APPROACHING FUNCTIONS VIA MULTIPLE REPRESENTATIONS: A TEACHING EXPERIMENT WITH CASYOPEE

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Abstract: *Casyopée is an evolving project focusing on the development of both software and classroom situations to teach algebra and analysis at upper secondary level. This paper draws on our current research in the ReMath European project focusing on the approach to functions via multiple representations. In this paper, we present the design of an experimental teaching unit for the 11th grade and some preliminary results.*

INTRODUCTION

The notion of function plays a central role in mathematics and for many authors technology can help students to learn about this notion especially because of the representational capabilities of digital environments. Recently, authors extended the range of representations by considering functional dependencies in a non symbolic domain. Falcade and al. (2007) proposed for instance to use Dynamic Geometry as an environment providing a qualitative experience of covariation and of functional dependency in geometry.

An aim of our team in the ReMath project is to develop a teaching unit taking advantage of a wealth of representations of functions offered by technology. In this aim, our software environment - Casyopée - has been extended, adding to the existing symbolic window a geometrical window with strong connections between them. Casyopée's symbolic window is a computer environment for upper secondary students. The fundamental objects in this window are functions, defined by their expressions and domain of definition. Other objects are parameters and values of the variable. Casyopée allows students to work with the usual operations on functions like: algebraic manipulations (factoring and developing expressions, solving equations ...); analytic calculations (differentiating and integrating functions); graphical representations; supports for proof …. The new window offers the usual dynamic geometry capabilities, like defining fixed and free geometrical objects (points, lines, circles, curves) and constructing others. It also offers distinctive features: geometrical objects can depend on algebraic objects and it is possible to export geometrical dependencies into the symbolic window, in order to build algebraic models of geometrical situations (Lagrange & Chiappini, 2007).

SOLVING A PROBLEM OF FUNCTIONAL DEPENDENCY WITH

CASYOPEE

In order to explain this extension, we expose now the type of problem whose resolution can take advantage of Casyopée, and how**.** This is an example:

Consider a triangle ABC. Find a rectangle MNPQ with M on [oA], N on [AB], P on [BC], Q on [oC] and with the maximum area

