USING HISTORY AS A MEANS FOR THE LEARNING OF MATHEMATICS WITHOUT LOSING SIGHT OF HISTORY: THE CASE OF DIFFERENTIAL EQUATIONS

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The paper discusses how and in what sense history and original sources can be used as a means for the learning of mathematics without distorting or trivializing history. It will be argued that this can be pursued by adopting a multiple-perspective approach to the history of the practice of mathematics within a competency based mathematics education. To provide some empirical evidence, a student project work on physics' influence on the development of differential equations will be analysed for its potential learning outcomes with respect to developing students' historical insights and mathematical competence.

INTRODUCTION

Fried (2001) argues that when history is used to teach mathematics the teacher must

> either (1) remain true to one's commitment to modern mathematics and modern techniques and risk being Whiggish, […] or, at best, trivializing history, or (2) take a genuinely historical approach to the history of mathematics and risk spending time on things irrelevant to the mathematics one *has to* teach. (Fried, 2001, p. 398).

Whig history refers to a reading of the past in which one tries to find the present.

The purpose of the present paper is to argue that this dilemma can be resolved by adopting (1) a competency based view of mathematics education, and (2) a multiple-perspective approach to the history of the practice of mathematics. Hereby, a genuinely historical approach to the history of mathematics can be taken, in which the study of original sources is also relevant to the mathematics one *has* to teach. To present some empirical evidence for this claim a student directed project work on the influence of physics on the development of differential equations will be analysed. The project belongs to a cohort of mathematics projects made over the past 30 years by students at Roskilde University, Denmark. Only one project is analysed in the present paper, but the reflections and discussions brought forward are based on knowledge about and experiences from supervising many of those projects.

First, mathematical competence and the role of history in a competency based mathematics education are presented. Second, a multiple-perspective approach to a history of the practice of mathematics will be introduced. Third, the chosen project work will be analysed and discussed with respect to specific potentials for the learning of differential equations within the proposed methodology. Finally, the paper ends with some conclusions and critical remarks.

MATHEMATICAL COMPETENCE AND THE ROLE OF HISTORY

In the Danish KOM-project (2000-2002) mathematics education is described in terms of mathematical competence. In this context mathematical competence means the ability to act appropriately in response to mathematical challenges of given situations. It can be spanned by eight main competencies (Niss, 2004). Half of them involves asking and answering questions in and with mathematics: (1) to master modes of *mathematical thinking*; to be able to formulate and solve problems in and with mathematics, i.e. (2) *problem solving* and (3) *modelling competency*, resp.; (4) to be able to *reason* mathematically. The other half concerns language and tools in mathematics: (5) to be able to handle different *representations* of mathematical entities; (6) to be able to handle *symbols and formalism* in mathematics; (7) to be able to *communicate* in, with, and about mathematics; (8) to be able to handle *tools and aids* of mathematics. In the discussion below, the possible learning outcomes of reading sources will be analysed with respect to these competencies.

History of mathematics is not one of the main competencies, but is included in the KOM-project as one of three kinds of *overview and judgement* regarding mathematics as a discipline. The first concerns actual applications of mathematics in other areas, the second, historical development of mathematics in culture and societies, and the third, the nature of mathematics as a discipline (Niss, 2004).

The KOM-understanding of the role of history in mathematics education has the honesty to history as an intrinsic part. In Danish secondary school this understanding of history is included in the curriculum (Jankvist, forthcoming). The objective of the present paper is to discuss in what sense such an understanding of history can be implemented in situations where the curriculum does not include history and does not assign time to teach history. Under such circumstances, history of mathematics is most likely going to play no role at all in the learning and teaching of mathematics unless it can also be used as a means to learn and teach subjects in the syllabus.

A MULITPLE PERSPECTIVE APPROACH TO HISTORY OF MATH

How can we understand and investigate mathematics as a historical product? One way is to think of mathematics as a human activity and of mathematical knowledge as created by mathematicians. This has been the foundation for many recent studies in the history of the practice of mathematics (Epple, 2000), (Kjeldsen et al., 2004).

To study the history of the practice of mathematics involves asking why mathematicians situated in a certain society, and/or intellectual context at a particular time, decided to introduce specific definitions and concepts, to study the problems they did, in the way they did it. In this line of thinking, mathematics is viewed as a cultural and social phenomenon, despite its universal character. Studying the history of mathematics then also involves searching for explanations for historical processes of change, such as changes in our perception of mathematics, our understanding of mathematical notions, and our idea of what counts as a valid argument.

