INTRODUCTION OF AN HISTORICAL AND ANTHROPOLOGICAL PERSPECTIVE IN MATHEMATICS: AN EXAMPLE IN SECONDARY SCHOOL IN FRANCE

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Abstract: To introduce an anthropological and historical perspective in mathematics from middle school is a challenge that we have tried to face for several years. We first present what we mean with "an anthropological and historical perspective in mathematics", our theoretical and didactical references, and our motivations for choosing the theme of irrationality. In the second part, we will present elements of three experimentations carried out with grade 8 (age 13-14) and grade 10 (age 15-16) pupils.

Key words: History of mathematics – Anthropological approach – Didactics of mathematics – Epistemology - Irrationality

I. MOTIVATIONS

In France, attempts to introduce an historical perspective in mathematics have been developed for several years, in particular, but not only, through the IREM Commission on History and Epistemology of Mathematics[3]. Some historical elements are also often introduced in textbooks (but most often without taking mathematical considerations into account). Beyond this, a crucial issue in a didactic perspective is the way it is possible to articulate historical elements with mathematical knowledge in teaching at the various levels of the curriculum. To approach historical texts mathematically most often necessitates an important effort to understand them, and the possibility of putting these texts in relation to the mathematical content for teaching is difficult and far from an evident choice, due in particular to the fact that the modern concepts are more efficient for solving the related problems. This could explain the rather common choice of limiting the introduction of history to informative aspects aiming mostly to motivate the students. Although this aspect should not be neglected, because it could allow us to modify the common representation of mathematics as timeless knowledge, it does not take into account the potential contribution of History of Mathematics for the learning of Mathematics itself. With Bkouche (2000), we consider that an historical perspective in the teaching of sciences « can be inserted less as a motivation than a *problematisation* » in the following meaning: "Epistemology of problems aims to analyse how the problems that lead humanity to elaborate this mode of knowledge that we name scientific knowledge have modeled the theories invented in order to solve these problems"[4].

II. THEORETICAL BACKGROUND

II.1. Anthropological foundations of mathematics

In continuity with Tardy (1997), we have chosen to situate the historical perspective in the field of Anthropology. Chevallard (1991) considers that Didactics of mathematics is the headland of the anthropological continent in the mathematics universe", that specifies its place in the field of Anthropology. In this perspective, he mainly studies the didactic transposition, i.e. the transformation undergone by mathematical knowledge when it is taught and used. For him, "present epistemology" studies the question of knowledge production while he considers Epistemology in the broader sense of Anthropology of knowledge.

In this paper, we refer to the sense of "present epistemology", including anthropological considerations, according to Kilani (1992) that Anthropology searches for relations between local knowledge or specific discourses on cultures to global knowledge or general discourse on humanity.

II.2. Genetic psychology and Anthropology

Genetic Psychology elaborated by Piaget questions Anthropology. In opposition to Piaget, present Anthropology does not consider hierarchy among different stages. The stages that Piaget has distinguished (practical intelligence; subjective, egocentric, symbolic or operative thought) cut across the questions of Anthropology on the relationship between culture and thought, leading to debate around myth and rationality, magic and science and the way to pass from one aspect to another. Anthropology states that operative and symbolic thoughts have different purposes; that they do not exclude each other, coexisting in a singular person as well as in a given society[5]. Moreover, it could be thought that Imagination as well as reason could play a role in scientific discoveries (Kilani, 1992)

Following Vergnaud, we can add that in mathematics activity, these different modes of thought are necessary and complementary.

> "Explicit concepts and theorems only form the visible part of the iceberg of conceptualisation: without the hidden part formed by operative invariants, this visible part would be nothing. Reciprocally, we are unable to talk about operative invariant integrated in Schemas without the categories of explicit knowledge: propositions, propositional functions, objects, arguments." (Vergnaud, 1991, p.145)[iii]

II.3. The epistemological model of « milieu » in the Theory of Didactical Situations (Bloch, 2002)

- ✓ **About the concept of milieu**

The concept of « milieu » plays an important role in the Theory of Didactical Situations (Brousseau, 1997). Several authors have reworked and developed this concept, which was one of the themes of The 11th Didactic Summer School in France in 2001. From our perspective, the models of *milieu* presented in this frame by Bloch is particularly enlightening. In the introduction to her course[7], Bloch indicates:

> "In this course, we aim to attempt a clarification of some fundamental concepts of Theory of Didactical Situations, and for this purpose to propose a reorganisation of the models of

> milieu of this theory to predict and analyse teaching phenomena. It is clearly an elaboration aiming to classify the theoretical elements related to the *milieu* according with their functionality (from knowledge; from experiment; from contingency)" (Bloch, 2002, p.2)[iv].

