ABDUCTION AND THE EXPLANATION OF ANOMALIES: THE CASE OF PROOF BY CONTRADICTION♠

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Some difficulties with proof by contradiction seem to be overcome when students spontaneously produce indirect argumentation. In this paper, we explore this issue and discuss some differences between indirect argumentation and proof by contradiction. We will highlight how an abductive process, involved in generating some indirect argumentation, can have an important role in explaining the absurd proposition, in filling the gap between the final contradiction and the statement to be proved and in the treatment of impossible mathematical objects.

Key words: proof, argumentation, abduction, proof by contradiction, indirect argumentation.

INTRODUCTION

The relationship between argumentation and proof constitutes a main issue in mathematics education. Research studies have been based on different theoretical assumptions, proposing different approaches and consequently different didactical implications (Mariotti, 2006). In some studies (see, for example, Duval, 1992-93), a distance between argumentation and proof is claimed, while in others, without forgetting the differences, the focus is put on the analogies between the two processes and their possible didactical implications (Garuti, Boero & Lemut, 1998; Garuti & al., 1996). As a consequence, the authors hold the importance for students to deal with generating conjectures, and highlight that this activity can promote some processes that are relevant in developing students' competences in mathematical proof.

Elaborating on this first hypothesis, concerning the continuity between the argumentation supporting the formulation of a conjecture and the proof subsequently produced, Pedemonte (2002) developed the theoretical construct of *Cognity Unity* in order to describe the relationship (continuity or break) between the argumentation process and the related mathematical proof in the activity of conjecture's production.

In this paper, we aim to investigate the relationships between argumentation and proof in the case of proof by contradiction. The reference to the framework of *Cognitive Unity* is of the interest for this study for the following reason. Although important difficulties have been identified in relation to this type of proof (see Antonini & Mariotti, 2008; 2007; Mariotti & Antonini, 2006; Antonini, 2004;

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Stylianides, Stylianides & Philippou, 2004; Wy Yu, Lin & Lee, 2003; Thompson, 1996; Leron 1985), in the literature we find evidence of arguments, spontaneously produced by students, that can be considered very close to proof by contradiction (see Antonini 2003; Reid & Dobbin, 1998; Thompson, 1996; Freudenthal, 1973; Polya, 1945). In fact, as reported by Freudenthal:

> "*The indirect proof is a very common activity ('Peter is at home since otherwise the door would not be locked'). A child who is left to himself with a problem, starts to reason spontaneously '... if it were not so, it would happen that...'* " (Freudenthal, 1973, p. 629)

We call *indirect arguments* the arguments of the form '*if it were not so, it would happen that…*'. Indirect arguments seem to be more like to appear in the solution of open-ended problems, as a natural way of thinking in generating conjectures, when one needs to convince oneself that a statement is true, or to understand because a statement is true.

Therefore, it is seems important to study differences and analogies between proof by contradiction and indirect argumentation, and this is what we are going to do in the following sections.

DIFFICUTIES WITH PROOF BY CONTRADICTION

According with the terminology of the model presented in (Antonini & Mariotti, 2008, 2007), given a statement S, that we called a *principal statement*, a proof by contradiction consists in a couple of proofs: a direct proof of another statement S*, that we call the *secondary statement*, in which the hypotheses contain the negation of S and the thesis is a contradiction (or a part of it); and a *meta-theorem* stating the logical equivalence between the two statements, the *principal* and the *secondary*. Here, we analyse two aspects and their relationships: the link between the *principal statement* and the contradiction achieved through the proof of the *secondary statement*; the treatment of impossible mathematical objects in both the argumentation and the proof.