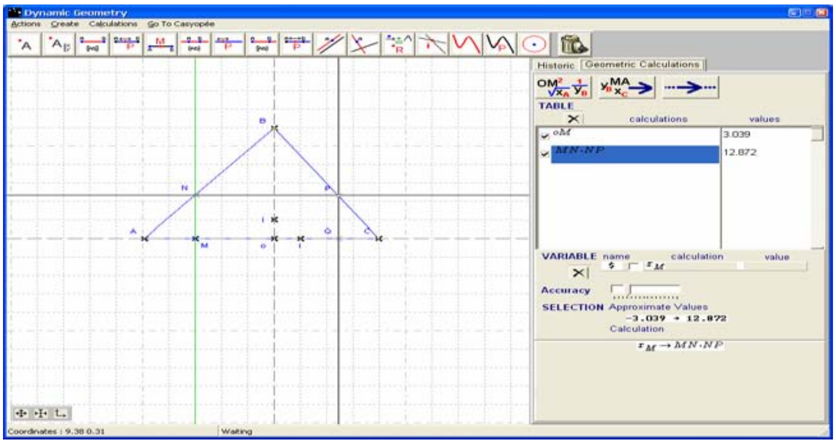

Fig. 1: The geometrical window of Casyopée

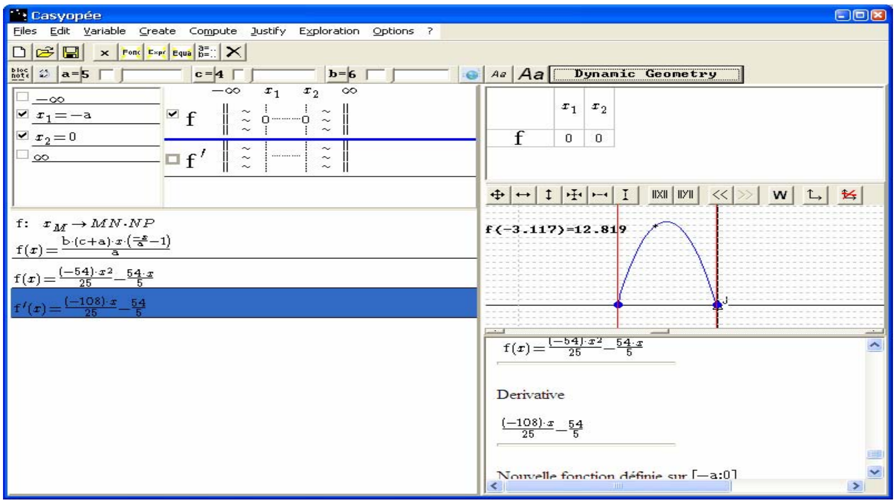

Fig. 2: The symbolic window of Casyopée

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Constructing a generic triangle ABC in the geometrical window can be done after creating parameters in the symbolic window. For instance, the points can be *A(-a;0), B(0;b)* and *C(c;0),* a, b and c being three parameters. Then one can create a free point M on the segment [oA] (o being the origin) and the rectangle can be constructed using dynamic geometry capabilities.

In the Geometric Calculation tab (Fig.1) one can create a calculation for the area of the rectangle MNPQ and then define an independent variable. Numerical values of calculations and of the variable are displayed dynamically when the user moves free points. The user can then explore the co-dependency between these values. If this co-dependency is functional (i.e., the calculation depends properly on the variable) it can be exported into the symbolic window and Casyopée automatically computes the domain and the algebraic expression of the resulting function. Otherwise, Casyopée gives adequate feedback.

After exporting into the symbolic window, one can work on various algebraic expressions of the function and on graphs. For instance, one can use properties of parabolas, or algebraic transformations or Casyopée's functionality of derivate to find the answer to the question. One can also use the graph of the function to conjecture about the area maximum.

QUESTIONS AND THEORETICAL FRAMEWORKS

As the above example shows, Casyopée offers very varied functionalities and representation of functions:

- means for creating generic dynamic figures,
- geometrical calculations to express a range of quantities that can be considered as dependant variables,
- possibilities of choosing an independent variable like a distance or an abscissa involving free points,… feedbacks about this choice of a variable,
- means to observe numerical covariation between points and calculations, or between an independent variable and a calculation,
- means to export a functional dependency between the chosen variable and a calculation to the symbolic window, resulting in an algebraic form of the function,
- means for treating this algebraic form in various registers.

The overarching question addressed by the Casyopée team is: how to exploit these varied functionalities of representation in order to develop students' understanding of a functional dependency, particularly by articulating a geometrical situation with its algebraic model?

To investigate this question, we built an experimental teaching unit at 11th grade. In this paper, we present first the frameworks that helped us to build this experiment and to interpret our observations. Then we present the experiment and we report on the observation of the last session where students used the wider range of representations.

The first framework is based upon the notion of "setting" introduced by Douady (1986). According to Douady, a *setting* is constituted of objects from a branch of mathematics, of relationship between these objects, their various expressions and the mental images associated with. When students solve a problem, they can consider this problem in different settings. Switching from a *setting to another* is important in order that students progress and that their conceptions evolve. Students can operate these *changes of setting* spontaneously or they can be helped by the teacher. The setting distinguished here are geometry and algebra,

We also rely upon the notion of *registers of representations* from Duval (1993). Duval stresses that a mathematical object is generally perceived and treated in several registers of representation. He distinguishes two types of transformations of semiotic representations: *treatments* and *conversions*. A treatment is an internal transformation inside a register. A conversion is a transformation of representation that consists of changing of a register of representation, without changing the objects being denoted. It is important that students recognize the same mathematical objects in different registers and they get able to perform both treatments and conversions.

