MODES OF ARGUMENT REPRESENTATION FOR PROVING – THE CASE OF GENERAL PROOF

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In light of recent reform recommendations, teachers are expected to turn proofs and proving into an ongoing component of their classroom practice. At least two questions emerge from this requirement. Is the mathematical knowledge of high school teachers sufficient to prove various kinds of statements? And does their knowledge allow the teachers to determine the validity of an argument made by their students? The results of the present study point to a positive answer to the first question in the framework of elementary number theory (ENT). However, the picture is much less positive with respect to the second one.

THEORETICAL BACKGROUND

The calls for enhancing students' abilities to prove and to refute mathematical statements appear prominently in various reform documents of different countries (e.g., Israeli Ministry of Education, 1994; National Council of Teachers of Mathematics [NCTM], 2000). In the NCTM document, reasoning and proof is one of five process standards for all grade levels. Still, there is a need to clarify what proof is in the classroom context. Stylianides (2007) made an attempt in this direction:

> Proof is a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:
>
> – It uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justifications;
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> – It employs forms of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom community; and
>
> – It is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of, the classroom community. (p. 107).

Stylianides' (2007) definition talks about the classroom community as the authority to determine the correctness of a proof. However, the teacher, as the representative of the mathematics community, has a special role in the endeavor. He needs to be attentive to both – the mode of argument for a

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given statement (such as general proof, counter example, supportive example), as well as the mode of argument representation (such as numerical, verbal or symbolic), to be able to determine the correctness of a justification.

To what extent are teachers prepared to implement proofs and proving as part of their classroom practice? Relatively little is known on teachers' subject matter knowledge in this area. Dreyfus (2000), following Healy and Hoyles' (1998) work with high school students, presented 44 secondary school teachers with nine justifications to the universal claim "The sum of any two even numbers is even". He found that most secondary school teachers easily recognized formal proofs, but had little or no appreciation for other types of justifications such as verbal, visual or generic ones. Knuth's (2002b) findings suggest that secondary school teachers recognized the variety of roles that proofs play in mathematics. Noticeably absent, however, was a view of proofs as tools for learning mathematics. Many of the teachers held limited views of the nature of proof in mathematics and demonstrated inadequate understandings of what constitutes proofs.

In a different study on in-service high school teachers' knowledge of elementary number theory (ENT), only a third of the 36 teachers provided counter examples to the (false) universal statement "All commutative actions are also associative" (Zaslavsky & Peled, 1996).

These studies focused solely either on universal or on existential statements. Tirosh (2002) presented the same group of elementary and middle school teachers with both universal and existential ENT statements. Tirosh and Vinner (2004) analyzed 38 prospective middle-school teachers' written answers to questionnaires on the issues of constructing and evaluating proofs and refutations in ENT. They found that about 20% of the prospective teachers incorrectly argued that some of the existence theorems in the questionnaires are false (e.g., "There exists a real number b so that $a + b < a$"). Furthermore, about half of the prospective middle school teachers incorrectly argued that numerical examples that satisfy existential statements are just examples and could not be regarded as mathematical proofs. These responses suggest that some prospective teachers develop a general view that a mathematical statement is true only if it holds for "all cases", a view which is adequate for universal statements but not for existential ones.

The present study addresses a high school teacher's knowledge with respect to universal and existential statements in the area of ENT. It aims to give a preliminary answer to the following two questions. Is the mathematical knowledge of high school teachers sufficient to prove ENT statements? And does their knowledge allow the teachers to determine the validity of an argument made by their students?

Note: the work of Tirosh (and Vinner) and Dreyfus differ from the work presented here in the population and the mathematical statements.

METHOD

Participants

A group of 50 high school teachers participated in the research. All teachers had some experience teaching in high school. Ms R was one of the teachers. Ms R was chosen as focus teacher for this study on the basis of her answers to the set of questionnaires below.

When participating in our project, Ms R had been teaching for five years in a high school, working with high-achieving students from a high socio-economic background. In parallel, she was studying for her Master's degree in mathematics education. The program included a number of mathematics courses and a number of psycho-didactical courses.