A way of answering such questions is to adopt a multiple perspective approach (Jensen, 2003) to history where episodes of mathematical activities are analysed from multiple points of observations (Kjeldsen, forthcoming). The perspectives can be of different kinds and the mathematics can be looked upon from different angles, such as sub-disciplines, techniques of proofs, applications, philosophical positions, other scientific disciplines, institutions, personal networks, beliefs, and so forth.

How can this approach be brought into play to ensure the honesty to history, in a teaching situation where the teacher wants to use history as a means for students to learn a specific mathematical topic or concept? It can be implemented on a small scale, by having students read pieces of original mathematical texts focusing on perspectives that address research approaches or the nature and function of specific mathematical entities (problems, concepts, methods, arguments), in order to uncover, discuss, and reflect upon the differences between how these approaches and entities are presented in their text book and the former way of conceiving and using them. In such teaching settings, the students have to read the mathematical content of the original text as historians, using the "tools" of historians, and answering historians' questions about the mathematics. For such tools, see e.g. (Kjeldsen, 2009).

Through activities where students work with historical texts guided by historical questions, connections between the students' historical experiences of the involved mathematics and their experiences from having been taught the text book's version, can be created in the learning process. When students read historical texts from the perspectives of the nature and function of specific mathematical entities, they can be challenged to use other aspects of their mathematical conceptions in new situations. So, it is of didactical interest to analyse historical episodes of mathematical research with respect to their potential to challenge students' mathematical conceptions.

A HISTORY PROJECT: PHYSICS AND DIFFERENTIAL EQUATIONS

In the following, the student directed project work will be analysed with respect to how and in what sense the students' work with original sources provided potentials for the learning of differential equations – without losing sight of history.

The educational context: problem oriented student directed project work

The project report on physics influence on the development of differential equations was written by five students enrolled in the mathematics programme at Roskilde University (RUC). All programmes at RUC are organised such that in each semester the students spent half of their time working in groups on a problem oriented, student directed project supervised by a professor. The projects are not described by a traditional curriculum, but are constrained by a theme (Blomhøj & Kjeldsen, 2009).

The requirement for this project was that the students should work with a problem that deals with the nature of mathematics and its "architecture" as a scientific subject such as its concepts, methods, theories, foundation etc., in such a way that the status of mathematics, its historical development, or its place in society gets illuminated.

Among the cohort of project reports, constrained by these objectives, this particular project was chosen, because the students happened to investigate differential equations, which are included in the core curriculum of advanced high school mathematics and mathematics and science studies in universities. Hence, the project work could be analyzed with respect to the issues addressed in the present paper.

Analysis of the project work: learning outcomes and the competencies

The students formulated the following problems for their project:

> How did physics influence the development of differential equations? Was it as problem generator? Did physics play a role in the formulation of the equations? Did physics play a role in the way the equations were solved? (Paraphrased from (Nielsen et. al., 2005, p.8)).

On the one hand, these are fully legitimate research questions within history of mathematics. They address issues about an episode in the history of mathematics seen from the perspective of how another scientific discipline influenced mathematicians' formulation of problems as well as the methods they used to solve the problems. On the other hand, these questions can only be answered by analysing the details of original sources that deal with this particular episode in the history of mathematics, studying how the differential equations were derived from the problems under investigation, how the equations were formulated, why they were formulated in that particular way, how they were solved and with which methods – issues which are also relevant for the learning and understanding of the subject of differential equations. Based on readings of three original sources from the 1690s, the students discussed these issues within the broader social and cultural context of the involved mathematicians, critically evaluating their own conclusions within the standards for research in history of mathematics. Hence, in this way of working with history in mathematics education history is neither Whiggish nor trivialized.

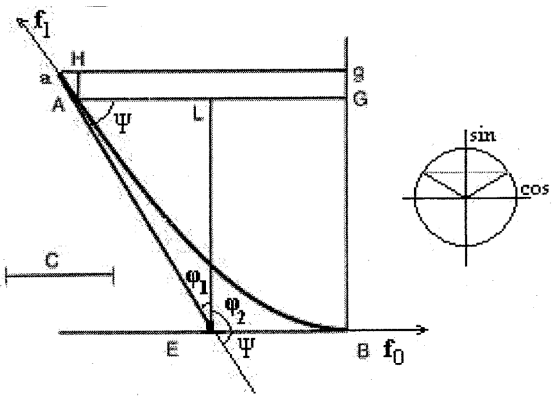
I will discuss three instances where the students – qua the historical work – were forced into discussions in which they came to reflect on issues that enhanced their understanding of certain aspects of differential equations in particular and of mathematics in general. The discussion will end with a short presentation of some of the learning outcomes with regard to the eight main mathematical competencies.

