Norwich -U.K.

We present the results of a questionnaire on equality we administrated to a large and vertical sample of Italian students. Some of the questions were devised to investigate the presence of relational thinking.

INTRODUCTION – THE SCENARIO OF THE RESEARCH

This paper emanated from an international study of arithmetical misconceptions in primary schools (Cockburn & Littler, 2008) part of which considered equality (Parslow-Williams & Cockburn, 2008). One way to detect whether a wrong answer can be attributed to a misconception or a slip (Schlöglmann, 2007), is to analyse the persistence of the same wrong answer through a range of school grades. Here we focus on a questionnaire on equality administered to 1,147 Italian seven to sixteen and a group of university students in their first year. (cf. table 1 below).

THEORETICAL FRAMEWORK AND THE AIM OF RESEARCH

It has been well documented that an understanding of equality is crucial to the development of algebraic thinking (Alexandrou-Leonidou & Philippou, 2007; Attorps & Tossavainen, 2007; Puig, Ainley, Arcavi & Bagni, 2007).). Here we focus on formal number sentences, building on the work of Molina, Castro & Mason (2007) and, in particular, *relational thinking* – a term that Molina *et al*. (2007) borrow from Carpenter, Franke & Levi (2003). The student employs *relational thinking* if s/he

> "makes use of relations between the elements in the sentence and relations which constitute the structure of arithmetic. Students who solved number sentences by using relational thinking (RT) employ their number sense and what Slavit (1999) called "operation sense" to consider arithmetic expressions from a structural perspective rather than simply a procedural one. When using relational thinking, sentences are considered as wholes instead of as processes to carry out step by step." (Molina *et al*., 2007, p. 925)

The term *relational thinking* here is the opposite of procedural thinking. Although it sounds similar to Skemp's (1976) *relational understanding*, i.e. "knowing what to do and why" (Skemp, 1976, p. 21), in this context it focuses on different aspects of learning. In our opinion *relational thinking* is very similar to *relational interpretation* of equality detected by Alexandrou-Leonidou & Philippou (2007) and very closely related to *conceptual knowledge*, as proposed by Attorps & Tossavainen (2007) as opposed to *procedural knowledge*. The latter adopted the framework of Sfard (1991) and focused on the mathematical properties of the equality relation, i.e. reflexivity, symmetry and transitivity and, using a sample of 10 qualified and 75 pre-service secondary mathematics teachers, concluded that a lack of understanding of these properties impairs the development of the concept of equation. In Italy the structural approaches to arithmetic and algebra, together with equations, are usually introduced in grade 9. Early structural approaches and equations are, however, in the curricula for grades 6, 7 and 8. In the light of the above, this study investigated

- whether there was evidence of relational thinking in grades 2 - 5;
- how the structural notions taught of pupils in grades 6 - 11 influenced the responses;
- misconceptions about aspects surrounding equality amongst the students.

METHODOLOGY

The questionnaire

All pupils were given a written questionnaire containing a series of equality problems. Our questionnaire comprised simple number sentences using similar questions and symbols to those found in the literature (cf. Radford (2000), Hejný & Slezáková (2007), and Behr, Erlwanger & Nichols (1980)).

Zan (2000) suggested that misconceptions may exist in a sort of 'grey' zone beneath the complete consciousness of the person. Our questions were intended therefore to be sensitive enough to reveal misconceptions and relational thinking without being too direct, since this can make the subjects aware of their errors, resulting in an immediate correction before they commit themselves to writing an answer.

