HISTORY, HERITAGE, AND THE UK MATHEMATICS CLASSROOM

Leo Rogers, Oxford University

Abstract

Since 1989 the UK mathematics curriculum has been dominated by a culture of testing 'core skills'. From September 2008, a new curriculum places the history of mathematics as one of its "Key Concepts' which is now a <u>*statutory right*</u>[1] *for all pupils. While the curriculum has changed, there has been virtually no relevant training for teachers, and while the testing regime remains in place, there seems little chance that pupils will obtain their entitlement. This paper examines the problem of teachers'scant knowledge of history of mathematics and proposes a new approach to introducing relevant materials together with a pedagogy which capitalises on recent research, to introduce the heritage of mathematics into our curriculum.*

1. THE NEW ENGLISH CURRICULUM

The first chapter of Fauvel and van Maanen (2000) considered the political context of the history of mathematics in school curricula. At that time, the UK curriculum[2] underwent radical changes, which produced a curriculum based on 'core skills' with modularised[3] lessons that enshrined traditional beliefs about 'levels' of knowledge and portrayed school mathematics as a collection of disparate topics rarely connected in any sensible way. Textbook design followed the topics, and test papers became *de facto* part of the curriculum, setting the norms for the new culture. The emphasis on utilitarianism and examination results produced little serious engagement with substantial mathematical thinking[4]. The latest Inspectors' report on our secondary schools shows that, as a consequence, too many pupils are taught formulas that they do not understand, and cannot apply:

"The fundamental issue for teachers is how better to develop pupils' mathematical understanding. Too often, pupils are expected to remember methods, rules and facts without grasping the underpinning concepts, making connections with earlier learning and other topics, and making sense of the mathematics so that they can use it independently." (Ofsted 2008: 5)

In contrast, the most recent version of the curriculum states that for the 11 to 16 age group, "Recognising the rich historical and cultural roots of mathematics" is one of its "Key Concepts" (QCA 2007)[5].

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[1] A 'statutory right' means that *by Law*, *all pupils at primary and secondary level* have the right to be taught about the "rich historical and cultural roots of mathematics".

[2] The UK mathematics curriculum applies to England and Wales. Due to government devolution Scotland and Northern Ireland have different curricula, regulations and examination systems.

[3] Modules purport to be convenient 'packages of knowledge' within the curriculum, with a well defined and limited range of knowledge. They are consequently easy to 'teach' and easy to pass.

[4] There are, of course, a number of exceptional teachers who have overcome these difficulties.

[5] *The Key Concepts* are: Competence, Creativity, Applications and Implications, Critical Understanding, and the *Key Processes* are: Representing, Analysing, Interpreting and Evaluating, Communicating and Reflecting. Applied to all pupils from age 11 to 16 (Key Stage 3 to Key Stage 4).

For the last fifteen years very few secondary school teachers have had the chance to discover the kind of contributions that history of mathematics could make to pupils' learning, and with the pressures of 'teaching to the test' it seems doubtful whether history of mathematics will make any impression in our classrooms while the examination structure remains the same[6]. So, what would 'recognising the rich historical and cultural roots of mathematics' mean in practical terms for our teachers?

Recently, colleagues have renewed their call for history of mathematics to be taken seriously as an essential part of the mathematics curriculum. Radford et. al. (2007) argue that an important sense of meaning lies within the cultural-epistemic conception of the history of mathematics:

"The very possibility of learning rests on our capability of immersing ourselves –in idiosyncratic, critical and reflective ways– in the conceptual historical riches deposited in, and continuously modified by, social practices. … Classroom emergent knowledge is rather something encompassed by the Gadamerian link between past and present. *And it is precisely here, in the unravelling and understanding of this link, which is the topos or place of Meaning, that the history of mathematics has much to offer to mathematics education*." (2007: 108) (italics mine)

In the terms described above, history stands in opposition to the utilitarian demands of the old curriculum, but having put history of mathematics into the curriculum, the government organization, QCA[7] have now revealed the pressing problems of resources and training. Changes need to happen not only in the classroom but also, and more importantly, in teacher training. So, *how can we provide material from the history of mathematics that can be integrated in a meaningful and effective way into the everyday activities of the classroom?*

2. NOT HISTORY BUT HERITAGE

Ivor Grattan-Guiness (2004) has made an important distinction between the History and the *Heritage* of mathematics. History focuses on the detail, cultural context, negative influences, anomalies, and so on, in order to provide evidence, so far as we are able to tell, of what happened and how it happened. *Heritage,* on the other hand, address the question "how did we get here?" where previous ideas are seen in terms of contemporary explanations and similarities are sought.

