# DESIGN OF A SYSTEM OF TEACHING ELEMENTS OF GROUP THEORY

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*In order to teach on the basis of the genetic approach, one should undertake an analysis consisting of the following two stages: 1) a genetic elaboration of the subject matter and 2) an analysis of the arrangement of contents including a consideration of various ways of representing it and its effect on students. The genetic elaboration of subject matter consists in the analysis of the subject from four points of view: historical, logical, psychological and socio-cultural. Also important is the epistemological analysis of the subject. We describe here the design of the system of study of the concepts of group theory.*

Keywords: tertiary mathematics education, teacher education, group theory, genetic approach, genetic teaching.

## 1. INTRODUCTION.

In this paper, we describe the design of the system of teaching of the concepts of group theory using the genetic approach. Recently, teaching of group theory was discussed in the number of papers, and modern textbooks on the subject appeared, see, e.g., Armstrong (1988), Burn (1985), Burn (1996), Dubinsky, Dautermann, Leron & Zazkis (1994), Dubinsky & Leron (1994), Leron & Dubinsky (1995), Zazkis & Dubinsky (1996).

However, in the textbooks created by M. Armstrong and R.Burn, only geometrical sources of group theory are emphasized and used for motivating the learning. Articles are mainly restricted to using constructivist teaching or APOS theory (Dubinsky & McDonald, 2001).

Our approach based on the genetic principle combines historical and epistemological elaboration of the subject matter with psychological and socio-cultural aspects and allows to construct effective system of teaching the subject.

In preparation of the system of teaching, we also use the principles of concentrism and of multiple effect (Safuanov, 1999).

The *principle of concentrism* requires the following means in teaching a subject: the *preparation* and, in particular, the *anticipation*; the *repetition* on the higher or deeper level and the *increase*; the *fundamentality* (the deep and strong study of the carefully selected foundations of a discipline).

The *principle of multiple effect* (on students) states that the essential educational result can be achieved not with the help of one means, but many, directed to one and the same purpose. For example, the following means of expressiveness may be used in teaching undergraduate mathematics: the *variation*, *splitting* (subject matter into smaller pieces), the *contrast*.

## 2. SYSTEM OF TEACHING BASED ON THE GENETIC APPROACH

In (Safuanov, 2005) the genetic approach in the teaching of a mathematical discipline (a section of a mathematical course, an important concept, or a system of concepts) is described. Its implementation requires two parts: 1) a preliminary analysis of the arrangement of the content and of methods of teaching and 2) the design of the process of teaching.

The preliminary analysis consists of two stages: 1) the genetic elaboration of the subject matter and 2) the analysis of the arrangement of contents, the possible ways of representation, and the effect on students. The genetic elaboration of the subject matter, in turn, consists of the analysis of the subject from four points of view:

historical;

logical;

psychological;

socio-cultural.

The purpose of the historical analysis is twofold: 1) to reveal paths of the origin of scientific knowledge that underlie the educational material and 2) to find out what problems generated the need for that knowledge and what were the real obstacles in the process of the construction of the knowledge.

For the construction of the system of genetic teaching, it is very important to develop problem situations on the basis of historical and epistemological analysis of a subject.

The major aspect of the logical organization of educational material consists in organizing a material in such way that allows the necessity of the construction and of the development of theoretical concepts and ideas to be revealed.

The psychological analysis includes the determination of the experience and the level of thinking abilities of the students (whether they can learn concepts, ideas and constructions of the appropriate level of abstraction); and the possible difficulties caused by beliefs of the students about mathematical activities. The analysis also has the purpose of planning a structure of the students' activities related to mastering concepts, ideas, and algorithms, of planning their actions and operations, and also of finding out the necessary transformations of objects of study.

One more purpose of the psychological analysis of the subject matter is finding out ways to develop the motivation for learning.

The socio-cultural analysis allows us to establish connections of the subject with the natural sciences, engineering, with economical problems, with elements of culture, history and public life; to reveal, whenever possible, non-mathematical roots of mathematical knowledge and paths of its application outside mathematics.

During the second part of the analysis, considering the succession of study, it is necessary, in accordance with the principle of concentrism, to find out, on the one hand, which concepts and ideas studied before should be repeated, deepened and included in new connections during the given stage, and, on the other hand, which elements studied at the given stage, anticipate important concepts and ideas, which will be studied more deeply later.

The principle of multiple effect on students requires also the search for the possibilities of multiple representation of concepts under the study, possibilities of using three modes of transmission of information (active, iconic and verbal-symbolical) and other means of effect on students (the style of the discourse, emotional issues, elements of unexpectedness and humor).

After two stages of analysis, it is necessary to implement the design of the process of study of the educational material. We divide the process of study into four stages.