We decided to avoid the issue of having both signs '+' and '-', in the same calculation, as an awareness of both algorithms was required to find the solution. The questionnaire was four pages in length [2]: 2a and 2s presented addition and subtraction problems respectively, using mainly single digit numbers; in 3a and 3s numbers were between 20 and 100. The first six questions on each page were designed to build confidence and involve two given numbers, one operational sign, '+' or '-'. On all pages a firm knowledge of symmetry of equal relation can help solve the first six questions; in 2a form, two of them focus explicitly on the symmetry of the equality relation. The next four questions have three given numbers, two operational signs (cf. Behr *et al.* (1980), Sáenz-Ludlow & Walgamuth (1998) and Alexandrou-Leonidou & Philippou (2007)). These were followed by 'open' questions [3] with two operational sign, two boxes and two given numbers, as $a+\square = \square+b$; $\square+a = \square+b$; $a-\square = b-\square$ (we have yet to come across such examples in the literature). These were intended to reveal the possible use of reflexivity of equality, the commutative property of addition and awareness of 0 and its formal properties. We also tested the presence of the 'commutative property' of subtraction. Other less common open and closed questions were devised to detect the possible awareness of the transitive property of equality, with two-equality schema such as $a\pm b = c\pm\square = \square$ or with three-equality schemas such as $a\pm b = c\pm\square = d\pm\square = \square$ and $a\pm\square= b\pm\square =c\pm\square = \square$. These can be solved correctly by direct calculation showing the *non-RT behaviour* or by the use of structural properties, applying the different *RT behaviours* of Molina *et al*. (2007). For the reflexivity of equality in forms 2a and 3a we included a question of the schema *a* = □.

The questionnaire instructions were intentionally open-ended: we asked "Can you complete these number sentences?", without specifying which type of number could be used (naturals, relative integers, rationals or reals), thus leaving the possibility that older students could apply their knowledge about the various numbers systems.

The sample

As we had to rely on volunteers teachers, our sample was determined by their response. The number of returned questionnaires was as shown in Table 1.

Grade	2	3	4	5	6	7	8	9	10	11	Univ.
No.	76	131	58	228	282	172	161	62	22	47	112

Table 1: The sample structure

The size of the sample (1,147 respondents giving 62,898 answers) and its breadth (11 different grades) allowed us to compare our data with the research literature; observe whether such findings might be extended to older students and detect any new phenomena. Due to the scope of the study, the conditions of the test administration were largely un-specified (time, day, duration of the test, surveillance during the proof, and so on) except in case of university students who were given 15 minutes to complete the questionnaire.

THE RESULTS

Interestingly, regardless of age, the majority of solvers only used natural numbers. Due to the lack of space we focus on sample questions (while retaining the original questionnaire 'numbering').

1. The first six questions on each page and symmetry of equal relation.

A-priori analysis. In the questions 2a. (b) 5+□=8 and 2a. (f) 8=5+□ the role of symmetry is evident, since the numbers involved are the same ('strict'). In other examples we can speak of a symmetry 'at large' for the structure of the number statements, but not for the numbers involved. This gave us the opportunity to examine whether some pupils were 'blind to the symmetric property of the equality' (Attorps & Tossavainen, 2007), in the 'strict' sense and/or the 'at large' meaning. For each pair the correct answers to both questions can be obtained by computation; in case of 2a. (b) and 2a. (f), the result is 3, for both. For this pair, a difference in the result or the lack of one answer can be attributed to an incomplete mastery of the formal property of equality. For the remaining pairs we presume that a right answer to one question of the pair and the firm awareness of equality relation symmetry may suggest a good strategy for solving the other question of the couple, even if the numbers are different: a solver of 79-□=25 who has trouble with 53=78-□, can think of this second task in the form

78-□=53 to find the right answer. A right answer of only one question of these couples can suggest an 'at large' non-application of the symmetry in the pair.

A-posteriori analysis. The case simplicity of 2a. (b) and 2a.(f) resulted in high success rates: 98,14% and 95,40% respectively. People responding differently to the two tasks, certainly gave an incorrect answer. Individuals who responded incorrectly, are highly likely to a lack of their understanding of symmetry. However, in the case of the other pairs, the situation is more complex since we cannot exclude wrong computations even if symmetry was being used. In table 2 we distinguish between the 'strict' symmetry non-application and the 'at large' non-application. Data in the latter case are obtained cumulatively for the other eleven pairs (sample no. 12,993).