Here we distinguish the geometric and the algebraic settings corresponding to Casyopée's two main windows. In these two settings, the functions modeling a dependency are different objects: a relationship between geometric objects or measures in the geometric setting, and an algebraic form involving a domain and an expression in the algebraic window. In the above problem, students have to switch from the geometric to the algebraic settings and back, to be able to use symbolic means for solving questions that were formulated in the geometric setting. As explained by Lagrange & Chiappini (2007), we expect that, working in the geometric setting, students would understand the problem and the objects involved, and that after switching to algebra, this understanding would help them to make sense of the objects and treatments in the algebraic setting.

Inside each of these two settings the functions can be expressed in several registers. In geometry, especially with dynamic geometry, functions can be represented and explored in different registers: covariations between points and measures, or between measures, or functional dependency between measures. In algebra, functions can be expressed and treated symbolically, by their expressions, by way of graphs and of numerical tables. Mastering these expressions and treatments, and flexibly changing of register are important for students' ability to handle functions and acquire knowledge about this notion.

A third framework is the *instrumental approach*, based on the distinction between artefact and instrument. An actefact is a product of human activity, designed for specific activities. For a given individual, the artefact doest not have an instrumental value in itself. It becomes an instrument through a process, called *instrumental genesis*, involving the construction of personal schemes or the appropriation of social preexisting schemes. Thus, an instrument consists of a part of an artefact and of some psychological components. The instrumental genesis is a complex process; it requires time and depends on characteristics of artefacts (potentialities and constraints) and on the activities of the subject (Vérillon & Rabardel, 1995).

In the case of an instrument to do or learn mathematics like Casyopée, the instrumental genesis involves interwoven knowledge in mathematics and about the artefact's functionalities. Artigue (2002) showed how this genesis can be complex, even in the case of simple task like framing a function in the graph window. More generally, the many powerful functionalities of CAS tools have a counterpart in the complexity of the associated instrumental genesis (Guin & Trouche, 1999). We are then aware that we must take care of students' genesis when bringing Casyopée into a classroom. Moreover, Casyopée offers a multiplicity of representations in two settings and in several registers. Understanding and handling these representations involves varied mathematical knowledge. Students have then to be progressively introduced to these representations, taking into account the development of their mathematical knowledge.

Constructing the sessions of the experiment, we also used the Theory of Didactical Situations as basis for designing tasks. According to this theory, learning happens by means of a continuous interaction between a subject and a milieu in an *a-didactical situation*. Each action of the subject in milieu is followed by a retro-action (feedback) of the milieu itself, and learning happens through an adaptation of the subject to the milieu. Thus, with regard to Casyopée use, learning does not depend only on the representational capabilities of this software, but also on tasks and on the way they are framed by the teacher. Within this perspective, we looked for situations in which students interact with Casyopée and receive relevant feedbacks. For example, to solve the above problem, students have to choose between different independent variables to explore functional dependencies in the geometrical window and to export a dependency into the algebraic window. In case the variable is inadequate, the feedback they receive is a message from Casyopée. In other cases, the algebraic expression automatically produced by Casyopée can be more or less complex, which is another feedback: too complex expressions have to be avoided in order to ease the subsequent algebraic work.

Concerning the methodology, we use *didactical engineering* (Artigue, 1989), a method in didactic of mathematics, to organize and evaluate the experimental teaching unit, and to answer the research questions. The treatments and interpretations of collected data based on an internal validation which consists in confronting *a priori* analysis of the situation with *a posteriori* analysis. This method produces an ensemble of structured teaching situations in which conditions for provoking students' learning have been planned.

THE EXPERIMENT

Our experimental teaching unit consisted of six sessions. It was experimented in two French 11th grade classes. It was organized in three parts. Consistent with our sensitivity to students' instrumental genesis, each part was designed in order that students learn about mathematical notions while getting acquainted with Casyopée's associated capabilities:

- ❖ The first part (3 sessions) focused on capabilities of Casyopée's symbolic window and on quadratic functions. The aim was that students became familiar with parameter manipulation to investigate algebraic representations of family of functions, while understanding that a quadratic function can have several expressions and the meaning of coefficients in these expressions. The central task was a "target function game": finding the expression of a given form for an unknown function by animating parameters.
- ❖ The second part (two sessions) aimed first to consolidate students' knowledge on geometrical situations and to introduce them to the geometrical window's capabilities. The central task was to build geometric calculations to express areas and to choose relevant independent variables to express dependencies between a free point and the areas. It aimed also to introduce student to coordinating representations in both algebraic and geometrical settings, by way of problems involving areas that could be solved by exporting a function and solving an equation in the symbolic window.
- ❖ Finally, in the third part (one session) of the experimental unit, students had to take advantage of all features of Casyopée and to activate all their algebraic knowledge for solving the optimization problem presented above.

