VISUAL PROOFS: AN EXPERIMENT

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The main goal of this paper is to start a preliminary study of the basic features of visual proofs in mathematics and their use in mathematics teaching. The investigation, based on college mathematics students, shows a very poor use of visual reasoning in mathematical tasks involving figures. Moreover, students’ use of visual semiotic systems is not spontaneous but seems to need some special training. Some of the ways of working students usually adopt when dealing with visual proofs have been identified, showing that most often diagrams are not seen as representations of complete processes, but rather as ready-made aids to solve problems.

INTRODUCTION

Many researchers have stressed the importance of visual reasoning in the learning of mathematics and have remarked that research in mathematics education has still a lot to develop about this topic (see e.g. Dreyfus 1991, Jones 1998, Presmeg 2006). In this perspective this paper focuses on visual proofs i.e. on proofs where the deductive steps are based on figures, diagrams or graphs. This means that the inferences are possible through just the reading of the figures. Although geometrical figures will be taken into account only, the expression ‘diagrammatic proof’ or ‘visual proof’ will be used in a more inclusive sense. At this regard, a number of works, such as Nelsen’s books (1993, 2001) have provided a wide selection of examples of visual proofs from different sources. In literature visual proofs are usually presented with no comments in verbal language (i.e. without words), but only based on diagrams, possibly equipped with numbers, letters, arrows, dots, or other signs and sometimes associated with symbolic expressions; the reconstruction of the proof is left to the reader. Nowadays visual arguments are far to be considered legitimate arguments for rigorous proofs probably due to the fact that they can easily misread and therefore lead to wrong inferences. Anyway their importance as an aid for the discovery of new results and the production of more formal proofs is widely recognized. In the last decades interest in visual proofs has grown up leading to both new mathematical investigations and applications to mathematics education. On the side of mathematical investigations above all we mention the work of Barwise and Etchemendy (1991) and further developments in the same line such as Jamnik’s study (2001). From the educational viewpoint the role of visual reasoning in mathematics teaching has been taken again into account and emphasized (see e.g. Dreyfus 1991, Dvora & Dreyfus 2004, Hanna 1989, Presmeg 1997, 2006).

The main goal of this paper is to identify the main difficulties in the use of diagrams in mathematics, in particular in the extraction of information. For this purpose some visual proofs have been taken into account. In the experiment I am describing some statements with the corresponding diagrammatic proofs have been given to mathematics sophomore and third year students. Such proofs have been presented without any explanation on the inference steps implicit in the figures. The work is also aimed to compare the processes involved in visual proofs to those involved in the standard ones. Diagrams are not relevant only in relation to visual proofs, but they can also support either standard proof processes (i.e. proofs based on a verbal or symbolic text) or problem solving. Indeed the heuristic role of figures is widely recognized both by mathematicians and by mathematics educators. Therefore some features of diagrammatic proofs will be taken into account, which might be relevant from the educational viewpoint and to explore the opportunities that they can provide in order to improve the approach to mathematical theorems.

THEORETICAL FRAMEWORK

The production or the understanding of a diagrammatic proof involves constructing and treating (detaching, reversing, superposing, translating,…) figures and extracting information from them. All these operations will make evident the inferential steps that make up a visual proof of a statement. Moreover a diagrammatic proof is developed for a particular value of the domain of validity of the theorem but anyway it represents the proof for all values of the domain (character of generality, Barwise & Etchemendy 1991).

We did not find in literature a theoretical framework closely focused on visual proofs in mathematics education. Although here we are focusing on visual proofs that are based on geometrical figures, we take into account some different works about visual reasoning and visualization that could help us to interpret difficulties about this topic.