	number of at least one wrong or missing answer		rate of symmetry non-application		rate of contemporary success	
	strict	large	*strict*	large	*strict*	large
Grades 2-5 [4]	*46*	891	*93.48%*	79,91%	*90.67%*	78.33%
Grades 6-8	*18*	1090	*83.33%*	74,86%	*97.07%*	83.90%
Grades 9-11	*4*	231	*75.00%*	62,77%	*96.95%*	83.94%
University		47		89,36%		93.01%
χ -test	*8.68E-6*	3.38E-94	*0.29*	1.27E-7	*3.29E-6*	3.04E-24
Global sample	*68*	2259	*89.71%*	75,92%	*94.51%*	82.61%

Table 2: The non-application of symmetry of equality.

Values of the χ-test less than 0.05 (0.01) show that difference among grade classes are statistically significant; the result 0.29 is consequence of small numbers.

Reference the sum of the numbers of all the wrong and missing answers to at least one of two tasks suggests that a lack of awareness of the formal property is the greater source of error.

2. The task 2a. (k) 5 + □ = □ + 7

A-priori analysis. The task is open with the choice of one of two missing numbers determining the other. The location of the boxes invites, possibly, the reflexive property of equality without the need for any sort of calculation e.g. $5+\boxed{7} = \boxed{5}+7$. The neutral role of 0 with addition could inspire the answer $5+\boxed{2} = \boxed{0}+7$. Other structural answers using the formal property of negative numbers (and 0) are $5+\boxed{0} = \boxed{(-2)}+7$ and $5+\boxed{(-5)} = \boxed{(-7)}+7$. Relational thinking offers a criterion for revealing a wrong answer: the given numbers are odd, therefore the two inserted numbers must have the same *even parity*. The repetition of a box could prompt (wrongly) younger pupils, in particular, into thinking that the numbers they are required to insert must be the same.

A-posteriori analysis. Of 1,143 students that were given this question, 1,057 responded, of which 842 gave the right answer (73.76%) suggesting that the task was relatively easy. Each answer given (right or wrong) used natural numbers. It is interesting to note the distribution of the structural answers by age of pupils. We suspect that the infrequent use of zero to solve the problems e.g. 5+$\boxed{2}$ = $\boxed{0}$+7 could be due to a 'fear' of 0 - i.e. the complex acknowledgment of 0 as a number - or, simply reflect that individuals were unacquainted with this mathematical character.

2a. (k) 5 + □ = □ + 7	correct response	presence of the answer 5+$\boxed{7}$ = $\boxed{5}$+7	presence of the answer 5+$\boxed{2}$ = $\boxed{0}$+7	commonest correct response (with frequency)
Grades 2-5	62.47%	7.26%	7.66%	5+$\boxed{4}$=$\boxed{2}$+7 (26.21%)
Grades 6-8	78.21%	6.86%	6.24%	5+$\boxed{3}$=$\boxed{1}$+7 (34.93%)
Grades 9-11	86.26%	2.65%	0.88%	5+$\boxed{3}$=$\boxed{1}$+7 (46.02%)
χ -test	4.79E-10	0.21	0.04	
Global sample	73.67%	6.41%	5.94%	5+$\boxed{3}$=$\boxed{1}$+7 (32.30%)

Table 3: The relational thinking presence and the commonest right answers to 2a.(k).

The commonest incorrect response was 5+$\boxed{2}$ = $\boxed{7}$+7 (with 18.14% of the 215 wrong answers). To interpret this we can consider the application of "Three First Numbers – TFN" and then "Answer After Equal Sign–AAES" modalities of Alexandrou-Leonidou & Philippou, (2007). The presence of two equal boxes, did not appear to be highly relevant as only the 8.37% of incorrect responses used the same number twice: 5+$\boxed{a}$ = $\boxed{a}$+7 (***a***=1 or ***a***=2 having the greatest frequency). The *even parity* criterion was found in all of the 842 exact answerers and in 26.98% of the wrong answers, giving a total rate of 85.15% of the answers. We have also an *echo effect*: when the given numbers are odd, the percentage of correct answers using a pair of odd numbers is 62.59%.

