THE AMBIGUITY OF THE SIGN $\sqrt{\ }$ [19]

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In this paper an educational problem is discussed deriving from the ambiguity of the radical sign, $\sqrt{\ }$, produced by its shift in meaning when passing from arithmetic to algebra. This problem is concerned with understanding difficulties that are linked to a particular tradition of teaching in which the radical sign is introduced by means of the square root notion. As a conclusion it indicates that any teaching proposal should take into account the distinction between root and radical.

Key words: roots, radicals, meaning, textbook

INTRODUCTION: The problem under investigation

The ambiguity of the sign $\sqrt{\ }$ as a consequence of the change in its meaning when passing from arithmetic to algebra often goes unnoticed by teachers and textbook authors. This lack of perception may be the cause of certain cognitive conflicts experienced by teachers and students.

This work takes its cue from one of these conflicts. It is a conflict expressed by a Spanish secondary school mathematics teacher called Patricia, on attempting to understand the definition of equivalent radicals. She states that the equality $\sqrt[6]{3^2} = \sqrt[3]{3}$ cannot be true, since in the expression on the left the index of the root is even, so that it has two opposing roots, two solutions, whereas in the expression on the right the index is odd so it only has one root, which means that the two expressions have a different number of roots.

The conflict expressed by Patricia leads to the difficulties and controversies related to the values, properties and rules of radicals, which are the ultimate aim of this work.

Examples of that, are the students opinion about the statement $\sqrt{\boldsymbol{25}} = \pm\ 25$. (Roach, Gibson and Weber, 2004), the value of $(-8)^{1/3}$ =-2 (Even and Tirosh, 1995; Goel and Robillard, 1997; Tirosh and Even, 1997), and the rule for multiplying imaginary numbers (Martínez, 2007).

THEORETICAL FOUNDATIONS

To support the work carried out, a theoretical approach has been adopted that has three fundamental references.

1. One of these looks at the cognitive side, taking into account the need to re-conceptualise signs that change meaning when passing from arithmetic to algebra (Kieran, 2006, p. 13).

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This happens with the sign $\sqrt{\ }$ which changes meaning, since it either indicates an operation, as happens in $\sqrt{4}$, or indicates the main root of this operation, as happens in the solution to the equation $x^2 - a = 0 \rightarrow x = \pm\sqrt{a}$.

There are examples of this double meaning to be found in the teaching tradition that appears in textbooks by such influential authors as Euler.

Euler considered that:

> 150. (…) the square root of any number always has two values, one positive and the other negative; that $\sqrt{4}$, for example, is both +2 y -2, and that, in general, we may take $-\sqrt{a}$ as well as $+\sqrt{a}$ for the square root of a (…). (Euler, 1770, p. 44)

In Euler's text the $\sqrt{\ }$ sign is used ambiguously. In $\sqrt{4}$ it is perceived as an indicated operation (finding the square root of 4) and it is associated to the set of two results, in this case +2 and - 2. In $+\sqrt{a}$ it is perceived as a result of the aforementioned process and designates one of the two roots of *a*.

This duality of meaning starts in arithmetic when introducing the $\sqrt{\ }$ sign in order to indicate an operation in an abbreviated way, *the fifth elementary operation*[20]. In arithmetic this number can be found and it is unique. Thus, for example, the square root of 4 is 2, which is written $\sqrt{4} = 2$.

Things change in algebra, since the square root of a $(a > 0)$ cannot be calculated, so that to indicate its value the expression $\sqrt{a}$ is introduced, which no longer represents an indicated operation but a result.

3. The second reference looks at the formal component. The mathematicians have decided to assign to the radical expression, $\sqrt[n]{x}$, $x \geq 0$, only one value, one of the roots of x, the root no negative, the one that they name principal root. With this restriction, the right thing is to write $\sqrt{4} = 2$, not $\sqrt{4} = \pm 2$.

> We agree to denote by $\sqrt{a}$ the positive square root and call it simply the square root of a. Thus $\sqrt{4}$ is equal to and not -2, even thought $(-2)^2=4$ (Lang, 1974. p. 10).

