APPROACHING PROOF IN SCHOOL: FROM GUIDED CONJECTURING AND PROVING TO A STORY OF PROOF CONSTRUCTION

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This paper presents some aspects of an ongoing research aimed at leading students (through activities of conjecturing, guided construction of proof and story making of the rationale of the proof) to become aware of some salient features of proving and theorems. Theoretical elaboration as well as an example of didactic engineering concerning Pythagoras' theorem will be outlined.

I INTRODUCTION

School approach to theorems has been a subject of major concern for mathematics education in the last two decades. Students' learning to produce proofs and their understanding of what does proof consist in (Balacheff, 1987) have been considered under different perspectives and with different aims: among them, how to make the students aware of the differences between proof and ordinary argumentation (Duval, 1991, 2007); how to favour students' access to the theoretical character of proof (M.A.Mariotti, 2000); how to exploit "cognitive unity" (which for some theorems allows students to exploit the arguments they produced in the conjecturing phase to construct the proof) in order to smooth the school approach to theorems (Boero, Garuti & Lemut, 2007); in what cases of cognitive unity do students meet difficulties in the passage from an inductive or abductive reasoning, to the deductive organization of arguments (lack of structural continuity: Pedemonte, 2007, 2008); what are the common aspects between ordinary argumentation and proving, and how to prepare students to proving by relying on those aspects (Douek, 1999a, 1999b; Boero, Douek & Ferrari, 2008). Previous research work helps us to formulate and situate some educational problems that arise in the school approach to theorems: how to tackle theorems for which cognitive unity does not work, or (if cognitive unity can work) when students meet important difficulties due to the lack of structural continuity? How to make the students aware of some salient characters of proving and proof? And how to lead them into some specific competencies of proving activity? In this paper we propose a possible way to tackle these problems in an integrated way. The idea is to guide students' constructive work on proving, then to help them focusing on the characteristics of the organisation of proof.

This paper presents a theoretical and pragmatic elaboration about how to deal with theorems for which cognitive unity does not work, and approach the rationale of a proof at first stages of proof teaching and learning. The theoretical elaboration also frames the accompaniment of students through two aspects of proving activity: exploration (in order either to find a statement, or to find reasons for validity of a statement); and organisation of reasons (or arguments), in the perspectives of producing a proof, or of understanding the links between statements or arguments in a proof. Our hypothesis is that the rationale of a proof can be approached early in the school context through “story making” situations, preceded by suitable activities of conjecturing and guided proof construction, and related classroom discussions. We provide an example of such a didactical engineering concerning Pythagoras' theorem.

II FRAMING PROOF CONSTRUCTION

Inspired by Lolli's analysis of proof production (see Arzarello, 2007), we consider proving as a cognitive, culturally situated activity engaging four modes of reasoning:

1) Heuristic exploration. It occurs when one tries to interpret a proposition or to produce a proposition or an example. One has in mind a target but the main focus is not on attaining the target through an acceptable mathematical reasoning. Any accidental event, writing, metaphor, may move the exploration activity. This type of reasoning is typically open to divergent paths.

2) Organisation of reasoning, making explicit the threads of reasoning holding propositions together. When a proposition seems pertinent, a calculation promising, a writing efficient, one searches for a convincing coherent link to a local goal or to the global one. The links may be theoretical reasons of validity. The intentional and planning characters are typical of this mode, and abduction is a good example of it. Deductive reasoning is not yet a priority. Such organisational intention may concern partial arguments or the whole of the argumentation aimed at proof construction.

3) Production of a deductive text following mathematicians' norms. Once ideas of proof are brought to light, they must be organised in a deductive reasoning.

4) Formal structuring of the text, to approach a formal derivation. This mode will not at all be approached in the school context we are considering.

These four modes could be considered as successive phases of a proof construction, as different moments with different intentions. But in fact, as reasoning modes, they seldom do appear separately. Not only the succession of modes can vary and loop, but even two or more of them may intervene very closely in one phase aiming mostly at exploring or at writing a deductive text, for instance.