3. The task 2s. (k) 6 - □ = 8 - □

A-priori analysis. This task is also open with the first number determining the second. Moreover, if restricted to natural numbers, the subtrahend must be less than minuend. The location of boxes may invite the following answer 6-$\boxed{6}$ = 8-$\boxed{8}$, a solution using 0 as result of both members of equality. Alternatively the neutral role of 0 when subtracting could be employed e.g. 6-$\boxed{0}$ = 8-$\boxed{2}$. For other aspects the a-priori analysis of this task is similar to the previous one. We expected a wrong relational thinking answer in the 'commutativity' of subtraction, i.e. the answer 6-$\boxed{8}$ = 8-$\boxed{6}$.

A-posteriori analysis. 1,056 students were given the question; 953 responded, 762

2s.k) 6 - □ = 8 - □	rate of success	rate of response 6-$\boxed{6}$ = 8-$\boxed{8}$	rate of response 6-$\boxed{0}$ = 8-$\boxed{2}$	rates of commonest right answer 6-$\boxed{2}$=8-$\boxed{4}$
Grades 2-5	61.09%	3.16%	2.76%	31.58%
Grades 6-8	76.06%	2.78%	1.05%	38.12%
Grades 9-11	80.15%	1.90%	3.43%	28.57%
χ-test	9.33E-7	0.24	0.24	0.09
Global sample	72.16%	2.76%	2.86%	35.17%

Table 4: The relational thinking presence and the commonest right answers to 2s. (k).

did so correctly (success rate 72.16%), suggesting that this task was relatively easy even if slightly harder than 2a. (k). Table 4 summarises the use of relational thinking. The *echo effect* appeared to be present as 56.17% of the right answers used pairs of even numbers. The *even parity* criterion is present in 87.20% cases. In this case the commonest correct answer is similar for all grades. Again we could argue that the commonest right answers were influenced by the fear of using 0 combined with the *echo effect*. The commonest wrong answer was 6-**2**=8-**2** (12.04% of the 191 wrong answers) and we could consider this kind of response motivated by application of TFN twice assuming that the second box is filled in first. Of the wrong answers, the structural, but incorrect, response 6-**8**=8-**6** was given in 4.19% cases. The value 0.09 of the χ-test show that the differences among grades classes are not statistically significant.

4. The task 3a. (k) □ + 21 = □ + 11

A-priori analysis. As above the task is open and has 'freedom grade one'. The location of boxes may invite the use of commutative property of addition, i.e. **11**+21 = **21**+11. Moreover the neutral element of addition could reduce computation e.g. **0**+21 = **10**+11. Questions 2a.(k) and 3a.(k) have the same quantity of given numbers and addition symbols, but the boxes are differently placed: in 2a.(k) reflexivity of equality is at stake while in 3a.(k) the commutativity of addition is involved.

3a.k) □ + 21 = □ + 11	rate of success	rate of response **11**+21 = **21**+11	rate of response **0**+21 = **10**+11	rate of commonest right answer **10**+21 = **20**+11
Grades 3-5	60.52%	9.22%	2.84%	26.24%
Grades 6-8	72.17%	10.76%	4.48%	28.92%
Grades 9-11	80.92%	11.32%	4.72%	31.13%
University	94.64%	11.32%	6.60%	42.45%
χ-test	1.04E-10	0.94	0.57	0.03
Global sample	73.03%	9.14%	4.51%	30.54%

Table 5: The relational thinking presence and the commonest right answers to 3a. (k).

A-posteriori analysis. This task was administered to 1,094 students from grade 3 to first year of University: 979 responded with 799 of them giving the right answer (success rate 73.03%), comparable with the success rate for 2a. (k). Here *RT* appears to become more evident with increasing age. The use of 0 as the neutral element in addition is similar to that in task 2a. (k) but the commutativity of addition is more prevalent. Multiples of ten - excluding 0 - were found in 53.82% of the correct answers. The *even parity* criterion occurred in 83.86% responses. In 3a. (k) question the *echo effect* was not evident as 60.33% of the right answers had a pair of even numbers. The commonest wrong answer is **32**+21 = **53**+11 (6.11%). We hypothesize that the first box is filled in when the task is interpreted as □ = 21+11, in a sort of "Left Side Sum-LSS" modality. The completion of the second box is suggested by AAES modality (Alexandrou-Leonidou & Philippou, 2007).