With this decision, the mathematical problem of the ambiguity of the radical sign disappears, but no the didactic problem. Students do not learn only what they are told; much of students' learning occurs when they attempt to make sense of the mathematical situations that they encounter (Roach, et al. 2004). To help students to make sense of the formal definition there are several options:

A) To avoid contradictions. If $\sqrt{4} = \pm 2$, then $\sqrt{4} + \sqrt{4} = (\pm 2) + (\pm 2) = \{-4, 0, +4\}$; $\sqrt{4} - \sqrt{4} = (\pm 2) - (\pm 2) = \{-4, 0, +4\}$ and $\sqrt{4} + \sqrt{4} = \sqrt{-4} + \sqrt{-4}$

[20] This consists of given a number, find another which when multiplied by itself gives the first.

B) To satisfy the requirements for the definition of operation of exponentiation to rational exponents. This definition should not depend on the representatives of numbers involved in the operation. We want $a^r = a^{\frac{m}{n}} = \sqrt[n]{a^m} = a^{\frac{km}{kn}} = \sqrt[kn]{a^{km}}$ (see Tirosh,& Even, 1997, p. 327). Nevertheless, if $\sqrt{4} = \pm 2$, then $\sqrt[6]{3^2} \neq \sqrt[3]{3}$. And, in general, $\sqrt[kn]{a^{km}} \neq \sqrt[n]{a^m}$, when *kn* is even and *n* is odd,

C) To satisfy the requirements for functions. The basic arithmetic operations addition and multiplication by a number different from zero establish bijective functions: $x \rightarrow x+a$, $x \rightarrow x \cdot a$, $a \neq 0$. These functions have unique inverse functions corresponding to the inverse operations. But, an operation like: $x \rightarrow x^2$ does not establish an injective function; because $x^2 = (-x)^2$. Consequently, the function $x \rightarrow x^2$ has to be confined to one of its branches to be inverted, $x \geq 0$. In the same way the inverse operation, $x \rightarrow \sqrt{x}$, has to be confined to positive domain, and range, in order to be unique.

2. The third reference takes on a psychological point of view, taking into account the dual operational/structural nature of mathematical conceptions and their role in the formation of concepts, indicated by Sfard (1991).

Sfard (1991) supports this theory with the fact that a mathematical entity can be seen as an object and a process. Treating a mathematical notion as an object leads to a type of conception called structural, whereas interpreting a notion as a process implies a conception called operational.

For Sfard, the ability to see a mathematical entity as an object and a process is indispensable for a deep understanding of mathematics, such that the “concept formation implies that certain mathematical notions should be regarded as fully developed only if they can be conceived both operationally and structurally” (p. 23).

It is worth pointing out that when referring to the role of operational and structural conceptions, Sfard conjectures that when a person gets acquainted with a new mathematical notion, the operational conception is usually the first to develop, whereas the structural conception follows a long and difficult process that needs external interventions (of a teacher, of a textbook), and may therefore be highly dependent on a kind of stimulus (of teaching method) which has been used (p. 17).

Pointing out that, the investigation on the conceptualization of the radical sign should be held in a revision of manuals and textbooks.

OBJECTIVES

Once the general problem to be studied has been pointed out, as well as the theoretical references, it is necessary to specify the general aims that are to guide the investigation's design and methodology:

1. To determine the characteristic aspects of teaching the radical sign, just as they are shown in textbooks today.

2. To diagnose mathematical knowledge with respect to the radical sign that some secondary school teachers have.
3. To explain teachers' possible conceptual and operational difficulties.

PATRICIA'S CONFLICT

The aims are linked to Patricia's conflict. Patricia is a high school mathematics teacher (in Spanish public education) and a student in a post-graduate programme. She presented the following conflict to her professor:

> In the textbook, the concept of equivalent radicals is defined as follows: "Two radicals are equivalent if they have the same roots" (and so I had learned). On the other hand, simplifying a radical by dividing the index of the radical and the exponent of the radicand by the same number, results (in theory) in a radical equivalent to the first. However, in a case like the sixth root of three squared, the cube root of three is obtained. As the index of the first radicand is an even number, two solutions exist (one being the opposite of the other) but in the second case, the index is an odd number and therefore there is a single root. Therefore, it cannot be said that these two radicals have strictly the same roots. So, are they equivalent?

