MATHEMATICS TEACHERS' REASONING FOR REFUTING STUDENTS' INVALID CLAIMS

Despina Potari*, Theodossios Zachariades* and Orit Zaslavsky**

*University of Athens, ** Technion - Israel Institute of Technology

This study investigates secondary school mathematics teachers' reasoning for refuting students' invalid claims in the context of hypothetical classroom scenarios. The data used in this paper comes from seventy six teachers' responses to a student's invalid claim about congruency of two given triangles and from interviews with a number of them. Some teachers responded to the claim by trying to refute it. Two main approaches to refuting the student's claim were identified: 1. by using known theorems; 2. by using counterexamples. Teachers' difficulties to generate correct counterexamples were traced. Moreover, a rather narrow meaning of the theorems and their use to refute invalid claims was manifested.

INTRODUCTION

Reasoning and proof are considered fundamental aspects of mathematical practice both in the practice of mathematicians and in the practice of students and teachers (Hanna, 2000). A large number of studies in mathematics education have explored students' justifications and proof strategies (e.g., Healy & Hoyles, 2000; Harel & Sowder, 1998). Refuting conjectures and justifying invalid claims requires reasoning that goes beyond the syntactic derivations of deductive proof which has been traditionally the focus of high school mathematics. It mainly involves the generation of counterexamples, the development of logical arguments that are grounded on exploration and experimentation, which are related to the construction of mathematical meaning and understanding. Balacheff (1991) discusses the diversity of ways of dealing with a refutation by referring to the epistemological work of Lakatos (1976) and to his own experimental study with high school students. Lin (2005) also demonstrates the complexity of the process by identifying the different types of arguments that secondary school pupils developed to refute false conjectures.

The process of evaluating and refuting students' claims is central to teacher practice. This often requires the teacher to give on the spot appropriate explanations that often involve the use of examples or counterexamples. Although the process of exemplification is highly demanding it has not been extensively investigated with regard to the teacher (Bills, Dreyfus, Mason, Tsamir, Watson & Zaslavsky, 2006). Desirable choice of examples depends on teacher's subject matter knowledge (Rowland, Thwaites & Huckstep, 2003) on her teaching experience (Peled & Zaslavsky, 1997) and on her awareness of students' prior experience (Tsamir & Dreyfus, 2002). The generation of examples and counterexamples in geometry gets a special meaning as the visual entailments of examples pose certain constraints (Zodik & Zaslavsky, 2008). In this paper, we investigate how teachers respond to students' invalid claims in the context of Euclidean geometry.

THEORETICAL BACKGROUND

We briefly present below the main theoretical constructs that framed our study. These include teacher knowledge, the process of refutation, and the nature and use of counterexamples.

The process of evaluating and refuting students' invalid claims strongly relates to mathematics teacher knowledge. Stylianidis and Ball (2008) studied the characteristics of teacher knowledge for reasoning and proof. Zodik and Zaslavsky (2008) also attempted to capture the dynamics of secondary mathematics teachers' choice and generation of examples in the course of their teaching. They offer an example-based teaching cycle with respect to teacher knowledge, the planning stage and the actual lesson.

The process of refutation has been mainly studied under the epistemological framework of Lakatos (1976) (e.g., Balacheff, 1991; Larsen & Zandieh, 2007). Lin (2005) developed a categorisation of students' refutation schemes. Accordingly, he distinguished between rhetorical arguments (reasons relative to the person spoken to), heuristic arguments (reasons taking into account the constraints of the situation), and mathematical proofs (the process of generating correct counterexamples).

Peled & Zaslavsky (1997) distinguished between three types of counterexamples suggested by mathematics teachers: specific, semi-general and general examples. Semi-general and general examples offer some explanation and ideas how to generate more counterexamples. Related to teachers' generation of counterexamples is the theory of personal *example spaces*, which encompasses examples that are accessible to an individual in response to a particular situation (Bill et al, 2006). Zazkis & Chernoff (2008) introduced the notions of *pivotal example* and *bridging example* and highlighted their role in creating and resolving cognitive conflict.

The study reported here is part of a larger study that investigates teachers' ways of responding to students' false claims. In this paper, we explore the different types of arguments that teachers use in dealing with an invalid claim in the context of geometry, an area where research is rather scarce.