This led her to propose the three following models: the *epistemological milieu* that concerns the cultural knowledge and their organisation, and the fundamental situations - the *experimental a priori milieu*, that concerns the researcher's work preparing for the relevant teaching situations, and the *milieu for the contingency* concerning the effective realisation of these situations. In this section we focus on the epistemological model.

- ✓ **About fundamental situations**

For Brousseau, a fundamental situation for a given body of knowledge ought to permit the generation of a family of situations characterised by a set of relationships between student and milieu permitting the establishment of an adequate relationship to this knowledge.

- ✓ **The need of a model of epistemological *milieu***

To give a definition of what could be an adequate relationship to a given body of knowledge is not as easy as it might appear at first sight. It is the task of a researcher who attempts to elaborate a model of epistemological milieu:

> "Such a model (written Mi_T) is elaborated taking in account the cultural mathematical knowledge, but is not restricted to it. To elaborate *milieus* consists in grouping problems that do not necessarily strictly obey the knowledge organisation, thus a conjunction of mathematical, epistemological, and referential practices is necessary. I will also add the identification of knowing. Thus, one has to take into account not only problems for which this knowledge is functional, but also the relationship between these problems, and as far as it is possible, the related knowing (possible actions, intuitions, personal and cultural references) that the student could be able to actualise in the situation. " (Bloch, 2002, p.5)

Our ambition, in this research, was not to elaborate a fundamental situation for a given notion (for us the notion of irrational number), but to attempt to enrich the net of relevant problems for the learning of this notion, leaning on a study (non exhaustive) of « the historical genesis of the knowledge concerning this concept and its ancient or contemporaneous occurrences, its functionalities in mathematics... » (Op. cit. p.7) as well as its links with other fields of human activity (philosophy; sociology; history; psychology; didactics …), all links that have to be taken into account in the elaboration of an epistemological milieu as defined above. This permits us to investigate the way to elaborate the milieu for a teaching situation aiming to integrate this historical genesis and this anthropological perspective. In other words, how to make possible that historical or cultural references, beyond their function of motivation, contribute in a genuine way to the teacher's project of the elaboration by students of knowing coherent and consistent with the involved knowledge. We will give further some elements that we have identified in this research.

III. AN EXAMPLE IN SECONDARY SCHOOL: IRRATIONALITY

III.1. Preliminary: a logical point of view

In a major work of Analytic Philosophy[v], the philosopher and logician Quine support the thesis that attributing a pre logical mentality to natives is wrong; in particular, rather than considering that they have contradictory believes, we have better to bet on an inadequate translation, or in a domestic situation[vi], on a linguistic disagreement. In other words, the irrationality or the incoherence of humans is less probable than a non adequate interpretation by the observer of the provided indicators. We have shown (Durand-Guerrier, 1996) an example of the domestic version in mathematics education in order to lift a suspicion of incoherence that might bear on students' responses[vii]. Matters concerning contradiction, rationality and irrationality are subjects of study for logicians, either those attempting to elaborate systems accepting contradictory propositions, due to the fact that such propositions are everywhere in ordinary life (e.g. Da Costa, 1977), or those developing theories taking in account simultaneously syntactic, semantic and pragmatic considerations in natural languages[viii]. In this perspective, the Model Theory developed by Tarski (1936) offers a relevant theoretical framework to deal with the questions of necessity and contingency, and to treat apparent contradictions (Durand-Guerrier, 2006, 2008).

The project of Granger (1998) is « to consider the sense and the role of irrational in some human works, in some major creations of human spirit, and particularly in sciences. »[ix] (Op.cit. p.10). From an author who has devoted his work to description, analysis and promotion of what is rational in human thought, this is not an apology of irrationality, but the testimony of an inscription in « the perspective of an open and dynamic rationality, in order to recognise and delimitate the role of what is positive in irrational. » (Op.cit. p.10). Indeed, Granger considered that « the irrationality, eminently polymorphic, draws in hollows, so saying, the form of rationality (…), and always supposes, at least for analysis, a representation of what it is opposing with." (Op.cit. p.9)

Accordingly, these short insights show that the crucial opposition in number theory between rational and irrational number, articulated by the opposition between coherence and contradiction, is a candidate for our exploration.