Patricia says:

(A) Two radicals are equivalent if they have the same roots.

Also Patricia makes reference to the following equivalency:

(E) $\sqrt[nk]{a^k} = \sqrt[n]{a}, k, n \in N^*, n \geq 2, a \geq 0.$

Applying the equivalency (E), Patricia obtains than: $\sqrt[6]{3^2} = \sqrt[3]{3}$. However, to her the sixth root of three squared has two opposed roots, “two solutions”, as the index is an even number and the cube root of three has a single root as the index is an odd number, which means that the two expressions do not have the same number of roots and so according to (A) they would not be equivalent.

Hypothesis in relation to this conflict

In order to try to explain the causes of conceptual and operative difficulties that give rise to Patricia's conflict, the following hypothesis has been formulated:

(H_1) The lack of perception of the difference between the operational and structural conceptions of the radical sign that Patricia expresses is the cause of her conflict.
(H_2) This lack of perception is a product of a traditional teaching proposal, which does not pay attention to the need to re-conceptualise the √ sign when passing from arithmetic to algebra.
(H_3) In an alternative teaching proposal, where the meanings of root and radical are formulated, the conflict expressed by Patricia is not expected.

METHODOLOGY

To verify the solidity of the hypotheses an exploratory study was carried out, as a step prior to a more rigorous inquiry in terms of methodology, still to be carried out.

This exploration is based on a revision of current and representative textbooks of two alternative proposed ways of teaching: the Spanish one, which introduces the radical sign in arithmetic, and the Rumanian one, which introduces it in algebra. The revision of textbooks is has been complemented by a questionnaire followed by an interview with two representative individuals, Patricia (Spanish) and Iulian (Rumanian), two typical high school mathematics teachers.

With the revision of textbooks an attempt has been made to identify characteristic features in the teaching of roots and radicals in Spanish and Rumanian textbooks, and to identify comments that may favour the ambiguity of the $\sqrt{\ }$ sign, and Patricia's conflict.

The questionnaire

The questionnaire consists of a paper and pencil test which included four tasks. The first one is based on the teaching proposal given in the Spanish textbooks. In the task it is considered, as in Euler's text, that the square root of any positive number has two solutions, one positive and another negative. However, to represent this set of results the symbolic form $\sqrt{4} = \pm 2$ is used as well as the rhetorical form: "the solution is double, positive and negative". The intention of this task was to know if the difference is perceived between the structural and operational conception. The task is:

> In the class of 9th grade, after introducing the theme of the roots and radicals, the students were asked to calculate the square root of four.
>
> One student wrote $\sqrt{4} = \pm 2$, justifying thus:
>
> "As the radicand is positive and the root's index is even, then the solution is double, positive and negative".
>
> Is this correct?

Task 1

The interview's design took into account the answers produced by Patricia and Iulian to task 1. If the answer was "No", then the interviewee was asked to justify why and if it was "Yes", then they were given the second task with the aim of bringing in a cognitive conflict, in order to study the students' reaction

The second task is based on substituting $\sqrt{4}$ for ± 2 in a context of calculation. With this the aim was to put the affirmative answer to the task 1 into conflict.

> If $\sqrt{4} = \pm 2$ then complete:
>
> $\sqrt{4} + \sqrt{4} = (\pm 2) + (\pm 2) = \ldots$
>
> $\sqrt{4} - \sqrt{4} = (\pm 2) - (\pm 2) = \ldots$
>
> Explain the answer.