Tools

In one of these courses, the participants' mathematical knowledge was analyzed through their written reactions to two questionnaires that dealt with six ENT statements. No time-limit was imposed for the work on the questionnaires. In this section we briefly describe each of the questionnaires.

Predicate / Quantifier	Always true	Sometimes true	Never true
Universal	**S1.** The sum of any five consecutive natural numbers is divisible by 5. *True/General proof*	**S2.** The sum of any three consecutive natural numbers is divisible by 6. *False/Counter example*	**S3.** The sum of any four consecutive natural numbers is divisible by 4. *False/Counter example*
Existential	**S4.** There exists a sum of five consecutive natural numbers that is divisible by 5. *True/Supportive example*	**S5.** There exists a sum of three consecutive natural numbers that is divisible by 6. *True/Supportive example*	**S6.** There exists a sum of four consecutive natural numbers that is divisible by 4. *False/General proof*

Table 1. Classification of the six statements

The Prove-Questionnaire was intended to identify the participants' production of proofs (validations and refutations) to various (true or false) statements. The questionnaire included six ENT statements (statements S1-S6 in Table 1). The statements were chosen to include one of three predicates (always true, sometimes true or never true), and one of two quantifiers (universal or existential). Clearly, the validity of a statement is determined by the combination of its predicate and its quantifier. Three of the statements are true (S1, S4, S5), and the other three are false (S2, S3, S6). Table 1 displays the six statements according to their quantifier and predicate; their truth value as well as a suitable proof method are also indicated. The participants were asked to examine each of the statements, to determine whether it is true or false, and to prove their claim.

The *True or False-Questionnaire* was intended to check the participants' identification of the correctness of 43 justifications for the six statements they had proven before, between six and nine justifications for each statement, using numerical, verbal or symbolic modes of arguments representations. For each justification, the participants were asked to determine whether it verifies (refutes) the statement, and to explain their evaluation. The justifications were presented as if they were written by students in various modes of argument representations.

In analyzing teachers' answers to the first and second questionnaire we related to the modes of argumentations as well as to the mode of argument representations.

RESULTS AND DISCUSSION

In this section we first present the participants' answers to the *Prove-Questionnaire*, with examples of Ms R's proofs. Then we discuss the participants' answers to the *True or False-Questionnaire.* Here we narrow the discussion to five justifications which relate to two statements – S1 and S6. We chose these two statements because they require general proofs. We present in detail the answers of Ms R to each justification, followed by a brief description of the results for all participants with regard to the same justifications.

Prove-Questionnaire

All the teachers produced correct proofs to each of the six statements. That is, the modes of argumentation the teachers chose for each statement were appropriate. Their proofs were presented in one of two modes of argument representation – symbolic or numeric (see Table 2).

All participants used the symbolic mode of argument representation for statements S1 and S6, which required a general mode of argumentation. About half of the participants produced numerical examples to refute the universal statements S2 and S3, and the majority of the participant provided a single numerical example to validate the existential statements S4 and S5. None of the participants provided several examples to prove or refute a statement. These findings indicate that the participants who used numerical examples knew when an example is sufficient for proving a statement.

	S1	S2	S3	S4	S5	S6
Numeric	---	50	44	72	80	---
Symbolic	100	50	56	28	20	100

Table 2: Percentages of modes of argument representation produced by the participants (N=50)

We present Ms R's proof for statement S1 which is a universal, always true:

> Let's denote five consecutive numbers by a, $a+1$, $a+2$, $a+3$, $a+4$. Their sum is: $a+a+1+a+2+a+3+a+4 = 5a+10$.
> $(5a+10):5 = a+2$. $a+2$ is a natural number for any a that is a natural number. Therefore the statement is true.

As we can see, the proof that Ms R provided related to all the cases in the domain, used correct inference rules, is concise, and thus exemplifies a sound proof.

Ms R's proof for statement S6, an existential, never true statement shows similar characteristics:

> Let's check: a is a natural number. $(a+a+1+a+2+a+3):4=(4a+6):4$
> We divide the last expression by 2, obtaining $(2a+3):2$. But, $2a+3$ is an odd number (the sum of even, $2a$ and odd, 3), and therefore is not divisible by 2. The statement is not true.

Again Ms R correctly identified the need for a general mode of argumentation, and used a symbolic mode of argument representation.

