TRANSFORMATION RULES: A CROSS-DOMAIN DIFFICULTY

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The learning of a symbolic system such as algebra relies on the learning of the use of transformation rules. The implementation of rules in a CAS (Computer Algebra System) for students' modelling has pointed out some questions that are at the junction of three research fields: informatics, mathematics and didactics. Each of these communities has its own perception of algebraic objects, founded on models or practices. The implementation of objects that live in school has questioned object reliability. In this paper, a parallel is proposed between difficulties of informatics implementation of transformation rules and novices' difficulties.

Keywords: algebraic calculations, rules, informatics implementation, students' difficulties.

An important part of school algebra rests on algebraic calculations, what Kieran calls the "transformational activity", which she distinguishes from the generational and global activities (Kieran, 2001). This activity focuses on changing the form of an expression or an equation in order to *maintain equivalence*. This includes, for instance, collecting like terms, factoring and expanding expressions. These are algorithmic tasks like the transformation of $(5+x)x+10+2x$ into $(5+x)(x+2)$. The conservation of equivalence relies on *correct rules* that allow substituting expressions by others. These rules will be called "transformation rules" in this paper. They are supported by the laws of the polynomial ring –commutative law, distributive law and so on. Rules produce objects of a particularly interesting form. Their use is guided by what the desired expression has to look like: reduced polynomial expression or factored polynomial expression. Bellard et al. (2005) call them the constituent rules of mathematic theory: "these rules constitute the base of the [transformational] activity, govern the motion and predetermine the permitted actions". Such mathematical rules are supposed to be accurate and self-sufficient.

Nevertheless, Durand-Guerrier and Herault (2006) stress the fact that rules are objects the usage of which is not so obvious: "the rule is not only a way to learn but it is also an object which has to be learned". It is, in fact, impossible to present a rule alone to students. Rules have to be transposed, adapted and as such lose a part of their accuracy. The implicit notions of rules are compensated by a necessary didactical contract (Brousseau, 1997): "it is an illusion to believe that one can produce the meaning in the mind of someone by indirect ways through the rule and examples" (Wittgenstein, Ambrose, & Macdonald, 1979). Durand-Guerrier and Herault (op.cit.) also point out the illusion to think that the use of a rule is plain, such as "rails that would guide unfailingly and in advance the way to be followed". Actually, it is an interpretation that allows these implicit details between the rule and its application to be overcome. But what are exactly the notions underlying the learning of a transformation rule?

Our research is in line with the identification of systematic errors that students commit when solving transformational exercises. A library of correct and incorrect transformation rules has been built for that purpose and an automatic diagnosis mechanism has been implanted in order to associate a sequence of applied rules to student's transformation (Chaachoua, Croset, Bouhineau, Bittar, Nicaud, 2008). The implementation of these rules has raised questions about the kind of representation of a transformation rule. Automating the process forces the researcher to clarify some implicit mechanisms for the expert: how does a rule work? In which way does it work? How is it matched? It has led to three crucial points about implantation difficulties:

- The reading direction of a rule;
- The notion of sub-expression;
- The generic status of a rule.

Each of these points is discussed in the next sections. We propose, in addition, to link these three points to three classical difficulties which novices may experience when doing transformational activities: the difficulty of understanding the symmetric aspect of the equal sign (see e.g. (Kieran, 1981)); the difficulty of the structural aspect of an algebraic expression (see e.g. (Sfard, 1991)) and the difficulty of applying a general rule to a particular case (see e.g. (Durand-Guerrier, Herault, 2006, p. 144)).

The choices made to raise difficulties in programming may shed light an improvement of the teaching of algebraic rules and may overcome students' problems. Indeed, the reading direction of a rule is essential for a deductive reasoning, the notion of sub-expression allows matching correctly a rule and the generic status of a rule is the power of algebra.

1. REPRESENTATION, READING DIRECTIONS AND REASONING PROCESS

Transformation rules can be represented by two kinds of writings: equality or implication. Both present advantages and have good reasons to be used. Yet, we will see that rules as implication form are interesting in that it highlights the reasoning process in the transformation activity.