First of all, according to Fischbein (1993), geometrical figures are mental entities (named also 'figural concepts') which possess conceptual and figural characters at the same time. In this frame, as other studies in geometry, we refer to figures as the mental entities which possess properties imposed by, or derived from axiomatic systems and to drawings as their (external) representations. A major problem in the use of diagrams and figures is the potential conflict between conceptual and perceptual features of figures. Fischbein's theory is very helpful at this regard. Fischbein argues that '…figural concepts constitute only the ideal limit of a process of fusion and integration between the logical and figural facets' (Fischbein 1993, p.150). In particular visual proofs involve some logical questions concerning the nature of deductions based on diagrams and figures. Actually, it is to be considered that visual proofs are bound to correspond to some extent to proofs in the standard mathematical sense. In this work I do not mean to question the rigorousness of diagrammatic proofs (on this topic see Barwise & Etchemendy (1991), Jamnik (2001), Hanna & Sidoli (2007), Allwein, G. & Barwise J. – Eds. (1996) and references therein) but I assume that they can be regarded as legitimate mathematical processes.

Another main difficulty encountered by students is due to the lack of coordination of systems of semiotic representations (Duval 1993). Working with a visual proof requires a continuous interplay between the semiotic system of figures and the semiotic systems involved in the statement, usually verbal texts or symbolic expressions. Like Duval, I assume that semiotic systems are not neutral carriers of meanings but can contribute to the construction of meaning themselves. This explains the attention I am going to pay to semiotic systems through this paper.

AN EXPERIMENT

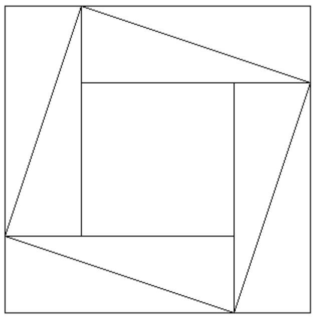
At the Università del Piemonte Orientale, in Italy, in the context of a course devoted to mathematical proof, we have given a group of 13 sophomore and third year undergraduate Mathematics students a number of tasks requiring to look at diagrammatic proof of some statement and to reconstruct such a proof (i.e. to describe how the proof could be extracted by the figure). The tasks have been administered as written tests and they were followed by interviews in order to better understand the arguments written by students.

The problems are the following:

Task 1.

The picture on the right represents a visual proof of the Pythagoras' theorem.

- Describe such a proof.
- Reconstruct the figure in the case that the legs of the right-angled triangle have the same measure.

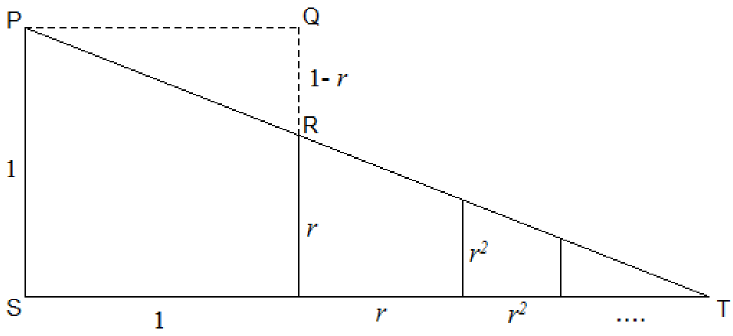


Task 2.

The picture on the right represents a visual proof of the theorem

$$\sum_{i=0}^{+\infty} r^i = \frac{1}{1-r} \quad \text{for } 0 < r < 1.$$

Describe such a proof.



The statements are in two different fields of mathematics: the Pythagoras' theorem and the geometric series. Pythagoras' theorem is customarily associated to visual representations, whereas the latter is less common (at least in Italy), as the convergence of the geometrical series is usually proven using a combination of algebraic and analytical arguments. So this visual proof is very unusual for Italian mathematics college students. The choice of theorems from different fields is aimed at finding common features and common difficulties related just to visual reasoning. As the results show, students find this kind of problems very difficult. The main difficulty is due to the fact that the drawing is a static object while a proof is made of an ordered sequence of inference steps. A drawing presents in a whole all written data and the reader has to choose the order of the construction and how to extract the information.

Analysis of task 1. Here the construction of the drawing may not present so many problems since it is not required a precise order of construction as far as one recognizes that there is a particular disposition of six right angle triangles. Troubles can arise when trying to find correspondences between the statement and the picture. This task is mainly based on visual arguments. Students could meet with difficulties in the identification of the area of the square built on the hypotenuse (Fig.1) and above all of the areas of squares whose sides are the legs of the right triangle (Fig.2) since they are not bounded with segments. Such a problem is related to the rearrangement of the figure.