[6] Recently, the government has decided to abandon the tests at KS3 (age 14), and plans have been published to include 'Interpretation and Analysis' of problems as part of the assessments from 2010.

[7] The Qualifications and Curriculum Authority, the Government sponsored body set up to maintain and develop the national curriculum and associated assessments, tests and examinations.

"The distinction between the history and the heritage of [an idea] clearly involves its relation to its prehistory and its posthistory. The historian may well try to spot the historical *foresight* - or maybe lack of foresight - of his historical figures, …. By contrast, the inheritor may seek historical *perspective* and hindsight about the ways notions actually seemed to have developed." and "…heritage suggests that the foundations of a mathematical theory are laid down as the platform upon which it is built, whereas history shows foundations are dug down, and not necessarily into firm territory." (2004:168; 171)

The interpretation of Euclid's work as 'geometrical algebra' has since shown to be quite misguided[8] as *history,* but as *heritage* is quite legitimate because it is the form in which some of the Arabs interpreted the Elements when they were creating algebra.

We have to be careful. Deterministically constructed heritage conveys the impression that the progress of ideas shows mathematics simply as a cumulative discipline. But, while mathematics does build on past achievements, and while we make stories about the links between the mathematics of the past to the present, *the mathematics of the past is not the same as the mathematics of now*. As Mathematics Educators we have a means of passing on our Heritage by bringing the links between the content we find in the curriculum to the attention of teachers and students. In this way it becomes possible to describe significant ideas in the history of mathematics in terms that teachers can use and pupils can understand without making impossible demands on their historical capability or on curriculum teaching time.

3. PROJECT AIMS AND OUTLINE

The project I describe is just beginning. It has arisen from the experiences of myself and other colleagues in presenting 'episodes' from the history of mathematics in workshop form to both teachers and pupils, so that interesting and worthwhile problems arise from interpretations of the historical context. Response to these classes has been encouraging, and has prompted wider experimentation. Some twenty teachers and teacher trainers around the country have joined an informal on-line workshop to experiment with the available materials and suggest ideas for the classroom presentation of current material and new topics to be explored. Initially, the principal interest is in providing Secondary teachers with materials for professional development that start from some of the important ideas in the existing curriculum, and to open up the possibilities

[8] Typically, this is done with Euclid II,4 and described as 'completing the square', but see the examples in Katz (2008)

of developing the concepts involved by finding ‘historical antecedents’ to support the connections between and motivations for these ideas and the possible links between them. Exactly what form this material may take is still under consideration[9]. Some examples can be found on the NRICH website[10] where themed historical ‘episodes’ are available with notes and pedagogical questions for teachers and pupils to explore.

While the web episodes are currently chronologically arranged, a more general idea is to produce a series of ‘concept maps’ that are intended to provide a topographical view of the significant features of a particular mathematical landscape[11](Burke & Papadimitriou 2002). A map can be examined and used from ‘inside-out’ and from ‘outside-in’, from following particular trails of thought to obtaining a broader overview of historical development. The ‘unravelling and understanding’ of the links between ideas, is the *topos* that Radford and our colleagues (quoted above) are talking about. The idea of a map is important here; it is intended to be a guide to how ideas might be connected, not a deterministically constructed list of events. In contrast, most curriculum activities are presented to teachers as a *narrative*, a list of topics to teach in a particular order, and often restricted to some imagined ‘levels of competence’ of the pupils. A map is there for teachers to have the freedom to make their own narrative. They have the responsibility for producing lessons, and it is up to them what parts of the map they want to use, and how they approach the pedagogical problems of dealing with the curriculum in their own classroom. The map can throw light on certain problems, it can suggest different approaches to teaching, it can help to generate didactical questions, but in the end it is there to be used or not, appropriately. The intention here is to develop ways in which the teacher, starting from a particular point in the standard curriculum, will be able to link a conceptual area with important developments in the history of mathematics through the use of ‘idealised’ historical problems and *canonical situations*[12] as part of the *Heritage* of mathematics. There is, of course, a considerable literature of historical and pedagogical material to draw on. The practical task is to find appropriate ways in which to link the source material with the curriculum opportunities.

[9] Today, many options present themselves: texts, posters, PowerPoint, DVD are all possibilities.

[10] The NRICH site is part of the UK Millennium Mathematics Project.

[11] I am indebted to my colleague Jeremy Burke for the use of this idea in his research, and for our conversation on 15 November 2008. I make no claims that such a map is (or even could be) ‘complete’.