METHODOLOGY

Seventy six teachers who were all candidates for a Masters in Mathematics Education programme participated in the study. Six of them were primary school teachers with an education degree, while the rest had a mathematics degree. Thirty of these were secondary school practicing mathematics teachers.

The teachers took a three hour exam as part of the selection process for the Masters programme. In this exam they had to respond in writing to five tasks in which they were asked to react to hypothetical teaching events. Four of these hypothetical events were related to the process of dealing with students' arguments and claims. Their written responses were analyzed from both mathematical and pedagogical perspectives. On the base of this analysis, 45 teachers were interviewed individually in order to explore further their reactions and justifications. Each interview lasted about 15-30 minutes. One researcher interviewed the teachers while another one took notes of the conversation. Since these interviews were part of the selection process, we refrained from using any audio or video recordings, in order to avoid negative effects on the candidates.

In this paper we analyse the data based on the test and the first set of interviews concerning one of the tasks.

The task

The task was the following:

> In a Geometry lesson, in grade 10, the teacher gave the following task:
> Two triangles ΑΒΓ and ΕΗΖ have ΒΓ=ΗΖ=12 and ΑΒ=ΕΗ=7 and the angles ΑΓΒ and ΕΖΗ equal to 30 degrees. Examine if the two triangles are congruent.
> Two students discussed the above task and expressed the following opinions:
> Student A: The two triangles have two sides and an angle equal. Therefore they are congruent.
> Student B: We know from the theory that two triangles are congruent when they have two sides and a contained angle equal. Therefore, the given triangles are not congruent.
>
> If the above dialogue took place in your classroom, how would you react?

The task refers to a hypothetical classroom scenario which focuses on issues of learning and teaching mathematics. Further discussion about the importance of this type of tasks as a research tool for exploring teachers' thinking can be seen in Biza, Nardi & Zachariades (2007). This task was based on an example discussed and analysed by Zodik & Zaslavsky (2007). Its mathematical content, the properties of the triangles and their congruence, is part of the Euclidian Geometry course taught in grade 10 in Greek high schools. In the task, student A expresses his belief that if two triangles have two sides and one angle that are respectively equal then they are congruent. He seems to over-generalize the theorem "if two triangles have two sides and the contained angle that are respectively equal then they are congruent."

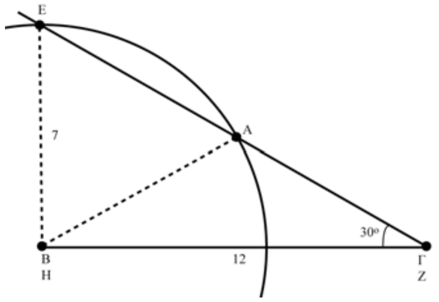

Figure 1: A geometric construction of a counterexample

There are at least three different approaches to refute the claim of this student. The first one is to provide a specific counterexample based on a geometric construction using a ruler and compass (Figure 1).

In this case we may continue and prove a general geometric theorem based on the geometrical construction, namely, that two sides (*a* and *b*, were *a*>*b*) and the angle (β) opposite the smaller side determine exactly two distinct triangles that are not congruent, except for a special case where $sin(\beta) = b/a$. In the latter case the triangle is necessarily a right-triangle, therefore it is uniquely determined, that is, all triangles with these givens are congruent. The second approach is to prove this general theorem and apply it to the specific given case. The third approach is the use of the sine and cosine laws in trigonometry. By applying the cosine rule for the given angle, we determine the third side, and find that there are two possible values for its length. By applying the sine law we find that there are two possible angles opposite the larger side (*a*) – an acute one and its supplementary angle. An interpretation of this calculation and the verification of the existence of triangles with these sides or angles lead to the conclusion that there are two (and only two) distinct non-congruent triangles satisfying the givens.

RESULTS

Classifying teachers' justifications

In this section, we present a classification of teachers' justifications based on their written responses. Out of the seventy six mathematics teachers three did not reply while eight considered the given triangles congruent. The remaining sixty five teachers acknowledged that the given triangles were not necessarily congruent. Sixty-three of them gave an explicit justification to their assertion. These justifications were grouped in categories which are presented in the tree diagram in Figure 2. The numbers in brackets indicate the number of teachers' responses that fall in each category.