Task 2

A third task is based on the restriction of the property of radicals in the case where *k* is an even number and $a < 0$, which requires the intervention of the module.

$$\text{(P)} \quad \sqrt[nk]{a^k} = \begin{cases} \sqrt[n]{a}, & k, n \in N, n \geq 2, a \geq 0 \\ \sqrt[n]{|a|}, & k, n \in N, k - \text{even}, n \geq 2, a < 0 \\ \sqrt[n]{a}, & k, n \in N, k - \text{odd}, n \geq 2, n - \text{odd}, a < 0. \end{cases}$$

Here, the intention was to confirm that the interviewee was taking into account the radical's formal definition, in a traditional problematic case. The hypothetic situation that is present is the following:

> In a class of 10th grade, after introducing the radicals theme, the students were asked to simplify:
>
> $$\sqrt[6]{(-8)^2}$$
>
> One student wrote: $\sqrt[6]{(-8)^2} = \sqrt[2\cdot 3]{(-8)^2} = \sqrt[3]{-8} = -2$
>
> and said: "I have applied the following rule: $\sqrt[nk]{\boldsymbol{a}^{mk}} = \sqrt[n]{\boldsymbol{a}^{m}}$. Is this correct?

Task 3

If the answer to the task was "No", then the interviewee was asked to justify why and if it was "Yes", then the fourth task was given with the aim of introducing a cognitive conflict, in order to study the student's reaction.

Task 4 imposes the strategy for calculating $\sqrt[6]{(-8)^2}$ that leads to a different result from -2. With this, the intention was to put the affirmative answer given previously to the task 3 into conflict, in order to again study the reaction of the interviewee.

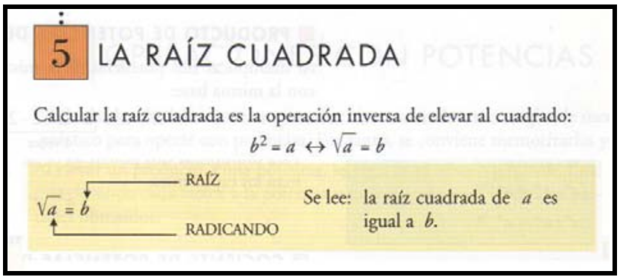
> If you consider:
>
> $\sqrt[6]{(-8)^2} = \sqrt[2\cdot 3]{(-8)^2} = \sqrt[3]{-8} = -2$
>
> then complete:
>
> $\sqrt[6]{(-8)^2} = \sqrt[6]{64} = ...$

Task 4

RESULTS OF TEXTBOOKS REVIEW

1. In the Spanish textbooks reviewed the sign $\sqrt{\ }$ is used to express the reverse operation of taking a number to the power of two (Figure 1):

> Calculating the square root is the reverse operation of calculating the power of a square: $b^2 = a \leftrightarrow \sqrt{a} = b$.



5 LA RAÍZ CUADRADA

Calcular la raíz cuadrada es la operación inversa de elevar al cuadrado:

$b^2 = a \leftrightarrow \sqrt{a} = b$

$\sqrt{a} = b$ — RAÍZ — RADICANDO

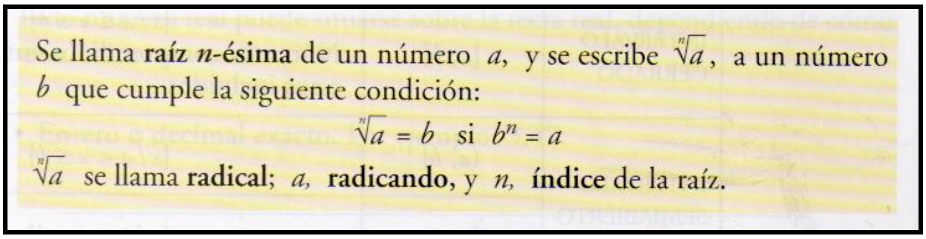
Se lee: la raíz cuadrada de *a* es igual a *b*.

Figure 1. 1º Secondary (7th grade), Anaya 2006, p. 52

The expression that has the $\sqrt{\ }$ sign is called a radical, that is to say the operation shown, and not the main root of said operation (Figure 2).

It is called the n[th] root of a number a, and is written $\sqrt[n]{a}$, where a number *b* meets the following condition: $\sqrt[n]{a} = b$ and $b^n = a$

$\sqrt[n]{a}$ is called radical; a, radicand, and n, the root's index.