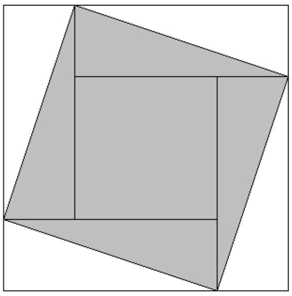


Fig.1

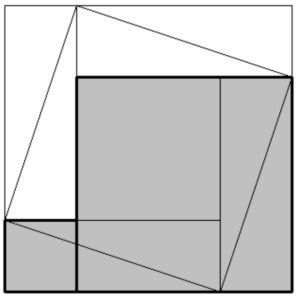


Fig.2

Therefore students could meet with difficulties from the perceptual side, as they might fail to spot the appropriate triangles or squares. In fact, as pointed out by Duval (1993) graphical sign can be either a help or a hindrance in understanding diagrams.

Analysis of task 2. In this case the reconstruction of the drawing itself is a difficulty. It requires the conceptualization that such a construction is made of infinitely many steps and that it proves that the series is convergent. All this requires a good conceptualization of the real numbers and their representation on the line. Moreover students could meet difficulties, not only with perceptual aspects, but above all with the lack of coordination of three different semiotic systems. One has to recognize that

$\sum_{i=0}^{+\infty} r^i = \frac{1}{1-r}$ is a proportion, then to translate it in the graphical system and finally to identify it in the given figure.

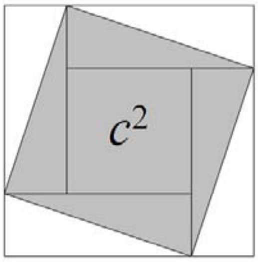
RESULTS

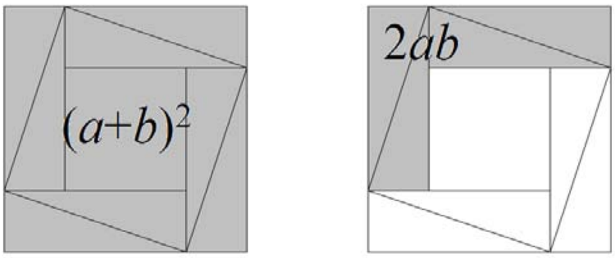
Task 1.

First of all in this task few students only provided an explicit description of the construction process of the figure**.** In this visual proof, the construction of the drawing is not related to the understanding of the proof since they succeed to achieve it even if with deductive arguments not based upon the whole picture but on some parts of it only. In particular, notice that some students do not feel the necessity to prove themselves that the tilted figure that looks like a square is indeed a square. In this case the perceptual facet is not controlled by the conceptual one. Second, all of them introduced letters *a*, *b*, *c* to indicate the measure of the sides of the triangle in order to find correspondence between the formula $a^2+b^2=c^2$ and the figure. Finally, students addressed the first task in three different but not necessarily separate ways:

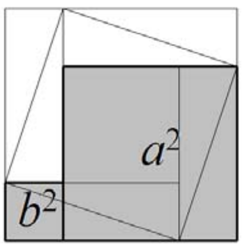
1. *Modifying the formula in order to find correspondence with the figure*

Some students tried to connect the formula $a^2+b^2=c^2$ to the figure and to identify just c^2 in the picture to the right. They were not able to do the same for a^2 and b^2. Then they wrote down $(a+b)^2-2ab=c^2$ most likely because they could find $(a+b)^2$ and $2ab$ in the picture too, as shown below:





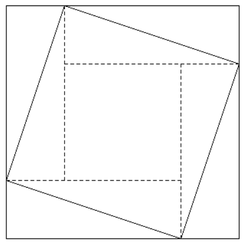
This way the students recognized the remaining area a^2+b^2 in the figure on the right. This kind of proof is mostly based on visual arguments except for the initial modification of the formula.