Out of 63 teachers, 18 justified their claim by drawing on mathematical theorems relevant to the problem and 45 asserted that a counterexample was needed to justify their claim.

<u>Reasoning based on known theorems</u>:

As mentioned above, this type of responses was manifested by 18 teachers. Only two gave a full valid proof. The rest gave invalid proofs that included proof-like arguments.

Valid proof. Interestingly, although the context is geometry, the two teachers who gave valid proofs based them on trigonometry. One of them (T64) used the sine rule, and the other (T32) used the cosine rule, as described earlier.

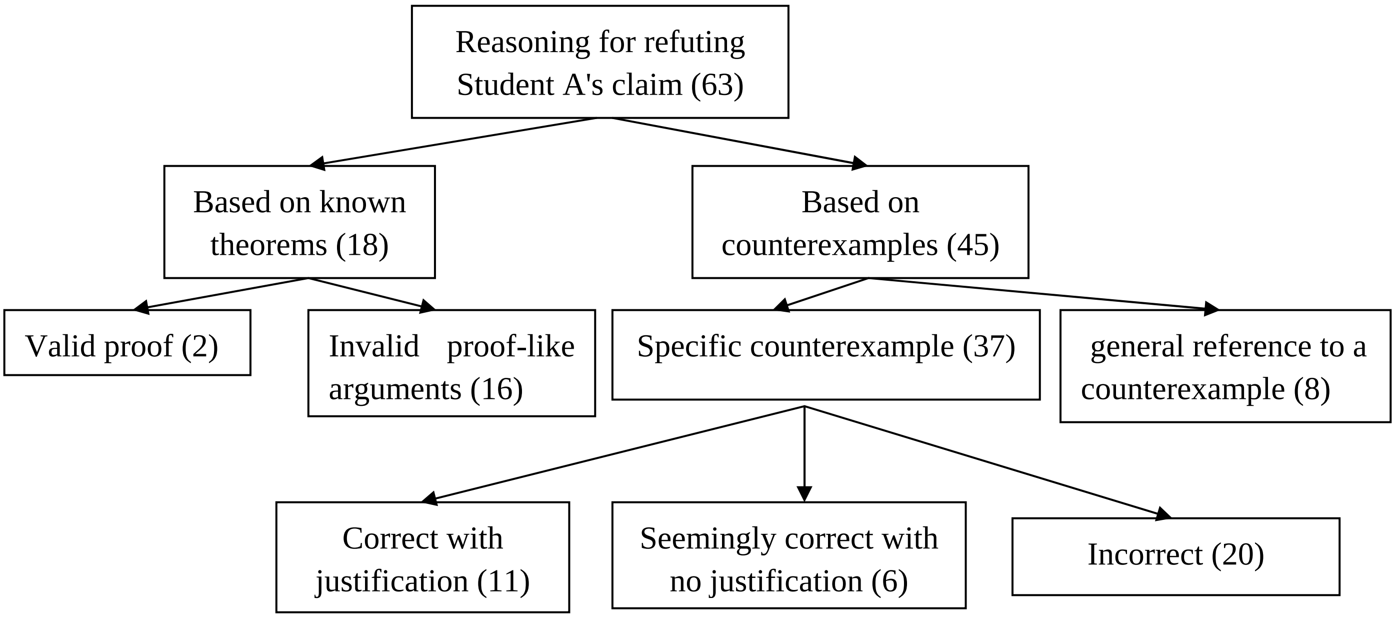

Figure 2: Mathematics teachers' justifications of the assertion that the two given triangles are not necessarily congruent

Invalid proof-like arguments. The remaining sixteen teachers provided invalid proof-like arguments to support their claim by maintaining that none of the known theorems about the congruence of two triangles applies in this case. The following example indicates the latter case: "Student A replied without considering the known criteria for congruence of triangles. I would encourage him to draw the two triangles so that to realise that these criteria cannot be applied" (T10). These teachers believed that this reasoning offers a valid proof for refuting student A's claim.

<u>Reasoning based on counterexamples</u>:

This type of responses was manifested by 45 teachers. Only 11 gave a specific counterexample with correct justification.