1: Johann's differential equation of the catenary problem. The catenary problem is to describe the curve formed by a flexible chain hanging freely between two points. The students read the solution that Johann Bernoulli presented in his lectures on integral calculus to the Marquis de l'Hôpital, supported by English translations of extracts (Bos, 1975). Bernoulli formulated five hypotheses about the physical system that, as he claimed, follow easily from static. For the students, of which none studied physics, to derive these assumptions was the first mathematical challenge in reading Bernoulli's text: "we had to derive most of them ourselves. We use 18 pages to explain what Johann Bernoulli stated on a single page" (Nielsen et. al., 2005, 19).

Below is one of the extract of Bernoulli's text (Bos, 1975, 36) that the students read. As can be seen from the text, Bernoulli used the five hypotheses to describe the catenary and the infinitesimals *dx* and *dy* of the curve geometrically and derived an equation between the differentials. The figure was produced by the students and is similar to a figure in Bernoulli's lecture, except from the sine-cosine circle.

Assuming these results, we go on to find the common Catenary as follows: Let *BAa* be the required curve, with its lowest point at *B*; its axis, the vertical line through *B*, is *BG*. The tangent to the curve at its lowest point is the line *BE*, which will be horizontal. Let the tangent at any other point *A* be *AE*. We draw the ordinate *AG* and a line *EL* parallel to the axis. Let $BG = x$, $GA = y$, $Gg = \mathrm{d}x$ and $Ha = \mathrm{d}y$.

Because the weight of the chain is distributed evenly along its length we may put that weight equal to the length of the curve *BA* which we call *s*. Thus since an equal and constant force will always be required at the point *B* (by hypothesis (iv)) whether the chain *BA* is made longer or shorter, let this force be of magnitude *a*, expressed by the straight line *C*. Let us now imagine the weight of the chain *AB* to be concentrated and suspended at the point *E* where the tangents *AE* and *EB* meet. Then (by hypothesis (ii)) the same force is required at *B* to support the weight *E* as was previously required to support the chain *BA*. Indeed (by hypothesis (v)) the ratio of the weight *E* to the force at *B* is the same as the ratio of the sine of the angle *AEB*, or of its complement angle *EAL*, to the sine of the angle *AEL*, that is the ratio of *EL* to *AL*. Therefore, whatever position on the curve is taken for the fixed point *A* (the curve, by hypothesis (iii), being always the same) the ratio of the weight of the chain *AB* to the force at *B* is the same as the ratio of *EL* to *AL*, that is $s:a = EL:AL = AH:Ha = \mathrm{d}x:\mathrm{d}y$, and inversely $\mathrm{d}y:\mathrm{d}x = a:s$.[1]



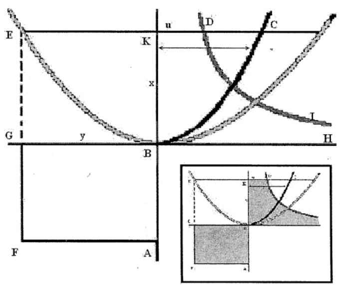
In their report, the students went through Bernoulli's text and filled in all the arguments. They were not familiar with this way of setting up differential equations from scratch so to speak, so the mathematization of the physical system was a major challenge for which they needed to consult some textbooks on static and to combine the physics with mathematical results about triangles and the sine-cosine relations.

Bernoulli's arguments do not meet modern standards of rigour and that created cognitive hurdles for the students. Didactical, it is important to identify such hurdles because they create situations where the students, during their struggle with understanding the mathematical content of the original text, can be challenged to reflect upon the differences between our modern understanding and the one presented in the source, thereby enhancing their own understanding of the concept of, in this case, differential equations and the mathematical techniques and concepts underneath. A concrete example of this is Bernoulli's use of the infinitesimal triangle. In the text above he used similar triangles, to argue that $s{:}a = dx{:}dy$ but, as the students pointed out in their report, *a* does not lie on the tangent but on the catenary. Bernoulli also used the infinitesimal triangle later in the lecture, when he reformulated the differential equation derived above, using that $ds = \sqrt{dx^2 + dy^2}$. Again – as pointed out by the students – *ds* is a part of the catenary, not the hypotenuse of a right angled triangle.