2. *Area computation*

This strategy is the most common in problems of this kind. It consists in calculating the area of the external figure in two different ways and then comparing the results to obtain the required relationship.

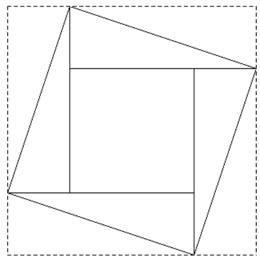
In the first problem they calculated the area of the square of side $a+b$ i.e. $(a+b)^2$ and then the same area as the sums of the five subfigures (four triangles and a square of side c) i.e. $4\frac{ab}{2}+c^2$. Comparing the two expressions they got the Pythagorean theorem through algebra. In this case they did not consider the dashed lines in the picture on the right.



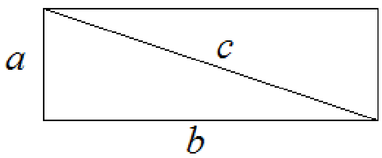
3. *Figures as plain tools*

The figure is not seen as a process embodying the proof of a statement but just as a tool that can be used to occasionally pick some piece of information useful to get a proof.

For example in this problem four students considered just the tilted square of side c and its five subfigures (four right-angled triangles and the square of side a-b). Actually they did not consider the dashed lines in the figure on the right. Comparing the area of the square of side c calculated as c^2 with the same area but regarded as the sum of the areas of the five subfigures one obtains the result as in point *2 (Area computation).*



Another student just considered the rectangle



defining a the short side and b the long one. Then she used a so called "circular argument" or "begging the premise" (cf. Weston, 2000), i.e. she used the Pythagoras' theorem to get $c=\sqrt{a^2+b^2}$ and hence squaring both sides she got the Pythagoras' theorem $c^2=a^2+b^2$.

Notice that also the answers in point *2. (Area computation)* denote that the figure is not seen as an autonomous process of proof.

Task 2.

The Problem 2 proved the most difficult one. Nobody succeeded in understanding this visual proof. So a hint was given to them while they were solving the task. It was told them that a fundamental tool for its comprehension was the similitude of triangles and in particular the proportionalities between corresponding sides of the triangles. After that some of them succeeded to recognize that $\triangle$PST and $\triangle$PQR are similar and they found the correspondence between the formula and the sides of triangles.

As a first result we have that students were able to match labels with the formula, and to understand the meaning of the dots '…'. As second finding we have that most students did not reconstruct the drawing. The reasons are three:

1. Students understood the need to reconstruct the drawing. Such construction is a necessary step in order to consider the visual proof as a process. Unfortunately they are not able to do such a reconstruction. One can see this outcome from the following excerpts:

   A: Consider a square of side of length 1 ($l = r^0$) PQMS and construct a right-angled triangle PST such that the shorter leg is $\overline{PS} = r^0$ and one finds that the longer leg $\overline{ST}$ is the sum of infinite segments having measure respectively r^0, r^1, r^2, …..

   (Student A understood that the measure of $\overline{ST}$ is not an assumption but a finding of the construction but he could not prove that result, as it became clear from the interview) or

   B: I can not understand how in the figure r^2 comes out from r.

2. Students considered figures just as plain tools. This is evident in task 2:

   C: …from figure I can see that $\overline{PS}$ measures 1, $\overline{ST}$ measures $\sum_{i=0}^{+\infty} r^i$,…

   Student C did not see that $\overline{PS} = 1$ is an assumption while $\overline{ST} = \sum_{i=0}^{+\infty} r^i$ is the result of a deductive steps and in particular it means that the series converges.

3. Students understood the need to reconstruct the drawing but they failed to do it since they considered it trivial.

Finally some students could conclude the proof using the help given to them, but we distinguish

- students who were able to prove that the triangles PST and PQR are similar because they recalled this notion;
- students who did not recalled this notion or never learnt it.

In this case the problem is that even if students had a good knowledge of similitude of triangles they failed to introduce such "new" tool which could not be directly extracted by a simple manipulation of the objects already appearing in the proof.