III.2 Our research hypotheses

Two main hypotheses are structuring our work. The first one is that the problematic of the articulations between various modes of thought, in particular the relationship between Science and Myth, Rationality and Beliefs, is relevant for the study of anthropological fundamentals of mathematics. The second one is that, through the intermediary of the genesis of mathematical knowledge, we will be able to achieve an anthropological mode of/way of thinking concerning mathematics and its links with the various modes of human thoughts.

III.3 The inscription of Irrationality in our investigation

The term Irrational (in Greek: alogon) has two main significations. First, it means « without a common measure; that cannot be measured as a quotient of two integers ». Second, it means « that is unable to insure the coherence of discourse; illogical ». For Granger (1998) the encounter of irrational numbers in Greece was an example of what he named « the irrational as an obstacle, starting point of the conquest of rationality anew ». This leads to two partly philosophical questions: what does the obstacle really consist of? How can we come to its resolution? Arsac (1987) claims that the encounter with Irrationality is at the origin of the transformation of mathematics in hypothetical deductive system. Of course, it is clear that the confrontation of Irrationality by itself is not sufficient to create anew the conditions of the apparition of the proof, but this invites us to turn toward an interdisciplinary approach to rigor, that we have modestly done in our work. If students of grade 8 or 10 are not a priori able to overcome the epistemological obstacle[14] (indeed, it would be necessary to work along two axes: Euclidean Theory of magnitudes; and a real number construction), our weaker hypothesis is that the confrontation of students with a mathematical or an interdisciplinary work about Irrationality could permit them to approach the question of the nature of this obstacle.

IV. OUR DIDACTIC INVESTIGATION

IV.1. General conditions for a didactical situation in our perspective

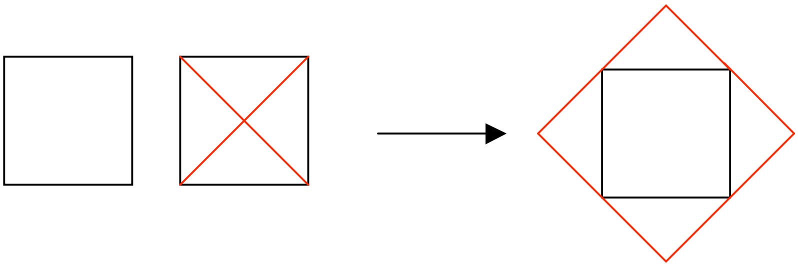
In coherence with our theoretical exploration, we propose conditions[x] that a didactical situation dedicated to the introduction of an historical and anthropological perspective for a given body of knowledge in mathematics in secondary school ought to fulfil.

1. The situation is based on a moment well identified in the genealogy of this knowledge. 2. The situation permits us to question the formidable efficacy of mathematics to act in the real world. 3. The situation fulfils the minimal conditions of a problem situation, in particular favouring framework changes (Douady, 1986)[xi]. 4. The milieu is rich enough to provide retroactions permitting to go forward in the situation and conditions for an intern validation. 5. From the situation, a contradiction between *a priori* beliefs and constraints from reality would emerge. 6. The situation permits us to end up in an institutionalisation of the concept involved in coherence with the curriculum, and of the specific contribution of mathematics to a more general problematic, linked most often to Human and Social Sciences.

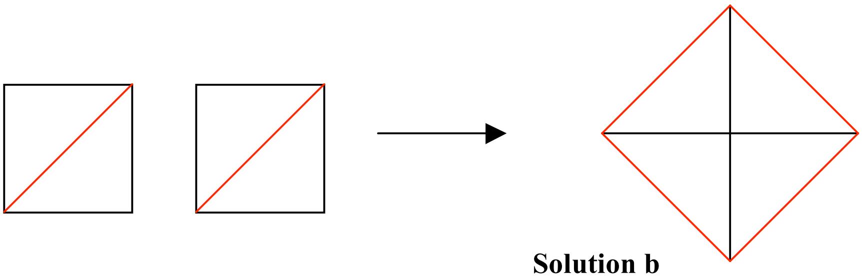
IV.2. Brief description of the experiment in grade 8

This experiment took place in December 2000 and January 2001, in an interdisciplinary project. It comprised four sessions in History course (on the 18th century); five sessions in French (Literature) course, on the theme of rational and irrational; and four sessions of mathematics that we describe below.