[12] By a *canonical situation* I mean a diagram, or a way of setting out a problem or process which is developable, has potential to represent more than one idea, and is presented to students to encourage potential links between apparently different areas of mathematics. See the Appendix for an example.

4. METHODOLOGY AND PEDAGOGICAL APPROACH.

Since the English curriculum now focuses more on what we call the 'process' aspects of learning mathematics, it may now become easier to incorporate the teaching of the 'key concepts' in such a way as to enable the history to emerge from the discussion of *canonical situations* (be they images, texts, or conceptual problems) introduced by the teacher. This approach also has the advantage of being able to link different areas of a standard curriculum, thereby enabling pupils to see connections between parts of mathematics that have been concealed by the traditions of official curriculum organisation. When the text-books and exercises are arranged so that their chapter headings conform to the same organisation as the curriculum, it is most unlikely that pupils will gain any idea that different areas of mathematics are connected at all. In this pedagogical strategy we are concerned with the *dynamics of production of the pupils' ideas* stimulated by episodes from the history of mathematics retold in heritage form. In principle, this is not new. I am advocating a methodology that is already available, which can bring mathematics education and the teaching of history of mathematics together. The principles are well-established, and the use of examples as a focus for discussion and exploration has been a tradition in teaching for many years. However, as Sierpinska (1994) has recognised:

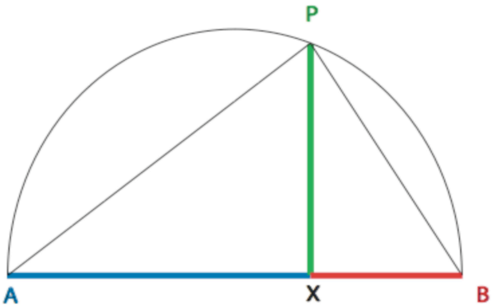
> "Pedagogues, of course, think of paradigmatic examples …. of instances that can best explain a rule, or a method, or a concept. The learner is also looking for such paradigmatic examples as he or she is learning something new. The problem is, however, that before you have a grasp of a whole domain of knowledge you are learning, you are unable to tell a paradigmatic example from a non-paradigmatic one." (1994; 88-89)

This problem is always present in the classroom, but there are many ways in which we try to alleviate the situation. Grosholz (2005) has demonstrated the role of 'constructive ambiguity' in Galileo's discussion of free fall,[13] and shows that ambiguity can play a constructive part in mathematics since it leads in this case to reading a particular diagram in more than one way. Galileo's argument was put forward in terms of proportions, geometrical figures, numbers, and natural language. He was then able to exploit Euclidean results and the arithmetical pattern of the diagram, but in reading the intervals as infinitesimals he led the participants heuristically to his analysis of accelerated motion. The use of ambiguity in mathematical heuristic is still alive today. Changing the

[13] Galileo (1638) Discorsi e Dimostrazioni Matematiche Day 3, Theorem 1, Proposition 1 (Dover edition p.173).

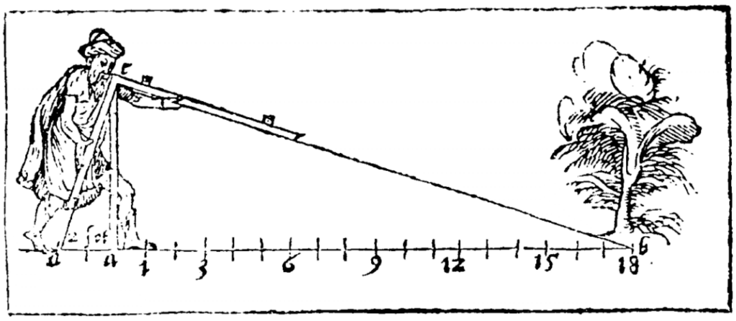
mathematical context by conceptualising new objects and the processes we use to deal with them, changes the ways in which arguments can be understood. This kind of ambiguity has been shown to provide useful material for classroom discussion.

The use of *canonical situations* is important in this context. A diagram can be interpreted in a number of ways, and this is where conceptualising new objects and new relationships becomes possible.



This diagram from Euclid VI, 13 is an example of a *canonical situation*. It describes a mean proportional between AX and XB and is fundamental to ideas of ratio and proportion. Pupils who know the properties of similar triangles can deduce that $AX.XB = PX^2$.

Starting from the properties of right-angled triangles, elementary knowledge of ratio and proportion and its early practical applications to measurement of all kinds of heights and distances can be developed.



Using dynamic geometry, it is easy to show how the product of the segments produces a square, and thus we have entry to the diagrams used by Viete and Descartes for demonstrating their quadratic solutions.