Methodologically, the consideration of these phases based on a cognitive analysis in terms of the four modes offers didactical tools to organise teaching-learning situations into sequences with clear didactical goals. As we refer to phases of predominant modes of reasoning, a didactical goal can be to lead the students to be aware of the processes they have to go through within a specific phase, essentially to favour students openness in exploration and their rational control in organising reasoning. But no exploration is blind nor any reasoning organising is totally controlled: when we analyse a phase of exploration activity, we ought to capture some reasoning organising activity, etc... (see sequence 1, in V).

The different modes and phases of reasoning involve several cultural rules of validity, and they affect the delicate game of changes from what is allowed or even needed for one mode, to what is allowed or needed for another. For instance abductive reasoning is typical of mode 2 but is not allowed in modes 3 and 4, and student will have to move from it towards deductive reasoning, which is not easy. We can also consider the use of examples (pertinent in mode 1 and 2 but not acceptable in modes 3 and 4), and the conscious handling and conversion of different semiotic registers according to different modes of reasoning (Morselli, 2007; Boero, Douek & Ferrari, 2008).

This analysis leads us to give a special role to argumentation both as an intrinsic component of reasoning, and as a didactical tool to manage the different modes of reasoning and the relationships between them in a conscious way, keeping into account specific cultural rules (to be mediated by the teacher).

III ARGUMENTATION IN PROOF AND PROVING

In this paper, an "Argument" will be "A reason or reasons offered for or against a proposition, opinion or measure" (Webster), including verbal arguments, numerical data, drawings, etc. An "Argumentation" consists of one or more logically connected "arguments". Proof itself is an argumentation. But other argumentations play an important role in proving. Mode 2 is specially based on argumentative activity: discussing the use of a theory or a mathematical frame to produce a step of reasoning relies on a meta-mathematical argumentation (Morselli, 2007). It is not really part of a proof, but is needed to produce it. Analogies may implicitly affect mode 1 reasoning or be explicit arguments in mode 2 (Douek, 1999a, 1999b).

For teaching and learning purposes, argumentation is a fruitful means to control the validity of reasoning (as the legitimate use of examples and, or transitions from one mode of reasoning to another with their different cultural rules).We are therefore interested in two levels of argumentation: as part of the proving tasks, specially for producing and organising arguments (mode 2); and in discussing procedures, as a means to assimilate and master elements of proving processes.

Convergent structure of argumentation in a proof

In general, an argumentation is made of more elementary ones that may be organised in various ways (converging towards a conclusion, or being parallel as when producing different explanations, etc.). In a proof, the elementary argumentations may form a linear chain, each conclusion being input as an argument for the following argumentation, thus forming one whole "line of argumentation". But in many cases of proof, argumentation may contain parentheses "blocks", or side argumentation branches that meet the main line to input a supplementary data or argument. A parenthesis might be considered as a secondary line of argumentation. This description underlines the possible hierarchical relations between various argumentations involved in a proof (Knipping, 2008), which is a difficult matter for students who are being introduced to proof (see the Example for a suggestion).

IV EDUCATIONAL ASPECTS

In the early stages of proof teaching and learning, students can be smoothly introduced to theorems and proofs by conjecturing and proving activities provided that cognitive unity works (Boero, Garuti & Lemut, 2007). In particular exploratory activity (Mode1) and justification (Mode 2) can be introduced at early stages. In a suitable educational environment, 7th and 8th graders are able to produce conjectures for non trivial arithmetic or geometric situations, and move (under a loose guidance by the teacher) towards constructing general justifications. Comparison of students' productions and classroom discussions about them, orchestrated by the teacher (Bartolini Bussi, 1996; Bartolini Bussi & al, 1999) allow students to appreciate some relevant cultural requirements of conjectures and proofs, like their generality and the conditionality of statements (Boero, Garuti & Lemut, 2007), and to become aware of processes favouring conjecturing and proving.

In the following we will focus on mode 2 reasoning, specially in the organisation of reasoning phases; then in the didactical engineering we will also consider mode1 more specially related to conjecturing.