General reference to a counterexample. Eight teachers only made reference to the need to give a counterexample by stating that they themselves or their students would give a counterexample. For example, T23 simply mentioned that "... to convince him (Student A) we could show him some triangles that have two sides and one angle equal but are not congruent" while T18 suggested asking the students: "... to experiment with the shapes and to make many different trials. So, Student A would see a good counterexample that would contradict his view".

Specific counterexamples: The remaining 37 teachers in this category, constituting half of the participants, gave a specific counterexample. Twenty of them provided *incorrect counterexamples*. For example, some sketched two triangles that appeared to satisfy the given conditions and claimed that these triangles were not congruent although in their drawing these triangles seemed congruent. Thus, we consider this to be non-appropriate examples. Other examples had too many constraints - thus were non-existent. For example, T72 drew two triangles that seemed symmetrical in his attempt to produce two triangles that were not congruent (Figure 3), however, they seemed congruent.

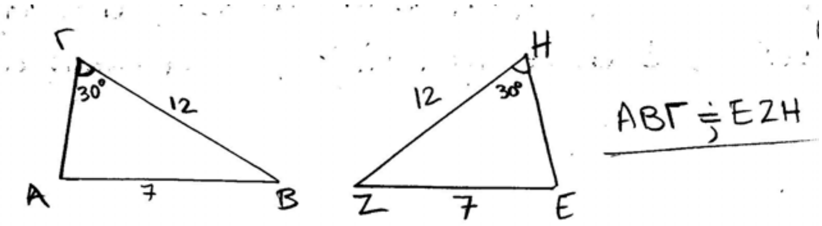

Figure 3: The drawing of T72

Some teachers considered the variation of a pair of angles but without giving specific measures. Others drew two triangles by attributing specific values to the angle contained between the two given sides. For example, T74 wrote: " I would ask the students to make two triangles ΑΒΓ and ΕΗΖ with ΒΓ=ΖΗ=12, ΑΒ=ΕΖ=7, the angles ΑΓΒ and ΕΗΖ equal to 30 degrees, the angle ΑΒΓ equals 90 degrees and the angle ΕΖΗ equals 45 degrees". In both cases, as it appeared also from the interviews and will be analysed further below, several teachers did not think about the existence of the suggested triangles and did not notice that they were suggesting non-existing cases.

Six teachers gave counterexamples that were *seemingly correct with no justification*. They drew two triangles which satisfied the given conditions for which the angles opposite to the sides of length 12 seemed supplementary, like in the appropriate counterexample. They claimed that this was a counterexample but did not give any justification for their claim. Finally, eleven teachers gave a *correct counterexample with justification* by constructing geometrically the two triangles that had the given elements and were not congruent. Some of them suggested to explore further with the students the situation and to formulate relevant theorems. For example, after the geometrical construction of a counterexample T33 wrote: "I would ask the students to try to prove that if one triangle has two sides and the angle opposite to one of these sides equal to the corresponding sides and angle of another triangle, then the corresponding angles that are not contained in the two sides are either equal or supplementary".

Emerging epistemological issues

The issue of existence of a (counter)example

As shown in Figure 2, over one third of the teachers (26) had not considered the problem of existence in their initial responses. In the interviews, the teachers who had not justified the process of constructing a counterexample as well as those who gave an incorrect example, were asked about its existence: "How do you know that the triangles you have drawn exist?". Some of them argued for the existence of their counterexample by inferring from a familiar theorem, recalling an image, or describing the drawing process. Following are some examples of their arguments:

"I have seen this counterexample in a textbook" (T32, recalling an image)

“If I remember well, there is a theorem that says that the non-contained angles are equal or supplementary”. (T40, inferring from a theorem)

“The sum of their angles is 180 degrees...They can be constructed...I can vary the angles” (T69, inferring from a theorem)

“I made them; I measured its sides with a ruler”. (T30, describing the drawing process)

When asked to consider the issue of existence, most of the teachers responded immediately that they had to check the existence of the suggested triangles. However, there were some who seemed to believe that the question about the existence of a triangle with specific properties had no meaning. Typical responses were:

“Yes, why can’t we? Do we have to prove it?” (T46)

“Is it possible not to exist?” (T60)

“I thought that it is sure that there are two triangles (satisfying the given conditions) which are not congruent. So, I opened a bit the angle and I moved the side to that direction.” (T73)

Another issue that emerged and was related to the problem of existence was the number of possible counterexamples. There were teachers who believed that there was more than one counterexample and in some cases they described a process of generating an infinite number of triangles (for example, T69, T73 mentioned above). This finding concurs with the findings of Zodik & Zaslavsky (2007).