Rules as equality, used in school

The first representation –a rule as an equality– is the usual one used in school. Rules can be called by different names in the textbooks: proposition, property, identity, equality and sometimes even theorem (Bellard et al., 2005). Whatever their name, rules are often coming in the form of equality. For example, the distributive law is presented as:

$k(a+b) = ka + kb$, where k, a and b are real. (*Eq1*)

There is a double meaning of the equal sign: that of "identity" or that of "relation". In transformation rules, the equal sign is of course used as "identity", whereas in equations, the equal sign is used rather as "relation". This well-known duality is a real difficulty for students. Presenting rules as identity can, on the one hand, be interesting to get students used to and, on the other hand, provoke confusion.

Such representations are declarative rather than procedural: this form of identity has no explicit reading direction since the equal sign has a double way: from left to right and vice versa. A learning of the way to use such a rule has to be taught. Whereas the process-product has been many times denounced (Davis, 1975) and that special exercises are proposed to students in order to grasp the equivalence notion, here is a case where the equality has to be used in one of the two ways. In fact, textbooks sense that, most of the time, it is necessary to distinguish the two ways by proposing two identities: not only (*Eq1*), which is used to expand expressions but also "$ka+kb=k(a+b)$" to factor. This kind of presentation requires a specific work to become operative: associate a reading direction to the equality for application, according to the aim.

Rules as implication, used in informatics

The second representation –a rule as an implication– is the one used in informatics. One calls "implication" what Durand-Guerrier, Le Berre, Pontille, & Reynaud-Feurly (2000) call "formal implicative", representation used in geometry:

$\forall x \in \Re,\ P(x) \Rightarrow Q(x)$.

Implemented rules are represented as *oriented* mechanisms, also called "rewrite rules" (Dershowitz & Jouannaud, 1990): $A \to B$, where A is *rewritten* in B. The object A produces the object B and B can not produce A unless an other rule $B \to A$ is considered. For example, the rules:

$k(a+b) \to ka+kb$ (*R1*) is used to expand,

$ka+kb \to k(a+b)$ (*R2*) is used to factor.

It is rather a necessity in computing modeling to represent rules as oriented ones than a choice. Indeed, it is not really possible to implement rules as identity. If a single rule is implemented both for expanding and for factoring, there will be some loop and ending problems. For example, with the single rule (*Eq1*), the expression "$3(x+4)$" would be transformed into "$3x+12$", then into "$3(x+4)$" and so on.

Even if it is a necessity, this kind of representation is interesting because its reading direction is explicit: given a real or a polynomial expression under the form "$ab+ac$", where "a", "b" and "c" are reals or polynomials, it can be rewritten into "$a(b+c)$". One can suppose that the use of rules as implication is easier because of its procedural aspect. The kind of representation has an impact on its use easiness, as we will show in the next section.

Impact of the reasoning process

Although geometry is a special introduction field for proof, the latter is not a prerogative of geometry. The "deep structure" (Duval, 1995) of the transformational activities can be presented as a ternary organisation proposed by Duval. A premise (here, a certain expression), a proposition (a transformation rule) and a conclusion (an other expression), as shown in Figure 1, constitute a deduction step. These steps follow on, the conclusion of the current step becoming the premise of the next one. Using Duval's classification (Duval, 1990), the algebraic calculation is formed by *deductive reasoning of steps explicitly concatenated in reference to a transformation rule*. Thereof, this activity can be viewed as a process of demonstration:

> "Demonstration would be defined to be, a method of showing the agreement of remote ideas by a train of intermediate ideas, each agreeing with that next it; or, in other words, a method of tracing the connection between certain principles and a conclusion, by a series of intermediate and identical propositions, each proposition being converted into its next, by changing the combination of signs that represent it, into another shown to be equivalent to it" (Woodhouse, 1801)

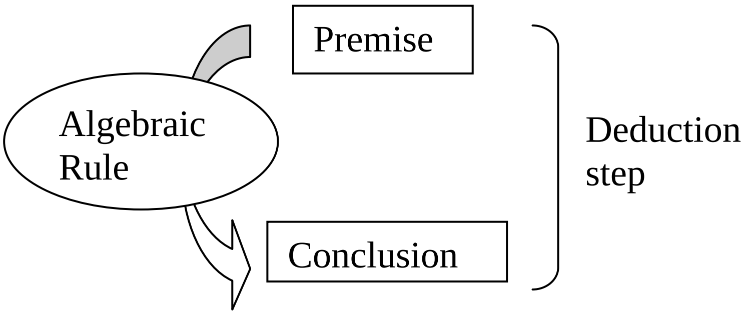

Figure 1: Deduction steps.