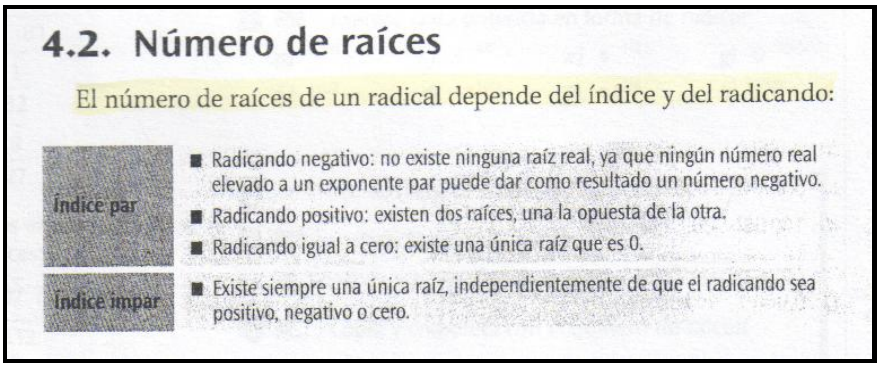
Se llama **raíz *n*-ésima** de un número *a*, y se escribe $\sqrt[n]{a}$, a un número *b* que cumple la siguiente condición:

$$\sqrt[n]{a} = b \quad \text{si} \quad b^n = a$$

$\sqrt[n]{a}$ se llama **radical**; *a*, **radicando,** y *n*, **índice** de la raíz.

Figure 2. 4º Secondary (10th grade), Anaya, 2006 b, p. 32

As a consequence it is considered that a radical has roots and that its number depends on the index of the radicand's sign (Figure 3).



4.2. Número de raíces

El número de raíces de un radical depende del índice y del radicando:

Índice par	■ Radicando negativo: no existe ninguna raíz real, ya que ningún número real elevado a un exponente par puede dar como resultado un número negativo. ■ Radicando positivo: existen dos raíces, una la opuesta de la otra. ■ Radicando igual a cero: existe una única raíz que es 0.
Índice impar	■ Existe siempre una única raíz, independientemente de que el radicando sea positivo, negativo o cero.

Figure 3. 3rd Secondary (9th grade), Oxford, U. P., 2007, p.32

So, equalities appear written as $\sqrt{36} = \pm 6$ (Figure 4).

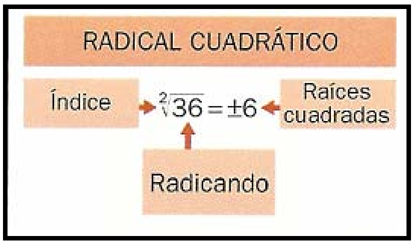


Figure 4. 3rd Secondary (9th grade), S. M., 2003, P. 36

The properties of the radicals are stated without mentioning their field of validity. So it is not taken into account that $\sqrt{a^2} = |a|, \forall a \in \mathbf{R}$ (Figure 5).

Propiedades de los radicales

- $\sqrt[np]{a^p} = \sqrt[n]{a}$
- $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
- $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$
- $(\sqrt[n]{a})^p = \sqrt[n]{a^p}$
- $\sqrt[m]{\sqrt[n]{a}} = \sqrt[m \cdot n]{a}$

Figure 5. 4º Secondary (10th grade), Anaya, 2006 b, p. 36

2. In the Rumanian textbooks reviewed, the sign $\sqrt{\ }$ is associated with the radical notion. The radical with an index two of a positive number a is defined as the positive solution of the equation $x^2 = a$ and is denoted by $\sqrt{a}$. (Figure 6)

4 Radicalul de ordinul 2 şi radicalul de ordinul 3. Proprietăţi

Radicalul de ordinul 2

Fie $a \geq 0$. Se poate arăta că există şi este unic $x \geq 0$ astfel încât $x^2 = a$. Privind acest număr ca fiind obţinut prin operaţia inversă ridicării la pătrat, acceptăm

Definiţia 4.1 *Numărul pozitiv x, cu proprietatea $x^2 = a$, se numeşte radicalul de ordinul 2 al lui a (rădăcina pătrată a lui a) şi se notează $\sqrt{a}$:*

$$\boxed{(\sqrt{a})^2 = a}, \quad a \geq 0.$$

1) $\sqrt{a} \geq 0$, pentru orice $a \geq 0$.
2) Nu se defineşte $\sqrt{a}$ pentru $a < 0$ întrucât $x^2 \geq 0, \forall x \in \mathbf{R}$. Astfel, $\sqrt{a}$ are sens numai pentru $a \geq 0$.
3) Deşi $2^2 = 4$ şi $(-2)^2 = 4$, avem $\sqrt{4} = 2$. Cu alte cuvinte, a scrie $\sqrt{(-2)^2} = -2$ este o greşeală!