In our opinion, the presence of two digit numbers had a double effect: the attempts decrease from 92.48% of 2a. (k) to 89.49% of 3a. (k) and this may be significant as the latter sample excluded 2nd graders but included first year university students. Secondly it may be that the presence of two digit number in this task activates a more attentive approach to the computation (the answers to other questions support this) and we could attribute to this attitude the greater presence of *RT*.

5. The tasks of type a = □.

A-priori analysis. Behr *et al*. (1980) include examples of the type $a = a$, with given numbers and so we incorporated 2a. (l), 9 = □, and 3a. (n), 42 = □. To solve them one needs only apply the reflexive property of equality. These are closed tasks and do not require computation.

We anticipated that the absence of operational symbols would be destabilizing and

	9=□ success rate	9=□ with operational signs	42=□ success rate	42=□ with operational signs
Grades 3-5	76.67%	35.00%	70.82%	43.48%
Grades 6-8	73.17%	59.68%	72.98%	39.60%
Grades 9-11	67.94%	86.67%	67.94%	57.89%
University			92.86%	100%
χ-test	0.10	2.11E-6	1.53E-5	0.02
Global sample	74.01%	54.70%	73.70%	44.77%

Table 6: Comparison of results of the tasks 2a. (l) and 3a. (n).

The result in no answer or the use of operational symbols (cf. Behr *et al*., 1980). location of the two tasks in their form allowed us to explore if there was a *tiredness effect*, influencing the rates of answer and success.

A-posteriori analysis. Task 2a. (l) was given to grades 2 - 11 (1,239) with 1,151 responding with 917 of correct (74.01%). The majority of incorrect answers (54.70%) express the result with operational symbols and the computation on the proposed numbers gives 9, showing a procedural interpretation of the sign =. The commonest answer of this kind is 9 = $\boxed{\mathbf{3^2}}$, in 44.53% of all 'operational' answers and was given by the majority of 6th graders and above.

Task 3a. (n), 42 = □, was given 1,190 grade 3-11 and 1st year university students, 1,049 responded with 877 of them giving the right answer (73.70%). The 'operational' answer rate is 44.77% and the commonest 'operational' responses were, globally, 40+2 (19.48%) and 21+21 (18.18%).

6. The task 2a. (m) 5 + 4 = □ + 6 = □

A-priori analysis. This task is the first which presents more than one equality sign. It is a closed task. The 'chain' of equality asks for the transitive property of equality.

Wrong answers suggest a lack of awareness of it. The most probable incorrect response is 5+4 = **9**+6 = **15** (cf. Alexandrou-Leonidou & Philippou, 2007).

A-posteriori analysis. 1,104 students responded with 718 giving the right answer (62.82%). As was expected the commonest wrong answer (70,47%) was 5+4 = **9**+6 = **15**. This suggests either the pupils filled in the second box before completing the first or that they worked step by step from left to right. In either cases such results bring into question their intuition as Semadeni (2008) states:

> "The transitivity of equality: "if A = B and B = C then A = C" was regarded by Fischbein (1987, pp. 24, 44, 59) as intuitively true. Piaget et al. (1987b, p.4) regards transitivity as an example of a systematic type of necessity…Transitivity is part of the deep intuition of equality (for numbers, for geometric points, for sets), involved in a multitude of deductive inferences." (p.10)

7. The task 3s. (m) 48 - □ = 47 - □ = 46 - □ = □

A-priori analysis. This task is complex: it is open-ended, involves two-digit numbers, three subtraction signs and three equalities. Despite having four boxes to fill, it has 'freedom grade one'.