Representing rules as implications could allows the user to follow this reasoning process explicitly, as shown in Figure 2.

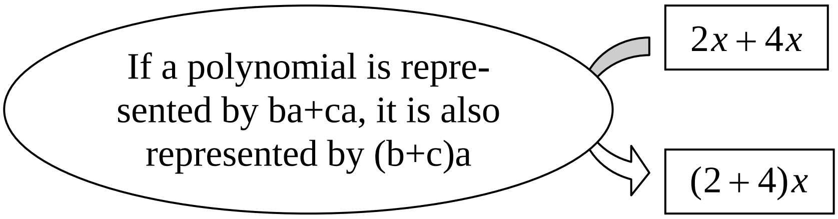

Figure 2: Example of the reasoning process in algebra. The level of making explicit a demonstration and the granularity of a deductive step evolves with the level of the student. Here, for example, we have omitted to explicit the commutative law. As Arsac notes: "any demonstration is shortened from another demonstration" (Arsac, 2004).

Splitting an identity into two implications conceals the fact that rules are equivalent but clarifies the *way of application* and, above all, it allows following the Duval's structure of a deduction step. This is the *modus ponens* mode: "*if p, then q*, now *p*, then *q*”. The representation form of a rule has an impact on its use easiness but it lets the difficulty to know to which object the rule can be applied.

2. MATCHING AND SUB-EXPRESSIONS NOTION

An unrefined syntactic unification between the premise of a rule and a part of an expression does not produce an algebraic behaviour. With an unrefined unification, a rule as $x + x \to 2x$ would transform “$5\boldsymbol{x} + \boldsymbol{x}$” into “$5\ \mathbf{2}\boldsymbol{x}$”, which has no sense (what is the operator between “5” and “2”?) nor the expected result. This is a well-known mistake committed by students: substitute an expression by another by working only on a syntax level and taking no account of semantics. Mastering substitution needs knowing the notion of what a *sub-expression of an expression* is.

The definition of an expression from the rewrite rule theory of Dershowitz (1990), in which rules are applied on *sub-objects*, underlines the notion of sub-expression, thanks to its recursive definition. Let us consider a set of symbols of terminal objects (e.g., integers), a set of symbols of variables (e.g., {*x*, *y*, *z*}), and a set of symbols of operators (e.g., +, –, ×, ^, sqrt, =, <, and, or, not). An *algebraic expression* is a finite construction obtained from the following recursive definition:

- a symbol of terminal object
- or a symbol of variable
- or a symbol of operator applied to arguments which:
    - are algebraic expressions,
    - are in the correct number (correct arity [1]),
    - and have correct types [2].

With this definition, matching a rule R to an expression E would consist of finding a sub-expression E1 of E, replacing E1 in E by the expression that produces R. For example, in “$5x + x$”, the algebraic (sub) expressions are “$5x$”, “x” (two times), “5” and “$5x + x$”. The expression “$x + x$” is not a sub-expression of “$5x + x$”. To deal with this problem, the internal representation of expressions in computer algebra systems (CAS) is a tree representation, in which the structure of the expression is explicit, as shown in Figure 3.

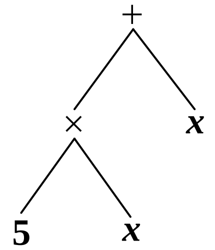

Figure 3: Tree representation of the expression $5x + x$.

The necessity of the tree representation appears also in school curricula. Although school approach of expressions is foremost syntactic -algebraic expressions are defined as “writings including one or more letters”-, new French curricula encourage making students work on tree representations. As they claim, tree representation allows pointing out the structural aspect of an expression as defined by Kieran:

> “The term *structural* refers, on the other hand, to a different set of operations that are carried out, not on numbers, but on algebraic expressions. [...] the objects that are operated on are the algebraic expressions, not some numerical instantiations. The operations that are carried out are not computational. Furthermore, the results are yet algebraic expressions.” (Kieran, 1991)

This structure notion is essential to deal with matching difficulties. It enables understanding why such rule like $k(a+b) \to ka+kb$ (*R1*) can be applied on sub-expressions of expressions such as $3+4(x+1)$. Nevertheless, is it sufficient to understand that this rule can be applied also on expressions such as $4x^2(x+1)$ or $4x^2(x+1+x^2)$? Either in informatics or at school, we will see that most of the time, one needs to precise as many rules as there are cases.