Below, we give some insight on how we are currently exploiting this experiment with regard to our question about Casyopée' potential for multi-representation. We limit ourselves to the final session for which the problem and the students' instrumental genesis should allow to take full advantage of this potential. We draw some elements of a priori analysis of this session and we compare with the a posteriori analysis of the functioning of a two student team.

THE SITUATION IN THE FINAL SESSION: ELEMENTS OF *A PRIORI* ANALYSIS

Tasks

The problem is presented by the teacher by animating a figure in Casyopée's geometrical window:

> *Let a, b and c be three positive parameters. We consider the points A(-a;0), B(0;b) and C(c;0). We construct the rectangle MNPQ with M on [oA], N on [AB], P on [BC] and Q on [oC]. Can we build a rectangle MNPQ with the maximum area?*

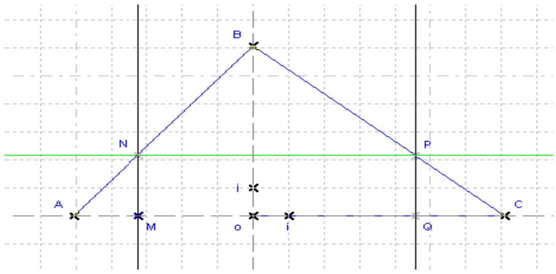

Fig. 3: The figure built in Casyopée

The tasks proposed to students are then:

- The construction of the rectangle MNPQ: students are required to load a Casyopée file with the parameters' definition and the triangle, then to complete the figure by building the segments [oA], [AB], [BC] and [oC] and to create the free point M and the rectangle's vertexes.
- To create a geometrical calculation for the area of the rectangle MNPQ: this can be obtained by the product of the lengths of two adjacent sides, e.g. MNxMQ
- To explore the situation by moving the point M on the segment [oA].
- To prove the conjecture by algebraic means.

The teacher also asks students to write the proof, indicating their choice of variable and using results displayed by Casyopée. Finally, students are expected to visualize the answer in the geometrical window.

Covariations and representation of functional dependencies

This situation involves two settings and different registers. Students can conjecture the answer to the question by exploring numerical values of the area in the geometrical setting. They can explore the variation of the area in different ways corresponding to different registers of representation. First, they can observe co variation between the point M and the area, looking at the values of the calculation they created for the area of the rectangle, noting that when M moves from A to B the value grows then decreases, with a maximum value when M is the middle of [oA]. They can also observe co variation between a measure involving the free point M and the area. For instance, they can observe together the values of the distance oM and of the area. Finally, they can choose an independent variable involving M and observe the functional dependency between this variable and the area.

In the algebraic setting students can apply different algebraic techniques to the algebraic form of the function in order to find a proof. Exporting a function with Casyopée, one obtains a more or les complex algebraic expression reflecting the calculation’s structure. Students then need to expand this expression to recognize a quadratic function. They can then apply their knowledge about these functions to prove the maximum. It is possibly not easy for them, because of the three parameter involved.

They can also use the graphical representation in this algebraic setting to explore the curve, complementing the exploration they did in the geometrical setting: the parabola is familiar to the students and they can easily recognize a maximum.

The situation is partly a-didactical. In each setting, students interact freely with Casyopée and use the feedbacks to understand the situation. Nevertheless, some key points like passing from a co variation to a functional dependency are expected to be difficult for students, although the corresponding action (choosing an independent variable) has been presented in the preceding sessions. Passing from one setting to the other is expected to be far from obvious for students. The corresponding actions in Casyopée (exporting a function in the symbolic window, interpreting a symbolic value in terms of position of a point) have also been presented before, but it is the first time that students have to do it by themselves.