In reading texts, Barbin (2008), has shown how text considered as a message to an audience can motivate a discussion about the intention and meaning of the author, and how it can be used as a means of encouraging pupils to consider the ways it could be interpreted and understood.

There is no sure way of posing problems or offering examples, but once done, then the learner's response has to be respected and managed carefully. We have become used to the principles of heuristic teaching, but Brent Davis claims that heuristic listening is also important:

"*Heurisitic Listening* …… is more negotiatory, engaging, messy, involving the hearer and the heard in a shared project [which] is an imaginative participation in the formation and transformation of experience through an ongoing interpretation of the taken-for-granted and the prejudices that frame perceptions and actions." (Davis, 1996: 53)

When we engage in mathematical problems we inevitably construct our own examples to help us illustrate the ideas involved, and use these examples as material for personal contemplation or discussion amongst our peers. If we do this as adult mathematicians, why should it be different for pupils? Why is it not possible to develop this idea of self-construction in the classroom?

In England, there has been a tradition of producing materials for teachers and pupils that focuses on an individual's learning process and encourages active engagement in, and discussion of mathematical problems[14]. Recent examples like Watson and Mason (1998) and Swan (2006) encapsulate this tradition and provide practical guidance to help teachers develop pupils' powers of constructing mathematics for themselves in the classroom:

"Our interest is in using mathematical questions as prompts and devices for promoting students in thinking mathematically, and thus becoming better at learning and doing mathematics. … We hope our work will show how *higher order mathematical thinking* can be provoked and promoted as an integral part of teaching and learning school mathematics…" (Watson & Mason 1998: 4)

Such publications display ideas for situations that are generic and offer ways for teachers of promoting 'Learner Generated Examples' *applicable at all stages* of teaching and learning mathematics. The materials are prepared to promote the kinds of activities that focus on ambiguity, raise doubts about interpretations, and encourage the learner (and the teacher) to develop a security with mathematical ideas that enables them to engage in intelligent questioning and active discussion of the problems concerned. A number of teachers are already engaged in this pedagogy that raises pupils' learning above mere acquisition of skills, and helps pupils to develop their own cognitive tools and achieve a higher order of mathematical activity.

5. THE MATERIALS: BRIEF DESCRIPTION AND EXAMPLES.

[14] This kind of material was introduced by the Association of Teachers of Mathematics, and has been its enduring hallmark. It is the result of a tradition of collaborative research and writing where texts and other materials have developed a particular type of pedagogical practice by offering examples of classroom work which require discussion, involve heuristic forms of reasoning, analogy and inference, and encourage the learner to create and verify their own examples.

Completing the Square is one of the drafts that has been used in a number of classrooms[15] and covers is a traditional area of the curriculum showing some of the connections between the stages to the solution of quadratic equations. It comprises a series of links from one period to another, stressing the transformation of the ideas from simple surveying to 'cut and paste' problems in Mesopotamia, and more sophisticated procedures of 'dissection and re-arrangement' in India and China, and how the problems were transposed and represented within the more abstract ideals of classical geometry in Greece. The conceptual blending of different traditions by the Arabs in the 9th and 10th century introduced algebraic concepts which found their way into Europe and resulted in the attempts to find solutions of different types of equations. The materials provide plenty of opportunities to discuss the development of geometrical and number concepts and the way these were represented in text and diagram form (ratios, proportions, integers, fractions, rationals non-rationals and eventually 'imaginary' numbers). Key ideas like geometric visualisation and the different forms of representation, appropriate notation, and whether a particular procedure is 'allowed' in a given context, can be discussed, and show how finding representations for 'impossible' numbers like $\sqrt{3}$ or π can have a liberating effect in allowing new ideas to flourish. And, of course, there is the ever-present idea of 'infinity' to be explored. The material has been gathered from published research and expert analysis[16] to identify and characterise significant moments in the evolution of particular ideas. In these examples we have not only translations into 'modern' language, but something of the pedagogical interpretations, so that these might be brought into the modern classroom and used in creative ways. The material is designed so that it can be used in 'episodes' in the normal course of teaching in school. Included are notes and references to the historical background, and 'pedagogical notes' aimed to help teachers raise questions and see where the material can be used in their classroom. In this way, selections can also be used as a basis for teachers' professional development both in the historical and mathematical sense.

There are optional entry (and exit) points to the material that allow considerable flexibility in its use. These 'episodes' apply to particular topics (or lack of them) in the English mathematics curriculum, and are

[15] This material has been used in whole or in part, with various groups of pupils from age 10 to 18, with teachers, teacher trainers, and with graduate teacher trainees. I gratefully acknowledge their feedback, which has been most useful.