3. GENERIC STATUS OF RULES

The third idea which emerges of rules implementation turns on the generic status of a rule: how a rule such as (*Eq1*) or (*R1*) can be sufficient to apply to the expressions “$7(3+x)$”, “$-7(3+x)$”, or even “$7(3+x+x^2)$”? How to deal with the matching of “$a+b$” with “$3+x+x^2$”? It is, with no doubt, the principal difficulty for novice users of rules: the application of a general rule to a particular case. It is, in fact, the same in informatics. Although the two first points –reading direction & matching problems– have been easily resolved in informatics, it has not been the same for this third problem.

The entry by rewriting rules –and thus a syntactic presentation– leads to some new problems. Let us study again the case of (*R1*). For experts, it is not really this rule that is used but much more the single distributive law. With this last one, experts can expand any product of polynomials. In informatics, one needs rules to be implemented and so, the exact structure of an expression has to be specified. For (*R1*) implementation, “k”, for example, has to be defined: is “k” a real, a product such as a monomial or a sum? It is not possible to just say “given a polynomial k”. Indeed, to transform “k”, its structure has to be specified. For example, if “k” is negative, the sign of the entire expression is changed. The main operator of the expression becomes “minus” and not “times”: the entire internal tree representation is changed, as shown in Figure 4. The same difficulty is found when “$a+b$” is a sum of three terms: it can change the mechanism of the implementation of the rule. Without genericity, one needs to distinguish cases like “$k(a+b)$” from “$k(a+b+c)$”. To deal with that, the concept of distributive law has been implemented. Let us consider two lists and an operator, the distributive rule can be written as:

$$(a_1, a_2, ..., a_n)\Delta(b_1, b_2, ..., b_n) \rightarrow (a_1\Delta b_1, a_1\Delta b_2, ..., a_1\Delta b_n, a_2\Delta b_1, a_2\Delta b_2, ..., a_n\Delta b_n).$$

We do not have to specify the length "n" of the lists.

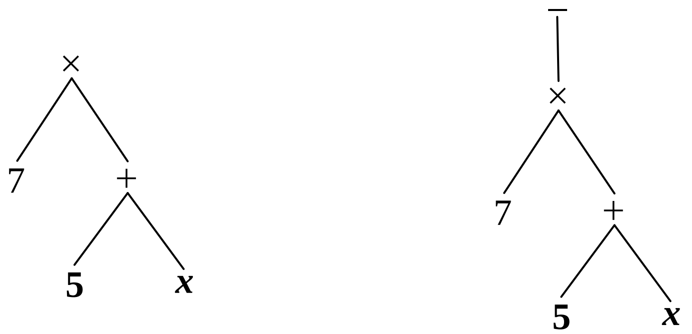

Figure 4: The single change of the real 7 into -7 changes the entire structure of the tree representation of the expression. On the left, the expression 7(5+x); on the right, the expression -7(5+x).

Another example is very representative of this problem: the rule of monomials addition, which can be written as $ax + bx \rightarrow (a \oplus b)x$, where $\oplus$ is the calculated sum operator. Such rule is not so easy to implement. If we ask the premise to be a sum of two products, this rule will not apply to expressions such as "$ax + x$" because "x" is a single argument and not a product: an automatic mechanism does not recognize "x" as the product of "1" and "x". To deal with this problem, some concepts have been implemented like the *monomial concept*. We have implemented the added fact that a monomial can be either a product of a real and a variable –of explicit degree or not– or a single variable –of explicit degree or not. Thus, expressions such as "$4x^5$", "$4x$", "x^1" or "x" are read as monomials, and the rule $ax + bx \rightarrow (a \oplus b)x$ can be easily implemented: one needs just to specify that the premise has to be a sum of two monomials.

The same problem occurs at school: the polynomial notion is not taught in France [3]. The variable "k" from the rule (*Eq1*) is then defined as a real, so are "a" and "b". Understanding that "a" can be itself a sum, or even a sum with variables, requires a real work. How do French textbooks deal with this problem?