In spite of their usefulness to initiate students into conjecturing and proving, in those cases in which cognitive unity works well, with no difficulties due to the lack of structural continuity (Pedemonte, 2007, 2008), the peculiar argumentative structure of a proof does not emerge as an object of reflection for students. Indeed the fact that both easy-to-prove theorems must be proposed for a smooth approach to theorems, and that the students themselves are able to enchain the arguments in an autonomous way, make artificial and rather empty the discussion about the specific argumentative arrangement of those arguments. However students must be enabled to move from theorems for which cognitive unity works to theorems (like Pythagoras') for which proof cannot consist in the deductive arrangement of arguments produced by conjecturing. For other theorems students can meet difficulties in moving from creative ways of thinking (abduction, induction) typical of conjecturing to deductive arrangements of the produced arguments (Pedemonte, 2007). In both cases proving needs a strongly guided activity; and teachers' guidance can even initiate students into the mechanisms inherent in the Mode 2 reasoning, and open the perspective of Mode 3. Drawing from theoretical reflections, we make the hypothesis that the inherent argumentative activities could be promoted through debates (with real others) about arguments and their relations on one side, and story making on the other.

The debate

Classroom debates, if well oriented and guided, stimulate efforts of expression and explanation. These efforts, in turn, favour the consciousness of the logical rules and their range of validity. For instance, discussing a statement may bring students to methodological and meta mathematical reflections such as: producing an example to support the statement can be an efficient step in the exploratory phase, but is not a valid argument when organising a general mathematical justification; some semiotic registers (like drawing) are crucial for exploration, and may be for organising reasoning, but insufficient to produce a suitable argument in a deductive reasoning. Such discussions question cultural rules of mathematical reasoning and mathematical knowledge too. Also the relation between arguments and the construction of lines of argumentation (mode 2) can be discussed in a debate, which draws students' attention to the goal of the line of argumentation in relation to its steps.

Making a story

> Logic is concerned not with the manner of our inferring, or with questions of technique: its primary business is a retrospective, justificatory one - with the arguments we can put forward afterwards to make good our claim that the conclusions arrived at are acceptable because justifiable conclusions.

This quotation from Toulmin (1974, p. 6) inspires the hypothesis that in order to grasp the rationale of a proof, students may make an individual story from the ideas and calculations involved in a reasoning that validates the statement. We emphasise the story that connects steps and fragments with reasons, in order to serve the conclusion, and not particularly the story of how the steps occurred in one's mind (Bonaffé, 1993), nor of how learning has evolved through time (Assude & Paquelier 2005). The goal is that the students recognise the involved lines of argumentation, their possible hierarchical relations, and their role in the logical combination that produces the proof. At least at first stages of proof learning, these individual story makings need to be prepared by suitable tasks of guided construction of proof and by related debates putting into evidence some crucial "steps" of Mode 2 reasoning.

In our theoretical construction, debates and story makings should be considered together and arranged as a dynamic system of complementary situations. Individual story making involves students in an active personal reconstruction of the rationale of a proof, while a debate on the work done in individual tasks of conjecturing and guided proving (and story making as well) offers both openness to other possible combinations and regulation. We expect this system to draw students' attention to the "components" of the story. The deductive structure of the proof (through mode3) will consist of a particular relating of the pertinent components of a story.

Students need to be gradually initiated in both activities, possibly before the activities on theorems in order to establish a suitable didactical contract (Brousseasu, 1986). However story making, in the case of theorems, shows particularities that need a careful mediation through sequencing suitable tasks.

Before illustrating the above theoretical reflection by an example, let us present the main activities we wish students to develop and their co-ordination, and give methodological precisions concerning the planned experiment:

- To associate exploration and conjecturing to enhance mode 1 (without excluding other modes).

- To stimulate proving the conjecture(s); either cognitive unity can work and thus the students are able to produce a proof, or it cannot work and the teacher offers a task to guide them towards the proof.

- To engage the dynamic system of collective debate / individual story-making, starting from discussing some of students' productions, to enhance mode 2 (without excluding other modes). In case cognitive unity could not work, students would not be in a good condition to understand the proof nor to learn much of it, and this dynamic becomes particularly crucial.

The analysis of the experimentation should concern: Students' engagement in the proposed activities; and the evolution and the differences between the various individual productions. Observing the discussion (or its video registration), we need to track: How student's individual production reappears in the collective discussion; how the student hold his/her position in front of other's, and if some elements of consciousness awakened during discussion. However, some students may not take active part in the debate. The final individual production of story telling that follows should help completing the analysis. Comparing this individual production with the previous one (proof construction or proof reconstitution), one can see if the debate helped to bring to consciousness the necessity of some types of reasoning and the necessity of avoiding some others. The form of storytelling may reveal hierarchies of the types of reasoning and more particularly the linearity of the argumentation.
Another slightly different proving situation might follow to examine: transfer of the various reasoning competencies; the various methods; and the level of awareness of the variability of rules of validity.