During the interviews, we observed that some of these teachers started to think about ways of constructing appropriate counterexamples. For example, T39 sketched two triangles and commented: “If we draw on the board two triangles ΑΒΓ and ΕΖΗ with the given elements and the angles ΑΒΓ and ΕΖΗ to be acute - one smaller than the other – it is easy to verify by using transparent paper that the two triangles are not congruent”. In the interview she formed a new hypothesis that: “if in both triangles (satisfying the given conditions) all angles are acute, they are congruent while they must be different if one triangle is acute-angled and the other obtuse.” Later in the interview, she used the sine-rule trying to prove her hypothesis. However, she did not manage to construct geometrically the suggested triangles. On the other hand, T32 had given as a counterexample two triangles, one right-angled and the other isosceles. In the interview she initially recalled a known theorem “that one pair of angles can be equal or supplementary” and finally she gave a correct geometrical construction of the counterexample.

The issue of over-reliance on familiar criteria

Another issue that emerged from our data was the use of theorems for justifying refutable (invalid) claims. A number of teachers believed that the non applicability of the known relevant theorems implied that the claim was wrong. In particular, some teachers concluded that the two triangles were not necessarily congruent as none of the three commonly used criteria about the congruence of triangles could be applied. For example, in his written response T11 reminded the (hypothetical) student these three criteria, and added that "the problem statement does not satisfy the criterion S-A-S ... so from the given data we cannot conclude that the two triangles are congruent". The above argument was the only one that the teacher gave for justification. T47 also expressed a similar view in his written response. In the interview, although this belief was challenged by one of the researchers, it seemed to be rather strong as demonstrated in the following extract:

T47: The two triangles are not necessarily congruent
R: How do you know this?
T47: We cannot apply the criterion S-A-S.
R: Ok, a known criterion cannot be applied. But how do you know that there is no other way to prove the congruence of the two triangles?
T47: We cannot prove the congruence with the criteria we teach.

In the above cases, the teachers seem to base their reasoning on the principle that we can infer that two triangles with given properties are congruent only if these properties satisfy one of the three commonly used criteria (S-S-S, S-A-S and A-S-A). So, since these conditions were not explicitly given in the task, some teachers (falsely) inferred that these triangles cannot be congruent, while others claimed that they were not necessarily congruent. Although the latter claim may reflect legitimate logical inference, it may also have flaws and lead to wrong conclusions. For example, if the length of shorter side of the two triangles of our problem were 6 instead of 7, then the above kind of reasoning would lead to the conclusion that the two triangles are not necessarily congruent, while in fact, in this case the two triangles are right-angled and thus are indeed congruent.

It should be noted that even though the above way of refuting invalid claims is not a mathematical valid proof, in some cases it can be used as a tool for posing conjectures. For example, in his written response T21 initially stated that the known criteria could not be applied and then gave a geometrical construction of the counterexample.

CONCLUDING REMARKS

From the seventy six mathematics teachers of our study only thirteen refuted correctly the invalid claim of student A, eleven by constructing a counterexample and two by using theorems. Some of the characteristics of teachers' reasoning that were identified in this study are similar to those reported by Lin (2005) in the case of students. For example, there were teachers who confirmed the invalid claim, others who suggested the possibility of a counterexample without generating it, and few who actually constructed a counterexample accompanied by a mathematical proof.

In our study, teachers seemed to draw on their personal example spaces in order to generate counterexamples. It should be noted that in the case of Student A's (false) claim, a carefully thought through construction of an appropriate counterexample is needed. Teachers who just randomly sketched two triangles were not able to come up with an appropriate counterexample. In this problem, there is only one counterexample. This counterexample can be seen according to Mason and Pimm (1984) as a generic example, in the sense that it can reflect and lead to the general case (as illustrated in Figure 1). Similarly, in terms of Zaslavsky and Peled (1997) this counterexample has a high explanatory power. However, thinking of it for the first time turned out to be a strong demand on the teachers.