To answer this question, we have used the concept of praxeologies from the Chevallard's anthropological theory of didactics. Let us remain that Chevallard proposes to describe any human activity by a quadruplet which enables an activity to be cut in types of task, which can be solved by techniques –a way of doing–, which can be explained by a rational discourse, "logos" (Chevallard, 2007) [4]. Our work in progress (Croset, 2009) shows that French textbooks distinguish three types of task for expanding expressions:

"$k(a + b)$", "$k(a - b)$" and "$(a + b)(c + d)$".

Cases like "$-(a+b)$" or "$+(a+b)$" are discussed in another part of textbooks ("how to remove brackets?"). Some textbooks propose even more distinctions: they discern also "$(a+b)k$" and "$(a-b)k$".

On the one hand, it seems that textbooks decide to specify many cases although all these tasks are explained by a single "logos": the distributive law. The fact that textbooks need to precise many cases points out the well-known difficulties of students to apply a general rule to particular cases. On the other hand, all possible cases cannot be specified. Textbooks do not specify types of tasks as "$k(a+b+c)$" or "$k(a+b)(c+d)$". Understanding the structure of the expression is supposed to be sufficient to deal with all these forms. Nevertheless, we have not found such work and reflection about the generality of rules. Only a few textbooks precise links between the three types of task described above. Explanations such as using $k(a+b) \rightarrow ka+kb$ to expand "$(a+b)(c+d)$" are not common. Neither are presented the iteration concept to expand "$k(a+b)(c+d)$" whereas our work (ibid.) shows that students' mistakes occur specially in this sub-type of task.

The second problematic example about monomials revealed by the computing implementation occurs also in students' difficulties: recognizing "x" as a monomial is not an easy task for a novice. A novice's common mistake is precisely to transform "$ax+x$" into "ax" because of the lack of the explicit coefficient "1" ahead of the "x": when "a" is added to "nothing", it remains "a" [5]. The concept of monomial is not taught currently in French curriculum. We speak about "like terms" but few textbooks precise that "x", "$1x$", and "x^1" are "like terms" which can be collected.

The force of algebra lies in the writings generic status. Its interest is lost if all cases are presented. To avoid that, a specific work on concepts such as distributive law or monomial could be proposed to novices, just like it has been done for the computing implementation.

4. CONCLUSION

The learning of the transformational activity cannot be restricted to memorizing rules. This requires a specific work about the application of rules. Our research focusing on automatic student modelling has brought to light three important difficulties concerning the application of transformation rules, which have been compared with similar novices' difficulties: knowing that a rule has a reading direction allows students to follow a reasoning process when they transform algebraic expressions; knowing the structure of an expression permits a correct matching; finally, having a good perception of the generic status of rules allows students to apply a general rule to a particular case. All these elements are necessary conditions for learning the algebraic symbolic system. Our paper has described the parallel between informatics implementation difficulties and the ones met by novices. One can wonder if the way to deal with the first ones could be used to deal with the second ones.

Regarding these three points, rules have been looked at from a technical point of view. Another point of view would be considering what experts' criteria are to control their transformations: substitute numerical values to equivalent expressions in order to verify the equivalence; in other words, being aware that equivalent expressions denote the same object. Similarly, another interesting point of view is to explore how to *choose* the appropriate rule. We have seen that a rule is general but the choice of a rule is crucial to obtain the form that one needs. The *raison d'être* of a rule, the strategic process and elements that guide an expert in choosing this particular rule, and not another one, have not been discussed here, despite the fact that informatics is also interested in such questions. We can expect that a parallel would be again possible between novices' strategic difficulties and the implementation ones.

NOTES

1. The arity of an operation is the number of arguments or operands that the operation takes. For example, addition is an operation of arity 2, sqrt is an operation of arity 1.

2. For example the expression " $\sqrt{5x=3}$ " has not a correct type.

3. A recent study has compared the algebra learning in France and in Vietnam (Nguyen, 2006). It shows that algebraic expressions found in French textbooks rely on the notion of polynomial function whereas the ones that can be found in Vietnamese textbooks rely on the polynomial notion.

4. The reference (Chevallard, 2007) is not the best one for the notion of praxeology but it presents the advantage of being written in English. French reader can see also (Bosch & Chevallard, 1999).