The link between the contradiction and the *principal statement*

The link between the final contradiction and the principal statement is a source of difficulties for students (see Antonini & Mariotti, 2008). It can happen that such difficulties are openly shown when they appear astonished and disoriented after the deduction of an absurd proposition. This is the case for example of Fabio, a university student (last year of the degree in Physics), who explains very well this type of difficulty:

> Fabio: *Yes, there are two gaps, an initial gap and a final gap. Neither does the initial gap is comfortable: why do I have to start from something that is not? […] However, the final gap is the worst, […] it is a logical gap, an act of faith that I must do, a sacrifice I make. The gaps, the sacrifices, if they are small I can do them, when they all add up they are too big. My whole argument converges towards the sacrifice of the logical jump of exclusion, absurdity or exclusion… what is not, not the direct thing. Everything is fine,*

> *but when I have to link back… [Italian: "Tutto il mio discorso converge verso il sacrificio del salto logico dell'esclusione, assurdo o esclusione… ciò che non è, non la cosa diretta. Va tutto bene, ma quando mi devo ricollegare..."]*

Fabio identifies two gaps (he speaks also of a "*jump*"!) in a proof by contradiction: an initial gap and a final gap. According to our model, the initial gap corresponds to the transition from the statement S to the proof of S*, and the final gap corresponds to the opposite move, from the proof of S* to the conclusion that S is proved. The perception of these gaps makes Fabio feel unsatisfied, as if something were missing. In fact, he can accept the proof but he is not convinced, as he says it is "*an act of faith that must be done*".

The treatment of impossible mathematical objects

It may happen that, at the beginning of a proof by contradiction, some of the mathematical objects have some characteristics that are absurd and strange, in an evident way. These mathematical objects are proved to be impossible in some theory. For this reasons, difficulties can emerge in the treatment of these absurd objects. As discussed in (Antonini & Mariotti, 2008; Mariotti & Antonini, 2006) difficulties may occur in the construction of the proof of S*, but difficulties may also emerge after the proof of S* is achieved, when absurd objects have to be discarded. In fact, at the end of a proof of S*, once a contradiction is deduced, one has to realize that some of the objects involved do not exist; actually, they have never existed. As explained by Leron:

> "*In indirect proofs […] something strange happens to the 'reality' of these objects. We begin the proof with a declaration that we are about to enter a false, impossible world, and all our subsequent efforts are directed towards 'destroying' this world, proving it is indeed false and impossible. We are thus involved in an act of* mathematical destruction, *not construction. Formally, we must be satisfied that the contradiction has indeed established the truth of the theorem (having falsified its negation), but psychologically, many questions remain unanswered. What have we really proved in the end? What about the beautiful constructions we built while living for a while in this false world? Are we to discard them completely? And what about the mental reality we have temporarily created? I think this is one source of frustration, of the feeling that we have been cheated, that nothing has been really proved, that it is merely some sort of a trick - a sorcery - that has been played on us.*" (Leron, 1985, p. 323).

Our research interest is in exploring whether and how these difficulties may be overcome when students spontaneously produce indirect argumentation. Two elements seem important to take into account: on the one hand the indirect argumentation as a product and its differences with a proof by contradiction, on the other hand the processes involved in producing the argumentation (see also Antonini, 2008). In this paper we focus on the hypotheses that in many cases the students try to fill the gap between the contradiction and the statement in order to re-establish a link and at the same time to give a new meaning to the "objects of the *impossible world*", so that they can be modified without being discarded.

THE ABDUCTIVE PROCESS

Abduction is one of the main creative processes in scientific activities (Peirce, 1960). Magnani defines abduction as

> "*the process of inferring certain facts and/or laws and hypotheses that render some sentences plausible, that explain or discover some (eventually new) phenomenon or observation; it is the process of reasoning in which explanatory hypotheses are formed and evaluated*" (Magnani, 2001, pp. 17-18).

The main characteristic of abduction is that of deriving a new statement that has the power of enlightening the relationship between the observation and what is known.

Many studies in mathematics education have dealt with abductive processes in students thinking: in problem-solving activities (Cifarelli, 1999), in generation of conjectures (Ferrando, 2005; Arzarello et al., 2002; Arzarello et al., 1998), argumentation and proofs (Pedemonte, 2007).

In this paper, through the analysis of a case study, we will show how an abductive process could assume a fundamental role in the production of indirect argumentation. Through an abduction a new statement is produced that has no logical need but allows one to make sense of the absurd and strange proposition and, in this way, to overcome the gap between the contradiction and the *principal statement.*

A CASE STUDY

The following open-ended problem was proposed to Paolo and Riccardo (grade 13), two students that, according to the evaluation of their teachers, are high achievers.

Problem: What can you say about the angle formed by two angle-bisectors in a triangle?