[16] For example, over the years I have been able to access the specialised work of many researchers on Ancient, Classical, Mediaeval and Renaissance mathematics. Now we can find substantial examples of much of the ancient mathematical material collected and specially written up in Katz (2007).

each recognised as an interpretation of a particular context in our heritage. The historical process can be described *in terms communicable to a modern school audience* and furthermore, the teaching is specifically designed to focus on the pupils' *contemplation and discussion of the problems*, and engagement in a dialogue with the material. Using the pedagogy described above, we have a real chance to recognise "the rich historical and cultural roots of mathematics" in our classrooms.

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UK Web Based Documents

a) The NRICH site is at http://nrich.maths.org/public/

b) Pedagogical sources:

Changes in Mathematics Teaching Project: http://www.cmtp.co.uk/

Deep Progress in Mathematics

http://atm.org.uk/reviews/books/deepprogressinmathematics.html

c) Government documents:

QCA Website www.qca.org Maths_KS3_PoS.pdf and QCA Maths_KS4_PoS.pdf

http://curriculum.qca.org.uk/key-stages-3-and-4/subjects/mathematics/index.aspx

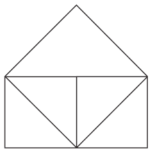
Ofsted UK (September 2008) Mathematics: *Understanding the Score*

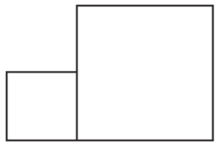
Download review of mathematics led by Sir Peter Williams published June 2008

http://www.ofsted.gov.uk/Ofsted-home/Publications-and-research/Documents-by-type/Thematic-reports/Mathematics-understanding-the-score

COMPLETING THE SQUARE (Some Samples)

1. Indian Area Methods.


(a)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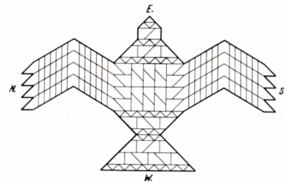
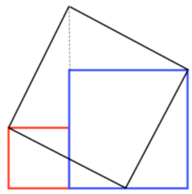

(c)

These diagrams and are inspired by practical Altar Building rules from the Sulbasutras, (15^{th} - 5^{th} Centuries BCE), (c) is the 'Kite Altar' still used in Kerala.

Challenge: It is easy to see how the combined areas of two equal squares can be found (a); with only a rope for measuring and drawing arcs, what about the combined area of (b)? Allow time for experiment and discussion of pupils' procedures. Ask pupils if they can find any more solutions. Does it work for any size of squares?

Cannonical Activity: Use square dot-lattice paper to draw squares with a dot at each corner and no dots on the edge. Find areas using the smallest square as the unit. Discuss methods of dissecting the squares to find equivalent areas and how these may be combined. Display diagram (d) and discuss 'transformation of areas'.


(d)

Explore the visual dynamic of diagram with software; extend to rectangles and other shapes; identify basic properties and justify procedures.
Link with ideas from Mesopotamian mathematics and Euclid Book II.

2. The 'Babylonian Algorithm'.

A number game: "I am thinking of two numbers, their sum is 7 and their product 12, what are the numbers?" Extend with increasing pairs of sum and product numbers, encourage pupils to discover the original numbers. Pupils to challenge each other, share results, and find a way of writing instructions or developing a notation.

Introduce a standard algorithm: 'Take half of 7, square it, subtract 12 from this square and find the square root of the result, then add and subtract this square root from half of 7.' Use this to test other pairs. If it works for integers, try it with simple fractions. This algorithm originates in Mesopotamia and variations of it are found in Al-Khowarizmi, Fibonnaci, Cardano and others.

Extensions what happens when the pairs are 7, 11 and 7,13? These simple variations give non-rational ($\sqrt{5}$) and complex results ($\sqrt{-3}$) respectively.

Note 1: I see no problem in introducing quite young pupils to ideas like this. The process of 'following the algorithm' with simple numbers allows pupils to arrive at results which mirror in the discovery of these 'impossible' numbers.

Note 2: In this context, we also have the opportunity of introducing an iterative solution method for finding square roots, linked to the famous Old Babylonian tablet YBC7289. Discussion about the number that when multiplied by itself can produce 2, can lead to pupils' experimenting and developing their own methods of 'trial and error'. This is also one of the important opportunities to contemplate how we can manage and understand an infinite process.

Note 3: Finding a suitable notation is an important part of mathematical history and communication. In most cases in school mathematics notation is given unmotivated to pupils. Situations where pupils are challenged to communicate ideas to their peers through such examples provide opportunities for exploiting historical analogy.