This mixed use of geometrical arguments and infinitesimals in deriving and reformulating the differential equation was very different from the students' text book experiences of differential equations. The fact that Bernoulli's method worked in this particular case, despite its lack of rigour, provoked a discussion among the students and their supervisor (the author) about Bernoulli's use of the infinitesimal triangle and his use of the infinitesimals, *dx* and *dy*, as actual infinitely small quantities. This made the students focus more systematically on the differences between now and then, questioning, at first, why we need to define a differential quotient as the limit (in case it exists) of difference quotients, then analysing the situation again to understand why Bernoulli's method worked fine for the catenary, and trying to picture situations where it would go wrong. This is an incidence where connections were created between the students' historical experiences and their experiences from modern mathematics which challenged them to examine their own understanding of the involved concepts. Through these discussions, the students built up intuition about infinitesimals and awareness about the reasons behind the construction of our modern concepts. Major differences were the lack, in the seventeenth century, of the concept of a function, of a limit, and the formalised concept of continuity. In this project work the historical texts provided a framework for discussions among the students and with their supervising professor, about what constitute the concept of a differential equation, and how we can read meaning into it. Through these discussions, which were triggered by the historical texts, the students came to reflect upon the concept of a differential quotient and the meaning of a differential equation on a structural level that went beyond mere calculations and operational understanding of the concepts. This is an example of what Jahnke et. al (2000) calls a reorientation effect of studying original sources.

2: Johann's solution of the catenary differential equation. Through some further manipulations Bernoulli reached the following formulation of the equation for the catenary $dy = adx/\sqrt{x^2 + 2ax}$ which he used to construct the curve geometrically. This puzzled the students and initiated discussions about, what it means to be a solution to a differential equation.

Let the normals *AK* and *GH* be drawn, meeting in *B*. Take $BA = a$ and describe an equilateral hyperbola *BC* with vertex *B* and centre *A*. Now construct a curve *DI* with the property that everywhere *BA* is the middle proportional between *KC* and *KD*, that is such that $KD = aa{:}\sqrt{(2ax + xx)}$. Now draw a parallel *AF* and take the rectangle *AG* equal to the area *HBKDI*. Prolong *DK* and *FG*, then their intersection point *E* will be on the required curve.[1]



As can be seen from the above extract (Bos, 1975, 41), Bernoulli interpreted the integral geometrically, as the area below a curve. The students added an illustration of this in their figure, as can be seen above, with the two shadowed areas which are not present in Bernoulli's figure. This way of solving the equation by constructing the curve forced the students into discussions about conceptual aspects of solutions to differential equations. It made them articulate what constitute a solution in our modern understanding, an articulation that does not automatically manifest itself from solving differential equation exercises from modern textbooks. In order to follow Bernoulli's construction, the students were challenged to think about and use integration differently than they would normally do when solving differential equations analytically. They were also forced to use the properties of the curve represented geometrically which they felt as a challenge. They were used to using the direct relationship between the analytical expression of a function and the coordinate system, to produce a graph. Here they went “the other way” and had to think of the curve as being represented by its graph instead of its analytical expression. Historically, they realised that what is understood by a solution to a differential equation has changed in the course of time.

3: Different solution methods of the brachistochrone problem. The brachistochrone problem is to describe the curve of fastest descent between two points for a point only influenced by gravity. Jacob and Johann Bernoulli published different solution methods to the problem in 1697. Johann Bernoulli interpreted the point as a light particle moving from one point to another. By using Fermat’s principle of refraction, he derived an equation for the brachistochrone, i.e. the cycloid, involving the infinitesimals dx and dy. Jacob Bernoulli considered the problem as an extremum problem using that, since the brachistochrone gives the minimum in time, an infinitesimal change in the curve will not increase the time.

The differences between Johann’s and Jacob’s solution of the brachistochrone illustrated for the students the power of mathematics. Johann’s solution was tied to the physical conditions of the problem and could not be generalised beyond the actual situation, whereas Jacob’s solution was independent of the physical situation and could be used on different kinds of extremum problems. Through the historical texts on the solution of the brachistochrone, the students experienced the characteristics of the nature of mathematics that makes it possible to generalise solution methods of particular problems. Thereby, they were able to understand why Jacob’s method could generate new kinds of questions that eventually led to a new research area in mathematics, the calculus of variations, and why Johann’s could not. For a didactical perspective on the brachistochrone problem see Chabert (1997).