The two main phenomena that emerged from our study should be of great concern: overlooking the question whether an example exists, and over-relying on familiar theorems and criteria. Our findings illustrate how these two phenomena may lead to invalid inferences. Similar to Zodik and Zaslavsky's (2008) findings, there were several instances where teachers considered a non-existing example as if it existed, and did not seem to be aware of this issue at all. The second phenomenon reflects teachers' beliefs that a claim can be refuted if "all" the relevant theorems that they know (mostly those that are included in the school textbooks) cannot be applied. This conception indicates a misleading epistemological view of theorems and their status in mathematical reasoning.

In this paper, we focused mainly on teachers' mathematical knowledge as reflected in their responses. Pedagogical aspects of their knowledge that emerged from our data have not been discussed here. These aspects may provide a more comprehensive account of what is entailed in dealing with students' invalid claims.

ACKNOWLEDGMENT

This research has been partially supported by ELKE of the University of Athens.

REFERENCES

Balacheff, N. (1991). Treatment of refutations: Aspects of the complexity of a constructivist approach to mathematics learning. In E. von Glaserfeld (Ed.) Radical constructivism in mathematics education (pp. 89–110). Hingham, MA: Kluwer.

Bills, L. Dreyfus, T. Mason, J. Tsamir, P. Watson, A. & Zaslavsky, O. (2006). Exemplification in mathematics education, in: J. Novotná, H. Moraová, M. Krátká, & N. Stehliková (Eds) *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education, Vol. 1,* 126-154. Prague, Czech Republic.

Biza, I., Nardi, E., & Zachariades, T. (2007). Using tasks to explore teacher knowledge in situation-specific contexts. *Journal of Mathematics Teacher Education, 10*, 301-309.

Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics,* 44 (1-2), 5-23.

Harel, G., & Sowder, L. (1998). Students' proof schemes: results for exploratory studies. In A.Schoenfeld, J. Kaput and E. Dubinsky (eds.), *Research in collegiate mathematics education 3* (pp.234-283). Providence, RI: AMS.

Healy, L. & Hoyles, C. (2000). A study of proof conceptions in Algebra. *Journal for Research in Mathematics Education*, 31, 396-428.

Lakatos, I. (1976). *Proof and Refutation.* New York: Cambridge University Press.

Larsen, S., & Zandieh, M.: 2008, Proofs and refutations in the undergraduate mathematics classroom. *Educational Studies in Mathematics*, 67, 205-216.

Lin, F.L. (2005). Modeling students’ learning on mathematical proof and refutation. In H. L. Chick and J.L. Vincent (eds.) *Proceedings of the 29th Conference of PME* (pp. 3-18). Melbourne.

Mason, J., & Pimm, D.: 1984, 'Generic Examples: Seeing the General in the Particular'. *Educational Studies in Mathematics, 15*, 227- 289.

Peled, I., & Zaslavsky, O. (1997). Counter-example that (only) prove and Counter-example that (also) explain. *FOCUS on Learning Problems in mathematics, 19*(3), 49 – 61.

Rowland, T., Thwaites, A. & Huckstep, P.: 2003, 'Novices’ Choice of Examples in the Teaching of Elementary Mathematics'. In A. Rogerson (Ed.), *Proceedings of the International Conference on the Decidable and the Undecidable in Mathematics Education*. (pp. 242-245). Brno, Czech Republic.

Tsamir, P., & Dreyfus, T. (2002). Comparing infinite sets – a process of abstraction: The case of Ben. *Journal of Mathematical Behavior, 21*, 1-23

Stylianides, A. J., & Ball, D. L. (2008). Understanding and describing mathematical knowledge for teaching: Knowledge about proof for engaging students in the activity of proving. *Journal of Mathematics Teacher Education, 11*, 307-332.

Zazkis, R. & Chernoff, E. J. (2008). What makes a counterexample exemplary?. *Educational Studies in Mathematics, 68*, 195-208.

Zodik, I. & Zaslavsky, O. (2007). Exemplification in the mathematics classroom: What is it like and what does it imply? In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education* (pp. 2024-2033), Larnaca, University of Cyprus.

Zodik, I. & Zaslavsky, O. (2008). Characteristics of Teachers' Choice of Examples in and for the Mathematics Classroom. *Educational Studies in Mathematics, 69*, 165-182.