- First session: construction of a square from a pair of superposable squares with sides of 10 cm, using a minimal number of cuttings with scissors; elaboration of a proof that the figure is actually a square.



Solution a



Solution b

- Second session: synthesis of the proofs elaborated in the first session; investigation in order to determine the area of the big square.
- Third session: enlightening of the fact that the length of the side of the big square is not a decimal number. Emergence of the following question: is it a rational number?
- Fourth session: elaboration of a proof that $\sqrt{2}$ is not a rational number. Information about the circumstances of this discovery; historical and anthropological aspects; links with what had been done in History and French courses.

In April 2001, an evaluation was made through a role-playing game (Pythagoras' Trial) organised by the three teachers involved in the experiment.

IV.3. Some results of the experiment in grade 8

The interdisciplinary work has permitted us to make the links explicit, although the students did not always perceive them. Concerning mathematics, it is necessary to find a balance between levels of difficulty on the one hand and interest and relevance of the problem on the other hand. This is the case in general for problem situations, but here due to the conceptual ambition it is more acute. Teachers do not wish their students to face difficulties; but the contents, although they do not really exceed the programmes, mobilize cognitive capacities hardly required in the ordinary school mathematical work. However, the effective experiment allows us to reveal that most students appreciated this type of problem and were able to provide rich and relevant arguments.

Students have dealt with the following mathematical notions: area of a square by cutting out shapes; property of areas to be additive; units; recognition of equality of two squares constructed by two different methods; calculations with decimal numbers, and rational numbers; interrogation of the results given by a calculator. Moreover, they have developed argumentation and deductive reasoning in geometry (for example, justify that a figure is a square), and in the numerical field (it is impossible that the square of a decimal / a rational number be equal to 2). Notice that the last proof is that one using the possible digits of the numerator and the denominator, and *reductio ad absurdum* (or infinite descent).

The analyses of the evaluation (Pythagoras' trial) on the one hand, and of three interviews with students on the other hand, give us *a posteriori* information. The development of the trial seems to indicate that students have understood the arguments; have discussed together, but did not have enough time for a right appropriation of the working of a trial. Here are some arguments developed buy students: "If the diagonal of the square is neither an integer, nor a decimal, nor a rational, he (Pythagoras) has not invented it, for this length existed." / "The accusation: it is serious not to reveal this discovery, it is a lost of time -The defence: he will not have been believed. -The accusation: but he had explication! In the end he will be believed; he had a theorem." / "If he revealed the irrational numbers, his whole previous theory would have been wrong. -these numbers are frightening - to say these numbers would have caused the end of the world ; it would have disturbed everything." (this student makes a distinction between ordinary people and scientists). / "When he (Pythagoras) said everything is number, he was not lying because at that time, he did not know about the existence of irrational numbers."

The students interviewed remembered precisely what had been done in the four sessions of mathematics. The link between Irrationality in Mathematics and in French and /or History courses is not done by all of them, but one of them summarized it saying "when we see the superstitions of humans, the sects, it may disrupt the world, and the number too, it may disrupt the world. There is a small link, but it is different."

This project provides an alternative to the aspect of "tools" generally devoted to mathematics. Although this aspect of "tools" is quite relevant, many teachers perceived it as a reduction of what mathematics really is. This project shows that school mathematics can also play its role, beside others disciplines, in the elaboration of elements of human culture, beyond the strictly technical aspects, that an excessive recourse to algorithms tends to reduce it to.

IV.4. Brief description of the experiment in grade 10

The experiment by an experienced teacher, took place in 2002-2003, and in 2006-2007 by a prospective teacher in the context of the professional dissertation in the Teacher Training Institute (IUFM) in Lyon. It comprised of five sessions:

- First session: Introduction of the problem of incommensurability through the following problem: given a square ABCD, is it possible to find a unit measuring both the side and the diagonal of the square; you may use calculator but not the key of square root. Students worked first in small groups; a square of side 12 cm had been provided; the synthesis was collective in the whole class.
- Second session: Working on the link between Incommensurability and GCD (Euclid Algorithm) in the whole class.
- Third session: Proof of the incommensurability of the diagonal and the side of a given square, by *reductio ad absurdum* in the geometric framework.
- Fourth session: Irrationality of $\sqrt{2}$; approximation by rational numbers.
- Fifth session: work on texts and documents; making of posters.