V AN EXAMPLE CONCERNING PYTHAGORAS THEOREM

Pythagoras theorem was chosen for two reasons: it is an important and early met theorem in school mathematics; and it is not difficult to get the conjecture through a loosely guided path, while the construction of a proof needs a strong guidance by the teacher (cognitive unity cannot work, because the geometric constructions needed for the usual proofs are not suggested by the work done in conjecturing). Teachers' guidance, classroom discussions and story making will allow students to approach the rationale of the proof and offer occasions for learning about proof and proving.

First sequence: "Discovering" Pythagora's theorem, expressing the conjecture and making sense of it

Students have not only to grasp the theorem, but also to develop some proving skills (though no proving activity is demanded in this phase) and prepare for the further work; thus the activity on Pythagoras' theorem is prepared by Task 1 (an individual production on another theorem), followed by classroom discussion:

> Task1: Consider the statement: "In a triangle of sides a, b and c, a+b is always smaller than c". Is it true? always? Why? Prepare yourself to explain how you checked it and why you think it is true, or it is not, or what makes you doubt.

No triangle is presented by the teacher; students are encouraged to draw some triangles for a check, if they did not do it spontaneously. This task aims at exploration through testing examples, and (specially in the discussion) at leading the students to express the rationale of the activity and to make visible the generality of the proposition they produce. An expression like “we wanted to see if it is true that... so we tried to verify it with four examples” is encouraged: such simple story making reflects an ability (and invites) to reconstruct the logical skeleton of the activity they went through. It bridges a Mode 1 reasoning with a Mode 2, and prepares Task 2.

> Task 2 (individual): Now if we consider the squares of the lengths, instead of the lengths themselves, the situation is different. See if a relation between the squares of the lengths of the sides of a triangle exists. Once you think you produced a valid statement (a "conjecture"), put it clearly in words to explain it to other students.

Right angled, acute and obtuse triangles, are presented on the worksheet. Afterwards a collective discussion guided by the teacher is engaged to share and discuss the conjecture(s) produced, and the ways followed to produce them; and to attain and share acceptable expressions of the conjecture(s) (according to mathematical standards). An incomplete conjecture or an erroneous one may offer fine opportunities to make explicit the important elements of the theorem (in particular the condition of validity of Pythagoras' theorem, i.e. the angle being right) and their role.

> Task 3 (individual): Write down the conjecture as now you think it should be. Explain it and illustrate it with some examples.

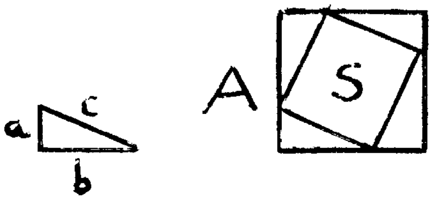
The teacher concludes with the standard formulation of Pythagoras' theorem.

Concerning proof learning, this first sequence aims at involving students in Modes 1 and 2: Exploring (drawing, measuring, calculating, induction when modelling and producing algebraic expressions, repeating procedures and modifying data) mostly in mode 1; and, mostly in mode 2, organising the exploitation of the gathered data, classifying them in order to find some rule, expressing results as general (everyday language being acceptable), etc; discussing and justifying propositions, and organising the steps of exploration in relation to a goal. Classifying and modelling are as much in mode 1 and mode 2. The explicit intentions of exploration and of organisation are satisfying sign in my opinion, as a main didactical goal is to enhance the processes students have to go through.

Second sequence: Guiding Pythagoras' theorem proof, and teaching/learning to organise the steps of reasoning into lines of argumentation

Given that cognitive unity cannot work, students are guided by means of individual and collective activities; then they reconstruct the lines of argumentation.

> Task 4 (individual): Here we study the proof of the theorem we have conjectured, you will be guided towards this proof. Consider a right angled triangle with sides a, b, c. We use it to build the square A (see below). Its central square S is of area c^2.



I) Can you describe how A can be obtained by using only our squared triangle? explain why S is a square (of area c^2)?

II) Try to write the area of A in two different ways (you may need to arrange the four identical square triangle differently). Find and explain the two ways.