True or False-Questionnaire – Ms R's explanations.

We now focus on the two statements that required general proofs, meaning that the general mode of argumentation should be used. Yet, such an argument can be displayed in at least two modes of argument representation – verbal and symbolic. Five sets of justifications, Ms R's judgments, and her explanations are presented. A short discussion follows each set.

Example 1: Verbal justification to statement S1 and Ms R's explanation

The given correct justification:

> Moshe claimed: I checked the sum of the first five consecutive numbers: *1+2+3+4+5=15* is divisible by 5. The sum of the next five consecutive numbers is larger by 5 than this sum (each number is bigger by 1 and therefore the sum is bigger by 5), and therefore this sum is also divisible by 5. And so on, each time we add 5 to a sum that is divisible by 5, and therefore we always obtain sums that are divisible by 5. Therefore the statement is true.

Ms R's judgment: Moshe's argument is not correct.

Ms R's explanation

> Moshe checked the case *1+2+3+4+5=15*, which can be accidentally true. In proving one needs to generalize, and therefore Moshe's justification is not correct.

From Ms R's explanation we can learn that she correctly identified the mode of argumentation needed for proving S1. Yet, she failed to notice the coverage aspect in Moshe's justification.

Example 2: Verbal justification to statement S1 and Ms R's explanation

The given correct justification

> Mali claimed: I first tried the first ten examples of 5 consecutive numbers:
>
> 1+2+3+4+5=15 2+3+4+5+6=20 3+4+5+6+7=25
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> 4+5+6+7+8=30 5+6+7+8+9=35 6+7+8+9+10=40
>
> 7+8+9+10+11=45 8+9+10+11+12=50
>
> 9+10+11+12+13=55 10+11+12+13+14=60.
>
> I saw that the statement is true for the first ten. All other sums of five consecutive numbers are obtained by adding multiples of 10 to one of the listed sums (for instance, the sum 44+45+46+47+48 is obtained by adding multiples of 10, 5 times 40, to the sequence: 4+5+6+7+8 that I checked before). Since multiples of 10 are also divisible by 5, the statement is true.

Ms R's judgment: Mali's argument is not correct.

Ms R's explanation

> Here also there is no generalization to all the natural numbers, and therefore this is incorrect. It is not a proof.

From Ms R's explanation in this case we can learn that Ms R is concerned with the mode of argumentation. She did not identify the cover aspect in Mali's correct verbal justification.

Example 3: Symbolic justification to statement S1 and Ms R's explanation

The given incorrect justification

> Ayala claimed: Among any five consecutive numbers, there is one that is divisible by 5. Let's look at a sequence of five consecutive numbers : $5x$, $5x+1$, $5x+2$, $5x+3$, $5x+4$ ($5x$ is divisible by 5). The sum of this sequence is: $5x+(5x+1)+(5x+2)+(5x+3)+(5x+4)= 25x+10$, and $25x+10$ is divisible by 5 for any x. Therefore the statement is true.

Ms R's judgment: Ayala's argument is correct.

Ms R's explanation

> x represents any number, and therefore the proof is general.

Ms R's explanation in this case relates to two important observations. x represents any number, and in this sense the justification is general. However, $5x$ represents a multiple of five, and thus the sequence 1, 2, 3, 4, 5, for instance, is not included. Hence, Ayala's justification is correct for only a subset of the cases that one needs to relate to in order to prove S1. Ms R failed to notice this flaw in Ayala's justification.

Example 4: Verbal justification to statement S6 and Ms R's explanation

The given correct justification

> Moshe claimed: I checked the sum of the first four consecutive numbers: $1+2+3+4=10$, ten is not divisible by 4. The sum of the next four consecutive numbers is obtained by adding 4 to this sum (each of the four numbers in the sum grows by 1, so the sum grows by 4). It is known that adding 4 to a sum that is not divisible by 4 will yield a sum that is not divisible by 4 either. And so on, each time we add 4 to a sum that is not divisible by 4, and therefore we always obtain sums that are not divisible by 4. Therefore the statement is not true.

Ms R's judgment: Moshe's argument is not correct.