Figure 6. 10th grade, Fair Parteners, 2005, p. 13

It is taken into account that $\sqrt{\boldsymbol{a}^2} = |\boldsymbol{a}|, \forall \boldsymbol{a} \in \mathbf{R}$, and the domain of validity of the radical's properties is specified. (Figure 7)

Proprietăţi ale radicalilor de ordin 2:

1) $\sqrt{a^2} = |a|, \quad \forall a \in \mathbf{R}$;
2) $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}, \quad a \geq 0, \ b \geq 0$;
3) $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}, \quad a \geq 0, \ b > 0$;
4) $\sqrt{a^n} = (\sqrt{a})^n, \quad a \geq 0, \ n \in \mathbf{N}^*$;
5) dacă $a \geq 0$ şi $b \geq 0$, atunci $a < b \Leftrightarrow \sqrt{a} < \sqrt{b}$ şi $a = b \Leftrightarrow \sqrt{a} = \sqrt{b}$;
6) $\frac{2}{\frac{1}{a} + \frac{1}{b}} \leq \sqrt{ab} \leq \frac{a+b}{2}, \quad \forall a, b \in \mathbf{R}, \ a > 0, \ b > 0.$[1]

Figure 7. 10th grade, Fair Parteners, 2005, P. 13

FINDINGS AND CONCLUSIONS

As for the first objective, the review of textbooks shows that there are substantial differences in dealing with the $\sqrt{\ }$ sign. Specifically, it can be said that in the Spanish textbooks studied, the conception associated with this sign is operational, whereas in Rumanian texts it is structural.

As regards the second objective, Patricia and Iulian's mathematical knowledge with respect to the radical sign shows significant differences.

In tasks 1 and 2, Patricia identifies $\sqrt{4}$ with the set of two solutions (2 and -2), and does not see the radical as the positive root when the index is even. In the interview, to emphasize this in task 2, she indicated that in reality there are not two solutions, but there are contexts in which it is replaced by +2 and others in which it is replaced by -2.

Iulian does not agree with $\sqrt{4} = \pm 2$, arguing that the radical of an even index of a positive number belongs to the interval $(0, \infty)$ and specifies that, in any context $\sqrt{4}$ represents a number, that is, the positive square root of 4.

In task 3 and 4, Patricia does not take into account that $\sqrt{a^2} = |a|, \forall a \in \mathbf{R}$. On the other hand Iulian correctly applies the restriction of the property of radicals and he realizes the error that the hypothetical student commits.

In conclusion, it can be said that Patricia has procedural knowledge of the $\sqrt{\ }$ sign, whereas Iulian has structural knowledge, and that these conceptions are consistent with what is shown in the textbooks studied.

As for the third objective, this part of the work was restricted to Patricia's conflict, the answers to the questionnaire and the interviews that provide indications suggesting the validity of the hypotheses.

(H_1), Patricia does not distinguish between operational and structural use of radical sign.

(H_2), the review of Spanish texts evidences that the teaching proposal reflects the ambiguity of the radical sign, used in the expression $\sqrt{4} = \pm 2$, and does not use the formal definition of radicals, so that it is plausible to think that they encourage the appearance of Patricia's conflict.

(H_3), in the revised Rumanian texts, the formal definition of the radical sign is observed, so that it is possible to think that they support Iulian's way of acting, which does not encounter the conflict that Patricia expresses.

Finally, the important educational implication that should be pointed out is that in any educational proposal that aims to avoid conflicts such as the one expressed by

Patricia, the formal definition of radical must be considered, and it must be guaranteed that students understand the reasons for this definition.

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