General discussion

One of the main findings of this work is that visual proofs are not seen as processes but the figures are just plain tools which help to find results. The investigation of the protocols highlights that the unsuccessful results of this kind of tasks are due not only to the semiotic system of figures or to the conflict between the conceptual and figural nature of visual proofs but it comes out that the concept of mathematical proof is not understood enough. This conclusion comes out above all from the fact that students do not feel the need for reconstructing the drawing. Moreover, in the first problem students used just some parts of the figure and not the whole of it, that is some students did not attribute values at every graphical sign, as it is explained in the analysis of the first task. Also this behaviour, in some cases, is due to a misunderstanding of the nature of the process of visual proofs. In fact the role of graphical signs and more in general of the perceptual learning of a figure is very important both in a positive and in a negative sense (Duval 1993). Perception can be a useful tool only if it is controlled by conceptual processes as pointed out by Fischbein.

Second it comes out that one of the main obstacles is the lack of geometrical knowledge: notions like similitude and congruence of triangles, Thales' theorem, etc. are hardly known, which severely prevents any attempt to work with the figure. This situation is found in the problem about the geometrical series. For example one of the fundamental steps for understanding this visual proof is to notice that the triangles Δ PQR and ΔPST are similar. No one spotted this geometrical fact. There might be two reasons of it. First, students have never learnt this or they have forgotten it. Second, they could not easily call to mind this notion, actually they knew something on similitude of triangles but they were not used to work with it. This means that students are not aware that there are some theorems, techniques, tools, which they can exploit when facing triangles. The fact is that Italian students work very little or do not work at all on the visualization of geometrical figures (for further details see Mariotti, 1998). Moreover, the time given for solving the task is not sufficient to remember or to reconstruct this notion. However, the necessity to use tools and constructions which are not directly related to objects at hand is a common feature in mathematical proofs, which do not refer to visual proofs only. Students could not overcome the difficulty of introducing such new elements in the visual proof we proposed them. Moreover, students were not even able to exploit the symbolic expression in the statement, since it would have required to represent it as a proportion, that in Italy is given prevalently by $\left(\sum_{i=0}^{+\infty} r^i\right):1=1:(1-r)$, and then into the figural system. The difficulty about the introduction of new elements, however, is not peculiar to visual proofs only. Indeed the first task does not present this problem. In this case all students succeeded in grasping the result even if in an improper way, for example using the figure just as a tool to extract information. Here one has just to manipulate the formula of the Pythagorean Theorem or manipulate its figure; there is no need to introduce new constructions, techniques, assumptions, tools, etc.

Finally the analysis of protocols shows that students prefer to work with algebra instead of using visual arguments coming from manipulation of figures. The visual proof in task one is only of visual nature but no one addressed it with just visual arguments. Just one student used prevalently visual arguments (see strategy 1), but even in this case there was a preliminary modification of the formula.

CONCLUSION

This explorative research outlines the lack of skills in visual reasoning by a group of Italian mathematics college students. This lack is due to different reasons: poor knowledge of certain basic mathematical tools, poor acquaintance with the use of figural representations, conflict between the conceptual and perceptual nature of diagrammatic proofs and sometimes poor understanding of the concept of mathematical proof itself. Besides, the research points out that it is very difficult to learn proofs without being able to pick and use some basic pieces of mathematical knowledge. In this context tasks like those presented in this work might help students to develop a correct use of deductive method when working with figural representation and not only in the field of geometry but also in other context as in the second task presented. Obviously graphical representations in different mathematical settings can present different features related to different concepts. For example, in the case of the geometrical series one has to take into account the graphical representation of real numbers and of their properties. According to Duval, the coordination of at least two different semiotic systems of representation of a concept can improve its understanding. In particular I think that the passage from verbal and symbolic representations into the figural one and vice versa could be very fruitful. Moreover I think also that problems like the second one can help to overcome the trend to deal with mathematical subjects in isolation. Since, as our result confirms, the use of graphical representation presents a lot of difficulties, its use requires a particular training in order to exploit its potential. In this perspective tasks like those presented in this work could help students to develop some important tools to approach also other mathematical problems such as standard proofs.

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