III) How can this help us to validate our conjecture?

A geometrical reasoning is expected to intertwine with an algebraic reasoning in order to attain the equality between the areas. If needed, some supplementary tasks can be inserted either for the whole group or for some students.

After students' individual work, the teacher orchestrates a collective discussion (Bartolini Bussi, 1996) concerning the reasoning that allows to prove the steps of argumentation and the calculations and why they are needed, and in particular, the connection between geometrical arguments and algebraic arguments. The interactions must be based on their own reasoning productions, theirs insights and their shortcomings. Therefore the teacher selects elements of students' production to provoke fruitful interactions. Two complementary levers can help maturing students' awareness of the reasoning organization "rules", and their specificity in contrast to exploration reasoning: analyzing elements of reasoning, and rising direct methodological questions in the debate. The aim is to favor the elaboration of some satisfactory reasoning about the quality of which the student may agree, and, on another hand, to characterize some insufficiencies found in some produced reasoning. The parts of debate concerning specific algebraic or geometric steps and some sort of gap filling reasoning (directly concerned by the activity) need to be intertwined with methodological reflection about the validity of a reasoning, its communicability, the bases on which it can be accepted by another (indirect, implicit activity in student's individual production). Open "methodological" questions may be: how exploration and induction had been produced (algebraic induction); which different rules allow a reasoning to be valid (in exploration, measures and experiment are welcome, in proving deductive reasoning is needed, here based on elementary geometry and on algebraic calculus); and, in reasoning organization, how to come to such reasoning, and why (in particular how exploring the disposition of the four rectangles may favour algebraic exploration). This double level of discussion concerning the activity, on one hand, and the meaning and mathematical rules of the activity, on the other, is theoretically developed by M.A.Mariotti (2000) based on M.Bartolini Bussi's mathematical discussion theoretical frame.

Task 4 is formulated and organised in a way to approach a story making of the proof.

The subsequent discussion of the organisation of the lines of argumentation and the insertion of "blocks" of arguments/calculations in the main line is meant to prepare students to write a “story”.

Third sequence: story making

> Task 5 (individual): Write down how you organized your steps of reasoning to reach a general justification of the conjecture, and justify why those steps are important

This task is particularly important for the students who were not productive in the previous sequence. It should allow them (as well as the others) to grasp and reconstruct the rationale of the proof. Here is the kind of arguments we hope the students produce:

> first *(block 1)* we calculate the area of A, then *(block 2)* we organised differently the calculation of the area (or we organised differently the disposition of the triangles) so that we found another algebraic expression of the are, because *(looking forward to the final goal)* surface measures of squares are written as algebraic squares. So we think that a^2, b^2 and c^2 will appear and will be related *(possibility to rejoin the main line)*. So, we can write the algebraic equality, and find the relation after transformations.

Mode 2 reasoning is needed for this task in block 2: students must go through an abductive reasoning ("how can I find a^2 and b^2 in this big square?") while deduction prevails in block 1 and will prevail afterwards, till the end.

It is important to notice with the students that the algebraic equality is the principal aim (and first to come to the mind, since it is near to the conclusion we want to reach) but that we have to begin with geometrical considerations, which are like parentheses besides the principal aim. Thus the reasoning is made of a principal line of argumentation and side parentheses involving geometrical reasoning and calculations, whose conclusions flow into the main argumentation line. Getting familiar with mathematical proof practices (like moving from a geometrical frame to an algebraic one, using geometry only for strategic purposes...) is a particular aspect of this work.

Difficulties inherent in the classroom implementation of the proposal

Comparing the proposal with the style of teaching of most teachers, and keeping into account my first experiences of work with teachers on this subject, I must say that teachers meet some difficulties in engaging in a coherent classroom implementation of the proposal. One difficulty consists in the fact that "To produce a conjecture" is a task that does not fit the most frequent didactical contract in our schools (statements are usually presented and illustrated by the teacher, and learnt by students who repeat and apply them afterwards; the same for proofs). Another is that teachers tend to identify student's task of reasoning and the task of explaining the rationale of a reasoning as bearing the same learning targets. And, finally, the presentation and management of the tasks in a way that guides students' work but does not prevents creativity is not easy; however, if creativity is not practised, there would be no sense in making a story out of a series of calculations.

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