What follows is an excerpt of their interview. After a phase of exploration, the students generated the conjecture that the angle S (figure 1) is obtuse. Afterwards, the students started to explore the possibility that the angle S might be a right angle.

61 P: As far as 90, it would be necessary that both K and H are 90 degrees, then K/2 = 45, H/2 = 45...180 minus 90 and 90 degrees.

62 I: In fact, it is sufficient that the sum is 90 degrees, that K/2 + H/2 is 90.

63 R: Yes, but it cannot be.

64 P: Yes, but it would mean that K+H is ... a square […]

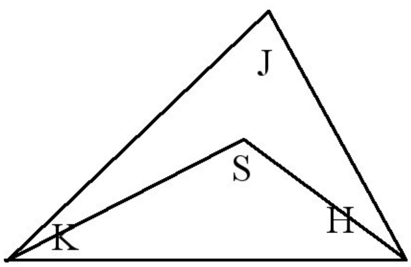

Figure 1: The angle between two angle bisectors in a triangle.

65 R: It surely should be a square, or a parallelogram

66 P: (K-H)/2 would mean that […] K+H is 180 degrees...

67 R: It would be impossible. Exactly, I would have with these two angles already 180, that surely it is not a triangle.

[…]

80 R: [the angle] is not 90 degrees because I would have a quadrilateral, in fact the sum of the two angles would be already 180, without the third angle. Then the only possible case is that I have a quadrilateral, that is, the sum of the angles is 360.

The episode can be subdivided in three parts: the development of a first argumentation (61-63), the introduction of a new figure, the parallelogram (64-67), the production of the final argumentation (80). This last argumentation is expressed by Riccardo, after the students are explicitly asked to write a mathematical proof.

The argumentation developed in the first part (61-63) is indirect: assuming that the angle between two angle bisectors of a triangle is a right angle, the students deduce a proposition that contradicts a well known theorem of Euclidean Geometry. From the logical point of view, the deduction of the contradiction would be sufficient to prove that this triangle does not exist, or, equivalently, that the angle S is not right, thus concluding the argumentation. But, although convinced that the angle S cannot be a right angle, the students do not feel that the argumentation is concluded and they look for an explanation for the anomalous situation. In fact, the subsequent part (64-67) seems to have the goal to complete the argumentation; in particular, the students seem to look for an explanation to the false proposition "K+H=180° ". An explanation is found by formulating a new hypothesis: the figure is not a triangle, it is a parallelogram. In this case, it is true that the sum of two adjacent angles (K+H) is 180. In search of an explanation the original triangle fades becoming for the students an indeterminate figure that have to be determined in order to eliminate the anomalous consequences. In 67, Riccardo makes clear that the figure can be transformed during the argumentation. His expression "*surely it is not a triangle*" means "this figure is not a triangle" and it must be something else. Differently, in a proof by contradiction, as the proof that could arise from the first part of the argumentation (61-64), the figure is well determined, it is a triangle and it is not possible to modify it. Once a contradiction is deduced, it is proved that this figure does not exist. In this case, the triangle would be part of the "false, impossible world" and it would have had a temporarily role: at the end of the proof we know that it does not exist. Actually, it has never existed.

When the new case is selected and because this new case can solve the anomaly, Paolo and Riccardo seem to be satisfied. In 80, Riccardo summarizes the argumentation in what for him is a mathematical proof. The fact that the angle S is not right is not proved by contradiction but is based on the analysis of different cases: triangle, square, parallelogram. The figure is determined, and it is not a triangle, as we have thought at the beginning of the argumentation. This argument seems very convincing for students, more than the argument based on deriving a contradiction.

The key point in the development of the argumentation is the generation of the new case that is the identification of the parallelogram. This process can be classified as an abduction, in fact an explanatory hypothesis is produced and evaluated, as Riccardo says "[the quadrilateral] it is the only possible case".