5. Haspekian (2005) proposes another explanation to this mistake: the difficult notion of neutrality of the multiplicative law. We think that, in our context, the mistake is more relative to a visual lack.

REFERENCES

Arsac, G. (2004). *Démonstration en géométrie et en algèbre, figure et calcul littéral: quels rapports ?* Retrieved September 15, 2008, from http://www.apmep.asso.fr/Lyon.html.

Bellard, N., Bronner, A., Boullis, M., Girmens, Y., Larguier, M., Lewillion, M., et al. (2005). *La règle dans tous ses états.* Montpellier: APMEP.

Bosch, M., & Chevallard, Y. (1999). Ostensifs et sensibilité aux ostensifs dans l'activité mathématique. *Recherches en Didactique des Mathématiques, 19,* 77-124.

Brousseau, G. (1997). *Theory of Didactical situations in Mathematics*. Dordrecht: Kluwer Academic Publishers.

Chaachoua, H., Croset, M.-C., Bouhineau, D., Bittar, M., & Nicaud, J.-F. (2008). Description et exploitations des traces du logiciel d'algèbre Aplusix. *STICEF, 14, numéro spécial « Analyses des traces d'utilisation dans les EIAH »,* 347-387.

Chevallard, Y. (2007). Readjusting Didactics to a Changing Epistemology. *European Educational Research Journal, 6*(2), 131-134.

Croset, M.-C. (2009). *Modélisation des connaissances de l'élève au sein d'un logiciel éducatif d'algèbre.* Unpublished doctoral dissertation, Joseph Fourier University, Grenoble.

Davis, R. B. (1975). Cognitive processes involved in solving simple algebraic equations. *The journal of Children's Mathematical Behaviour, 1-3*.

Dershowitz, N., & Jouannaud, J.-P. (1990). Rewrite Systems. In J. Van Leeuwen (Ed.), *Handbook of Theoretical Computer Science, Formal Models and Sematics* (Vol. B: Formal methods and semantics, pp. 243-320). Amsterdam, North-Holland.

Durand-Guerrier, V., Le Berre, M., Pontille, M.-C., & Reynaud-Feurly, J. (2000). *Le statut logique des énoncés dans la classe de mathématiques. Eléments d'analyse pour les enseignants*. Villeurbanne: Irem de Lyon.

Durand-Guerrier, V., & Herault, J.-L. (2006). Définition et règle, le mythe de la transparence en géométrie. *Interactions verbales, didactiques et apprentissage* (pp. 139-167).

Duval, R. (1990). Pour une approche cognitive de l'argumentation. *Annales de didactique et de sciences cognitives, 3,* 195-221.

Duval, R. (1995). *Sémiosis et pensée humaine: registres sémiotiques et apprentissages intellectuels*: Peter Lang.

Haspekian, M. (2005). *Integration d'outils informatiques dans l'enseignement des mathématiques.* Doctoral dissertation, Denis Diderot University, Paris VII.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*(3), 317-326.

Kieran, C. (1991). A Procedural-Structural Perspective on Algebra Research. In F. Furinghetti (Ed), *Proceedings of the 15th Conference of the International Group for the Psychology of Mathematics Education (PME), June 29-July 4, (pp. 245-251).* Assisi, Italy.

Kieran, C. (2001). The core of algebra: Reflections on its Main Activities. In K. Stacey, H. Chick & M. Kendal (Eds.), *The Future of the Teaching and Learning of Algebra: The 12 th ICMI Study* (pp. 21-34). Melbourne, Australia: Kluwer academic publishers.

Nguyen, A. Q. (2006). *Les apports d'une analyse didactique comparative de la résolution des équations du second degré dans l'enseignement secondaire au Viet-nam et en France.* Doctoral dissertation, Joseph Fourier University, Grenoble.

Sfard, A. (1991). On the dual nature of mathematics conceptions : reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, Vol 22,* 1-36.

Wittgenstein, L., Ambrose, A., & Macdonald, M. (1979). *Wittgenstein's Lectures, Cambridge, 1932-1935*: Blackwell.

Woodhouse, R. (1801). On the Necessary Truth of Certain Conclusions Obtained by means of Imaginary Quantities. *Philosophical Transactions of the Royal Society of London, 91,* 89-119.