Ms R's explanation

> Moshe chose an example, and on the basis of this example he concluded that there are no such four numbers. But maybe if he would have picked up four other numbers it could have been correct.

Once more, Ms R's reaction exemplifies her view that Moshe's verbal explanation is an example. Again she correctly determined that for this statement an example is not an appropriate mode of argumentation.

Example 5: symbolic justification to statement S6 and Ms R's explanation

The given incorrect justification

> Ayala claimed: Among any four consecutive numbers, there is one that is divisible by 4. Let's look at a sequence of four consecutive numbers : $4x$, $4x+1$, $4x+2$, $4x+3$ ($4x$ is divisible by 4). The sum of this sequence is: $4x+(4x+1)+(4x+2)+(4x+3) = 16x+6$. $16x$ is divisible by 4 for any x, while 6 is not divisible by 4. So, the sum $16x+6$ is not divisible by 4. Therefore the statement is not true.

Ms R's judgment: Ayala's argument is correct.

Ms R's explanation

> Ayala proved the claim for all four numbers, and hence it is not possible to show that there are four numbers, hence the justification is correct.

The same phenomenon as in example 3 is evident again in Ms R's reaction. On the one hand, it shows that she fully understands the mode of argumentation needed, but on the other hand she fails to recognize whether the given justification carries the general aspect needed.

It seems that for Ms R, the symbolic mode of argument representation, assures that the cover aspect of the proof is taken care of. Also, for Ms R, a verbal mode of argument representation is judged to be merely a numerical example.

One may wonder whether Ms R is unique in her judgments. Let's return to the entire population of 50 participants and check how many teachers made similar choices as Ms R.

For the first statement (S1), 34 percent of the participants rejected the correct verbal justifications (Examples 1 and 2), on the ground that they are not general, and at the same time accepted the incorrect symbolic justification (Example 3), on the ground that it is general. As Ms R, these teachers correctly identified the mode of argumentation needed for each statement.

For the last statement (S6), 26 percent of the participants rejected the correct verbal justification (Example 4), on the ground that it is not general, and at the same time accepted the incorrect symbolic justification (Example 5), on the ground that it is general. Also in this case, the teachers correctly identified the mode of argumentation needed for each statement.

Twenty percent of the participants were consistent in their answers, that is made the same choices as Ms R in the cases of the five justifications presented above.

SUMMING UP AND LOOKING AHEAD

The present study addressed the following two questions. Is the mathematical knowledge of high school teachers sufficient to prove mathematical statements from the field of elementary number theory? And does their knowledge allow the teachers to determine the validity of an argument made by their students?

Our findings indicate that the participants were able to produce correct proofs and refutations to the statements presented. While the teachers chose correct modes of argumentation for each statement, it was evident that they were concerned with this aspect in the second questionnaire.

The picture emerging from the *True or False-questionnaire* seems more complex. About a third of the teachers failed to identify as universal the general-cover aspects of the given arguments in verbal modes of representation. These findings substantiate similar findings reported by Dreyfus (2000), that teachers tend to perceive verbal proofs as deficient because they lack symbolic notations. However, Dreyfus (2000) found that teacher tended to reject verbal justifications. Our findings indicate that teachers had difficulties in understanding verbal justifications, but they did not reject them as such. Teachers’ difficulties with verbal justifications are particularly worrying in light of the results reported by Healy & Hoyles (2000), namely that high school students not only preferred verbal proofs due to their explanatory power but also that their verbal arguments were more often deductively correct than their arguments in other modes of representation, yet at the same time they expected to get low grades for such proofs.

A quarter of the participants failed to identify when symbolic justifications did not cover all cases in the domain. These findings substantiate findings reported by Knuth (2002b): "In determining the argument's validity, these teachers seemed to focus solely on the correctness of the algebraic manipulations rather than on the mathematical validity of the argument” (p. 392). When being presented with an algebraic justification, the teachers' focus was on the examination of each step, ignoring the need to evaluate the validity of the argument as a whole.

The everyday practice of teachers involves a constant evaluation of students’ justifications for statements. It is likely that verbal or symbolic justifications of the kinds presented in our study, will emerge during interactions with students. Therefore, it is important that teachers will be familiar with verbal justifications and able to judge their validity.

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