The assumption of the parallelogram transforms a false into a true proposition. This argument allows students to overcome some of the difficulties that might be raised by a proof by contradiction (figure 2). In particular:

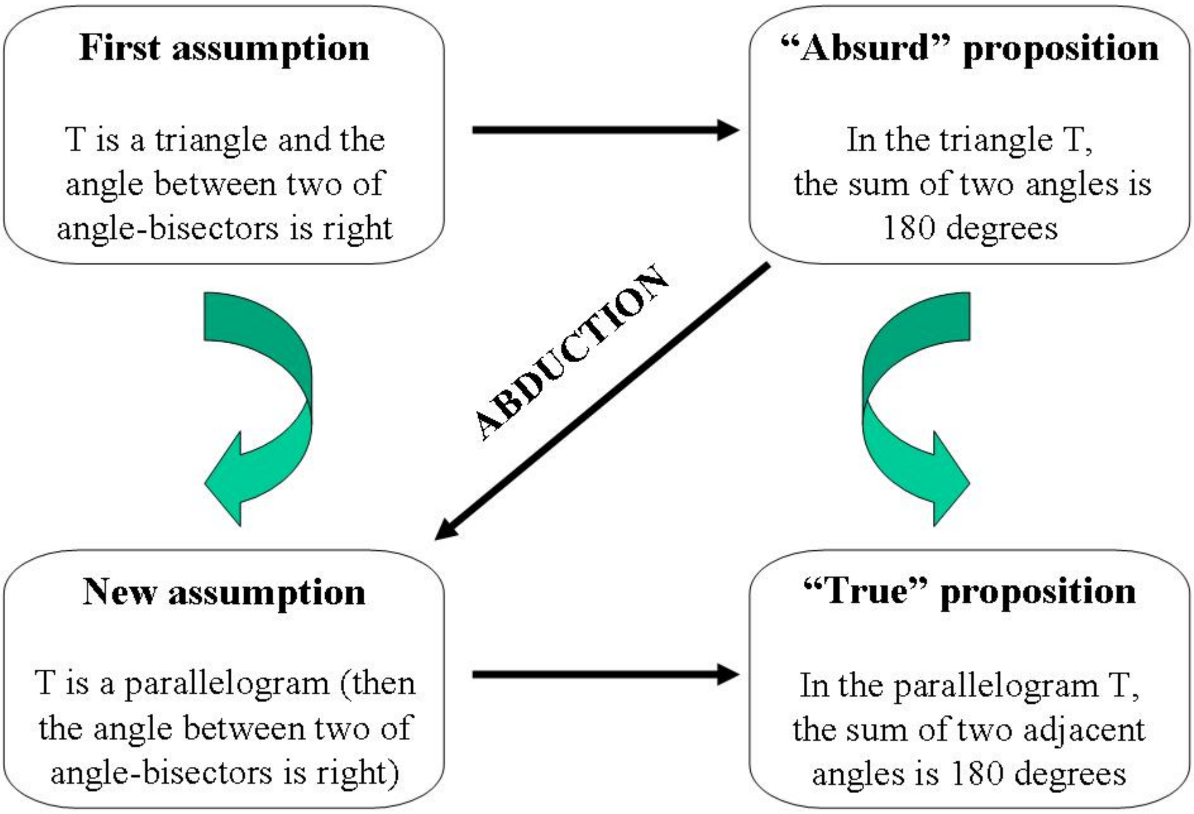

Figure 2: An abductive process in an indirect argumentation

- The false proposition - "in a triangle the sum of two angles is 180°" – becomes a true proposition related to the new explanatory hypothesis (in a parallelogram the sum of adjacent angles is 180°);
- The mathematical object (the triangle) is considered an indeterminate object that is identified only through the abduction with the goal to explain the anomaly. Then the mathematical object is changed and not discarded as it happens in a proof by contradiction. The problem of treatment of mathematical object at the end of proof by contradiction highlighted by Leron (1985) is bypassed.

- Differently to what happens with proof by contradiction, a link, that is not only logical, between the *secondary statement* and the *principal statement*, is constructed: it is not possible that S is right because otherwise the triangle would become a quadrilateral.

As the previous example shows, in geometry, the identification of the case that can explain the anomaly and allow getting out of the "impossible world" seems to be related to the transformation of figures. Most of the students asked to solve the problem of angle bisectors provided arguments based on transforming the triangle in a quadrilateral or in two parallel lines.