Development of mathematical competencies. In the discussions above of episodes where the students through their work with the original sources used other aspects of their mathematical conceptions in new situations and discussions, some learning potentials regarding differential equations and the mathematical concepts underneath have already been emphasised, especially in the discussion of the students’ work with Johann Bernoulli’s text on the catenary. A more systematic analysis of the students’ report with respect to the KOM-report showed that the students, in their work with the historical texts, were challenged within seven of the eight main competencies. The students’ awareness of the special nature of *mathematical thinking* (1) was especially enhanced in their comparison of Johann’s and Jakob’s solutions of the brachistochrone as discussed above. The students’ *problem solving* (2) skills were trained extensively and in different areas of mathematics. As mentioned in the discussion of their work with Johann’s solution of the catenary problem, the students’ had to fill in a lot of gaps in order to understand Johann’s results. Each of these gaps required that the students derived intermediate results on their own about similar triangles using trigonometry, and solved mathematization problems. Through their work with understanding the Bernoulli brothers' mathematization of the physical problems, parts of the students' *modelling* competency (3) were developed. The competency to *reason* (4) in mathematics was developed in all those parts of the project work where the students tried to make sense of the original sources by means of their own mathematical training and knowledge. (5) *Representations*: As exemplified in the discussion of the students' work with Bernoulli's construction of the solution to the differential equation of the catenary, the students were challenged so work with a representation of the solution to the differential equation that is different from the analytical representation given in modern textbooks. In the report, the students also solved the differential equation analytically and compared the analytical representation with Bernoulli's geometrical one. During their mathematization of the five hypotheses from static that Bernoulli took for granted, the students were trained both in working with different representations and in using the mathematical language of *symbols and formalism* (6). This competency was especially developed in the students' work with the two original sources on the brachistochrone problem in their struggle to understand Johann's mathematization of the path of the light particle and Jakob's use of the minimising property of the brachistochrone. The writing of the report (90 pages) in which the students, through a thorough presentation and analysis of the original sources, answered their problems for their project work within the historical context, developed their competency to *communicate* (7) in, with, and about mathematics in ways that go far beyond what normal exercises in solving differential equations requires. The competency to handle *tools and aids* (8) was not represented.

SOME CONCLUSIONS AND CRITICAL REMARKS

Based on their studies of the original sources and relevant secondary literature, the students concluded that physics did function as problem generator in the early history of the development of differential equations and played a decisive role in the derivations of the equations describing the catenary and the brachistocrone. They further concluded that physics played a significant role for Johann's solutions of both the catenary and the brachistochrone problem, but not for Jacob's solution of the brachistochrone problem. Jacob's arguments were not linked to the physical system; hence his method could be transferred to other problems of that type. This became the beginning of the calculus of variations. The students did not move beyond this in their project, but it is interesting to notice that the calculus of variation later became central in physics, providing an important feedback in the opposite direction.

The analysis of the chosen project has shown that, *if* we adopt a competency based view of mathematics education and evaluate learning outcomes not with reference to standard procedures and lists of concepts and results, but with respect to how and which mathematical competencies, the students have been challenged to invoke, and thereby develop, and *if* we let the students work with the history of the practice of mathematics studied from specific perspective(s) that address(es) significant issues regarding the mathematics in question, then history can be used as a means to teach and learn core curriculum subjects without losing sight of history.

The above claims are further supported through analyses of other historically oriented mathematics projects that have been performed by students at RUC. A project on the history of mathematical biology, where the students read an original source of Nicholas Rashevsky on a mathematical model for cell division is treated in (Kjeldsen & Blomhøj, 2009) and analysed with respect to learning outcomes regarding deriving and understanding the general differential equation of diffusion, the students' understanding of the integral concept, and development of the students' modelling competency. Other examples of projects with substantial learning outcomes of core mathematics, in university mathematics education, are "Paradoxes in set theory and Zermelo's III axiom", "What mathematics and physics did for vector calculus", "Generalisations in the theory of integration", "Infinity and "integration" in Antiquity", "Bolzano and Cauchy: a history of mathematics project", "The real numbers: constructions in the 1870s", and "D'Alembert and the fundamental theorem of algebra". In the present paper focus has been on how history can be used for the learning of core curriculum mathematics without trivializing it or using a whiggish approach to history. The learning outcome of the above history projects can also be analysed with respect to *Mathematical awareness*, as explained by Tzanakis and Arcavi (2000), which includes aspects related to the intrinsic and the extrinsic nature of mathematical activity. These projects can then also be seen as empirical evidence for some of the possibilities history offers as referred to by Tzanakis and Arcavi (2000, 211). With respect to the KOM-report these aspects relate to the three kinds of *overview and judgement*.

It can be raised as a critic that only certain perspectives of the history are considered, and that e.g. to gain insights into historical processes of change, episodes from different time periods need to be studied. In the above project work, the students did not experience the historical process of change, but they did experience that the understanding of the involved mathematics in the 17th century was different from our understanding. The students did not solve a huge amount of differential equations, and they did not learn to distinguish between different types of differential equations.

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