3s.m) 48-□=47-□=46-□=□	rate of success	48-**48**=47-**47**=46-**46**=**0**	48-**2**=47-**1**=46-**0**=**46**	commonest right answer rate 48-**3**=47-**2**=46-**1**=**45**
Grades 3-5	56.99%	2.73%	9.09%	20.00%
Grades 6-8	56.91%	4.29%	12.29%	27.14%
Grades 9-11	60.77%	0%	7.59%	48.10%
University	83.04%	2.15%	6.45%	61.29%
χ-test	3.64E-6	0.22	0.29	6.5E-12
Global sample	60.19%	3.16%	10.28%	33.54%

Table 7: The presence of relational thinking regarding 0 and the commonest right answers to 3s.m).

To solve these questions correctly an explicit awareness of transitive property seems to be required. The task allows simple solutions involving *RT* and formal properties of 0 in many ways: 48–**48** = 47 – **47** = 46 – **46** = **0**, or 48 – **2** = 47 – **1** = 46 – **0** = 46. It is also possible to apply negative numbers, or fraction and so on, but no one did.

A-posteriori analysis. 896 – out of a possible 1,050 - responded with 632 giving the right answer (60.19%). The commonest correct answer reveals that the learners are at different levels of understanding, growing with age, taking care of the additive decomposition of numbers by fives: 48 = 45+3, 47 = 45+2 and so on. The structural properties of zero were most common in first eight grades of schooling. 41 pupils gave incorrect answers (22.65%) applying the transitive property of equality only once. 47.51% of those who were incorrect responded 48-**1** = 47-**1** = 46-**1** = **45**.

CONCLUSIONS

The questionnaire enabled us to explore a phenomenology linked to relational thinking expressed by the reflexive, symmetric and transitive property of equality, the roles of zero respect to addition and subtraction and the commutativity of addition.

Our study is peculiar in the variety of schools and age range sampled. In this sense other similar experience known in literature took place in smaller school, segments. Another feature of our paper is that we are interested here in the right answers, even if sometimes we quote, also, wrong answers. Our research would have been more rigorous had we selected the sample statistically. Therefore our paper cannot be used for drawing general conclusions, statistically sound, about relational thinking, nevertheless in our feeling it might open a new trend of study about the equality, pointing out that this subject needs an attentive reflection regarding the way and the time in which the concept of equality is presented (in itself), let it grant that is introduced somewhere and somehow.

Overall primary school pupils were slightly better (even if in many cases differences are not statistically significant) than the older respondents in their application of relational thinking in specific tasks, but the presence of two-digit numbers appeared to hindered them. Nevertheless, a small but significant group demonstrated structural thinking provoking the question of how to extend such thinking to others. The transmissive teaching methods in Italy may explain why relational thinking does not appear to improve between grades 6 and 11 even if the structural properties of operations are taught explicitly, suggesting a parallel presence of relational and procedural thinking, independent from teaching. For symmetry our pupils confirmed the Attorps & Tossavainen (2007) results with prospective teachers.

There was a global score progression with increasing age. Addition questions were easier than subtraction; generally, pupils responded more appropriately to one digit answers than to two digit problems. Answering more complex questions under conditions of stress (e.g. tiredness) suggests that the students possessing a ‘reified understanding’ (described by Sfard (1991) as ‘being able to see something familiar in a different light’) of formal properties have an important tool which saves time and mental energies. Students who were aware of formal properties tended to cope better than others under conditions of complexity and stress. The prevalence of such knowledge was low however and in some cases appeared to decrease with age despite such topics being introduced in Italian Secondary School. Few participants (even from University) reificated the reflexive property of equality, and the function of zero in addition and subtraction. The commutative property of addition was more apparent. The more complex nature of the statements of symmetry and transitivity of equality do not necessarily indicate their presence, but only their absence. The sub-sample of university students appeared to have the awareness of these arithmetic tasks, but, surprisingly, more than 1/5 of the sub-sample responded to 3s. (m) incorrectly with more than 1/3 of them revealing a lack of a global view, answering 48-**1** = 47-**1** = 46-**1** = **45**, and of the transitive property of equality!