IV.5. Some results of the experiment in grade 10

In grade 10, the teachers considered that the first four sessions were rich for the following reasons. 1. They give a meaning and a legitimacy to proof, as said a teacher. « Indeed, some students have difficulties to understand the necessity of proof. When we propose a proof for a problem for which they know the result, they do not understand why they are proving. Here, a debate rose at the first session. Some were convinced of incommensurability of the side and the diagonal of the square, but others were not. The objective of the proof was to convince, to argue. Let us notice the role of *reductio ad absurdum* in the third session; however it is not involved *a priori* in the numerical field to prove irrationality, but in the geometrical situation that permits to prove this incommensurability; moreover this incommensurability has been studied experimentally in the first session (in a geometrical or numerical field, according with the process used by students), that permits us to pose the problem in a better way »; 2. "They make links between the numerical and geometrical fields. Some notions allowing solving the problem have got signification for students as GCD or Euclid algorithm." / 3. "These sessions have permitted an evolution of the vision that students had of mathematics: « we have discovered the fact that the construction of mathematics did not occur in a linear way but through ruptures ». So the students could change their mind that mathematics "*vont de soi*". As sometimes mathematicians face difficulties to apprehend some notions, students realise that their own difficulties were normal." / 4. "They allow various mathematical notions to be revised: GCD – Euclidean Algorithm – Pythagoras theorem – rational number …" / 5. "All students have been involved in this work (at school as well as for homework), and interested whatever their level."

CONCLUSION

We consider we have given some evidence (in an *existential* sense) that it is possible in grade 8 and 10 in France to do interdisciplinary work, structured around a mathematical notion, for which a study, even of the partial historical genesis allows us to show the anthropological dimension in the sense we have defined above. Irrationality appears as a paradigmatic theme, or even an *ad hoc* theme, of what we aim to develop. That other notions could permit such a work remains for us an open question, but it seems to us that it would be possible to find candidates towards themes common to mathematicians and philosophers, sociologists, historians, without forgetting artists; themes like propositions; infinity; emptiness; space-time; paradox; truth; necessity; transcendence….

We are aware that more work has to be done, particularly in defining relevant characteristics of the didactical situations in order to reach our learning objectives on the one hand, in identifying potential institutional "niches" depending on the curriculum on the other hand.

Another question concerns the way to elaborate and share with teachers situations aiming to integrate the historical genesis and the anthropological perspective for a given theme.

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Annex

Didactical conditions	**Implementation in 4ème (Grade 8)**	**Implementation in 2nd (Grade 10)**
Genealogy	The problem of the duplication of a square; Pythagoras' discovery of √2	Incommensurability of the diagonal of a square with its side; Euclid's algorithm about GCD.
Efficiency	Effective production of a square the area of which being 2 from a square with an area of 1	
Frames	Numerical/Geometrical	Arithmetical/geometrical Geometrical/numerical
"Milieu"	Effective realisations; success checking	Construction of a decreasing series of squares
Contradictions	"to double the area, you must double the side"	We know how to fix a measure to any measure of length
Institutionalisation	The length of the side of the square of area 2 in not a decimal number	Incommensurability of the diagonal of a square with its side; Irrationality of the number square of 2
Connections with human sciences	History, Philosophy (Menon' s dialog) Arts,	Respective positions and roles of rational and irrational numbers; questioning about the meaning of "to exist"

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[iii] Our translation
[iv] Our translation
[v] Quine (1960) Word and object
[vi] That means our co speaker
[vii] Durand-Guerrier (1996) pp. 276-280
[viii] The use of such a perspective in primary and lower secondary education can be found in Durand-Guerrier & al (2006)
[ix] Our translation
[x] You can see a table about them in the annex
[xixi] Framework changes refer to *Jeux de cadres*: framework is here to be taken in its usual meaning (algebraic, arithmetical, geometrical…); such changes are supposed to favour research process in problem solving and evolution of students' conceptions.