Students can choose their own independent variable between possible choices (oM, x_M, MN, MQ…) with consequences upon the algebraic expression of function. They can do it alone but it is expected that the teacher mediation will be necessary. It is also possible that they will want to change their choice of a variable in order to obtain a simpler algebraic expression of the function.

We expect a great variety of uses of representations reflecting students’ free interactions with the situation. Some students can stay a long time exploring co variations and need teacher mediation to go to functional dependency while others pass more or less quickly to the algebraic setting to consider the function. In this setting, some can prefer to explore graphs, while others prefer working on algebraic expressions. It is possible that some students find too difficult to apply algebraic techniques to the general expression (i.e. with parameters) and prefer to work by replacing these parameters by numbers. In any case, we expect that students will consider several representations, make sense of them and make links between them.

ELEMENTS OF *A POSTERIORI* ANALYSIS: THE CASE OF A TEAM

During the experiment, we observed selected teams of students. In this paper, we focus on a team of two students, which according to the observation in the first five sessions had a favorable instrumental genesis. According to their teacher they were good students.

The explorations in different settings and registers

Creating a geometrical calculation for the area of the rectangle, they typed MNxMP instead of MNxMQ by mistake. They moved M and observed growing numerical values of this calculation, while, for some positions of M the area was visibly decreasing. This first feedback allowed them to correct the geometrical calculation.

Like most students they had difficulties in choosing an appropriate independent variable, confusing the independent variable and the calculation. They needed help from the teacher to activate the correct button. They chose at first NP. They moved for a long time the point M and observed how numerical values of this variable and of the area MN×MQ changed. They found an optimal value and interpreted it: "(the optimum) is when N is the midpoint of [AB] I believe, and P is the midpoint of [BC]". The teacher asked them for a proof. A student suggested an equation in an interrogative tone. Actually, the problems solved in sessions 4 and 5 were about equalities of areas and have been solved by way of an equation.

The teacher guided them to export the function, but they found the resulting expression too complicated. Then they choose another independent variable MQ, and got the same expression after exporting again the function. Finally, they chose x_M as an independent variable, obtained the algebraic expression b(x-1/ax)(a+c-a(x-1/ax)-c(x-1/ax)) and expanded it into a quadratic polynomial.

Proving the maximum

The team graphed the function, recognized a parabola, and said that they do not know how to determine the maximum's x-coordinate. Then they wanted to apply an algebraic formula to get this x-coordinate and used Casyopée to expand the expression. For some reason they got a non parametric expanded expression, the parameters being instantiated. Then it was easy for them to obtain by paper/pencil a numerical value of the maximum's x-coordinate. Then they returned to the geometrical window, checked this result and generalized, saying that the maximum is for x_M=a/2.

They did not attempt to prove this generalized property by working on the parametric expression and then they only partially solved the problem. Other teams did, but had much difficulty to apply the formula to the parametric quadratic expression.

SYNTHESIS

The observation reported above is globally consistent with the a priori analysis. The students used more or less all registers of representation. The independent variable was recognized as the central feature of the solution, allowing connections between registers. Casyopée offered means for exploration and various feedbacks that helped this recognition. The students' instrumental genesis helped them globally to interact with Casyopée, but important actions like choosing a variable and exporting a function were still unfamiliar. They were influenced by the problems they solved before and it was difficult for them to have a clear approach of an optimization problem. Although they used parameters before and they understood the generalized problem, using parametric expressions was still difficult.

With regard to our question on how to exploit Casyopée's varied functionalities of representation, we can say that, in spite of remaining difficulties, the teaching experiment helped this team to develop an understanding of a functional dependency. We have of course not now a more definite conclusion and we are currently analysing the other teams' observation as well as productions after the experiment. We are especially sensible to the teacher's help to students. In the above observation, we saw this help in crucial episodes, like changing settings and we want to know whether this help was efficient for students' learning, beyond the solution of the problem.

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