Further researches are necessary to corroborate this hypothesis and investigate whether it can be extended to other context. In fact we hypothesize that also in contexts other than Geometry abduction can be for students the key to come out from the anomalous situation that occurs in proof by contradiction. In order to support this extension to other contexts, we report now a short episode concerning the algebra domain.

ABDUCTION AND PROOF BY CONTRADICTION IN ALGEBRA: AN EXAMPLE

In a questionnaire proposed to 68 secondary school students (grade 10, 11, 12) and 19 university students, a proof by contradiction of the incommensurability of the diagonal of a square with its side was presented. We aimed to investigate the recognition and the acceptability of this type of proof. In the presented proof, it is assumed that the ratio is a rational number, expressed by the fraction m/n where m and n are two natural numbers (with n different from 0). Then it is deduced that the number n is both odd and even. The students were asked to choose one of the following answers to explain what it is possible to conclude:

a) This is not a proof

b) There is a mistake in some passages, but I can not identify it

c) There is a mistake, that is (specify the error): ....

d) We have not proved anything, because being even or odd has nothing to do with what we wanted to prove

e) We have proved what we wanted, in fact:……………………………………

f) Other (specify):

The 25 per cent of the students gave the correct answers and the 35 per cent chose the answer d). This expresses the feeling that something is missing and let us suppose the need to see a link between the contradiction and the statement. A hint in this direction comes from one of the answers. One student (grade 12) marked the correct answer and explained:

"*we have proved what we wanted in fact one of the two numbers [the number n] is not a* *natural number and then the ratio is not a ratio between two natural numbers*"

The argument provided does not refer to what could be recognized as the *meta-theorem*, explaining the logical equivalence between the *principal statement* and the *secondary statement*, and thus rejecting the existence of the mathematical object m/n.

Differently, this student does not reject the initial assumption that the ratio is rational from the contradiction "n is even and odd", rather he changes the nature of the number n coherently (in his opinion) with the deduced proposition. If m/n is not a rational number, as we have believed before, everything is explained.

Inferring the explaining hypothesis that number n, odd and even at the same time, is not a natural number is the product of an abduction. The hypothesis that n is not a natural number can explain the anomaly "n is odd and even" and, at the same time, it offers a link between the deduced proposition and the principal statement: n is not a natural number and then the ratio m/n is not a rational number. A link between the contradiction and the statement is now established and the proof can be accepted.

CONCLUSIONS

Main difficulties with proof by contradiction are related to the link between the contradiction and the statement to be proved, to the treatment of the impossible mathematical objects during the construction of the proof and at the end, to the need of discarding the mathematical objects involved in the proof of the *secondary statement*. The feeling of frustration that may emerge at the end of a proof by contradiction, as clearly expressed by Fabio's words, is accompanied by the need of giving a meaning to the absurd proposition, the need of establishing a stronger link with the principal statement and adjusting the "false, impossible world".

The analysis of the episodes proposed above shows how abductive processes may be mobilized to produce explanatory hypotheses. The system of relationships represented in the diagram of figure 2 shows the key role of the abductice process and highlights some differences between indirect argumentation and proof by contradiction.

Interpreting these results in terms of Cognitive Unity leads us to point out the distance between indirect argumentation as it is spontaneously developed and the scheme of a proof by contradiction. In particular, it clearly appears the distance between the *meta-theorem* - providing the equivalence between the *principal* and the *secondary statement* - and the abductive process that might emerge in an indirect argumentation. The question rises whether and how such distance can be filled through an appropriate didactical intervention.

Of course, further investigation is necessary to better understand the differences between argumentation and proof by contradiction and to identify and analyse other processes that could be important for the production and the development of indirect argumentation.

We think that the comprehension of these processes is fundamental for teachers to identify, explain and treat students' difficulties with proof. We also believe that indirect argumentation, even if it presents significant differences with proof by contradiction, should be promoted, in particular through open-ended tasks. As Thompson writes:

"*If such indirect proofs are encouraged and handled informally, then when students study the topic more formally, teachers will be in a position to develop links between this informal language and the more formal indirect-proof structure.*" (Thompson 1996, p.480)

As regards the transition from the argumentation to proof by contradiction, further researches are necessary to identify the tools to construct the didactical activity to face the gaps and promote the acceptability of method of proof by contradiction.

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