NOTES

[1] The authors gratefully acknowledge the support of the British Academy (Grant no. LRG-42447) which provided a platform for this study.

[2] The questionnaire presents 54 questions divided in four forms: 2a, 2s, 3a, 3s (the digit refers the grade of primary school and the letter ‘a’ is for addition and ‘s’ is for subtraction). The integral version of questionnaire and the report of results are available at the web-site http://www.unipr.it/arpa/urdidmat/M2ip.

[3] When a solution is uniquely determined, e.g. 32 + 25 = □ + 16 = □ we use the adjective ‘close’; whenever the solver is free to choose the suitable numbers, e.g. 48 - □ = 47 - □ = 46 - □ = □ we use ‘open’.

[4] Italian children start school 6-years-old. Primary school comprises five grades; stage one of secondary school, grade 6, 7 and 8, and the final stage of secondary school 9 to 13.

REFERENCES

Alexandrou-Leonidou, V. & Philippou, G. (2007). Elementary school students’ understanding and use of the equal sign. In Pitta-Pantazi & Philippou (2007, pp. 825 – 834).

Attorps, I. & Tossavainen, T. (2007). Is there equality in equation? In Pitta-Pantazi & Philippou (2007, pp. 2250 – 2259).

Behr, M., Erlwanger, S. & Nichols, E. (1980). How children view the equals sign. *Mathematics Teaching*, *92*, 13-15.

Carpenter, T. P., Franke, M. L. & Levi, L. (2003). *Thinking mathematically: integrating arithmetic and algebra in elementary school*. Portsmouth: Heinemann.

Cockburn, A. & Littler, G. (Eds.). (2008). *Mathematical Misconceptions – A Guide for Primary Teachers*, London: Sage Publisher.

Hejný, M. & Slezáková, J. (2007). Schema $a \pm b = c$ as the basis of arithmetic structure. In Pitta-Pantazi & Philippou (2007, pp. 426 – 435).

Fischbein, E. (1987). *Intuition in mathematics and science. An educational approach*. Dordrecht : Reidel.

Molina, M., Castro, E. & Mason, J. (2007). Distinguishing approaches to solving true/false numbers sentences. In Pitta-Pantazi & Philippou (2007, pp. 924 – 933).

Parslow-Williams, P. & Cockburn, A. (2008). Equality. In Cockburn & Littler (2008, pp. 43 – 68).

Piaget, J., Ackermann-Walladao, E. *et al*. (1987). *Possibility and Necessity* (Vol. 2). *The role of necessity in cognitive development*. Minneapolis: University of Minnesota Press.

Pitta-Pantazi, D. & Philippou, G. (Eds.), 2007, *Proceedings CERME 5*, Department of Education, University of Cyprus.

Puig, L., Ainley, J., Arcavi, A. & Bagni, G. (2007). Working on algebraic thinking. In Pitta-Pantazi & Philippou (2007, pp. 810 – 815).

Radford, L. (2000). Signs and Meanings in students´ emergent algebraic thinking: a semiotic analysis. *Educational Studies in Mathematics*, *42*, 237- 268.

Sáenz-Ludlow, A. & Walgamuth, C. (1998). Third graders’ interpretations of equality and the equal symbol. *Educational Studies in Mathematics 35*, 153-187.

Schlöglmann, W. (2007). Student errors in task-solving processes. In Pitta-Pantazi & Philippou (2007, pp. 358 – 368).

Semadeni, Z. (2008). Deep intuition as a level in the development of the concept image. *Educational Studies in Mathematics 68*, 1 – 17.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*, 1-36.

Skemp, R.R. (1976), Relational understanding and Instrumental understanding. *Mathematics Teaching*, 77, 20 - 26.

Slavit, D. (1999). The role of operation sense in transitions from arithmetic to algebraic thought. *Educational Studies in Mathematics*, *37*(3), 251-274.

Zan, R. (2000). ‘Misconceptions’ e difficoltà in Matematica, *L’insegnamento della Matematica e delle scienze integrate*, *23A*(1), 45 – 68.