*1) Construction of a problem situation.* In genetic teaching, we search for the most natural paths of the genesis of processes of thinking and cognition.

*2) Statement of new naturally arising questions*

*3) Logical organization of educational material*

*4) Development of applications and algorithms*.

According to principles described above, we present here the design of a system for the teaching of the concepts of group theory.

## 3. THE PRELIMINARY ANALYSIS.

### 1) Genetic development of a material.

a) Historical analysis.

F.Klein, who had brought in the essential contribution to the development of the group theory due to "Erlangen program" of the study of geometry through the study of groups of geometrical transformations, argued that "the concept of a group was originally developed in the theory of algebraic equations" (Klein, 1989, p. 372). Thus, groups, in his opinion, have arisen as groups of permutations. However, such fundamental concept as a group had also other roots in mathematics. As indicated in "The Mathematical encyclopedic dictionary" (1988, p. 167), sources of the concept of a group are in the theory of solving algebraic equations as well as in geometry, where groups of geometrical transformations have been investigated since the middle of the 19-th century by A. Cayley, and in number theory, where in 1761 L.Euler "in essence used congruences and partitions into congruence classes, that in the group-theoretic language means decomposition of a group into cosets of a subgroup" (ibid.). However, abstract groups were introduced by S.Lie only at the end of the 19-th century.

The main conclusion from this historical analysis is that the theory of groups has grown out of the development of many diverse ideas and constructions in mathematics and serves to the generalization and more effective theoretical consideration of these ideas and constructions.

b) Logical and epistemological analysis.

For the introduction of the concept of a group, the preliminary knowledge of a lot of set-theoretical and logical concepts and constructions is necessary which can be seen from the detailed logical and epistemological analysis of the homomorphism theorem (Safuanov, 2005. p. 260). In turn, the group-theoretical concepts are used in the subsequent sections. Abelian groups are used in the definition of vector spaces, rings, ideals and fields. The cosets of a subgroup and quotient groups are used in the definition of cosets of ideals and quotient rings. The groups are used also in geometry, in the study of groups of linear, affine and projective transformations. At last, groups will further occur in useful for the future teachers special courses on Galois theory, on geometry of Lobachevsky etc.

From the point of view of epistemology, groups serve for the organization of ideas connected to permutations, bijections and symmetries, therefore, examples connected to these ideas will serve to the good formation of the concept of a group in students' minds.

c) Psychological analysis.

School graduates are not actually prepared for mastering such abstract concept as a group. They can not operate with general concepts of algebraic operations and even with mappings. Therefore, in particular, they can not freely investigate geometrical transformations and their compositions.

On the initial stage, in our view, it is inexpedient to motivate the introduction of the concept of a group by examples of sets of transformations (for example, translations or rotations), because, as the experience of teaching geometry to the first year students of pedagogical universities shows, the geometrical imagination of many students (and spatial imagination in general) is very poorly developed. One more serious complication is bad understanding of quantifiers. On the initial stage the weaker students perceive quantifiers formally, poorly understanding and confusing their sense; they try to learn formulas with quantifiers by rote, confuse the arrangement of quantifiers in the formulas. As a result, the sense of the definition of a group becomes deformed, when the students try to reproduce the definition: it turns out, for example, that for any element of a group there is a distinct neutral element or, on the contrary, for all elements of group there is a common inverse. For the elimination of these difficulties it is necessary to offer the students special exercises, performance of which would reveal the role of the arrangement of quantifiers.

As the majority of the school graduates perceive mathematics mainly as actions with numbers, it is necessary to use these representations at the initial stage of the construction of group-theoretical concepts. Besides, the school graduates remember such rules as associativity and commutativity of addition and multiplication, and these properties anticipate associativity and commutativity of group-theoretical operations.

According to the activity approach (Leontyev, 1981, p. 527-529), in order to operate with group-theoretical concepts (for example, groups, subgroups, cosets), it is necessary that intellectual operations (say, finding out the structure of a group, construction of cosets of a subgroup etc.) were carried out at first as actions, i.e. as purposeful procedures. It accords also to Ed Dubinsky's APOS (action - process - object – scheme) theory of the learning of concepts. Therefore it is necessary to plan skills which should be acquired by students at intermediate stages of learning group-theoretical concepts. It is necessary to design actions, which should precede mastering these skills. For example, before the study of the general way of construction of cosets (as results of the "multiplication" of the entire subgroup to an element of a group), the students should get experience of construction of concrete cosets of finite and infinite subgroups.

One more remark of the psychological character. It is well-known that the concept of a group isomorphism is narrower than the concept of a homomorphism and, moreover, in some sense more difficult, as it includes rather complex requirement of the bijectivity of a mapping. However, the teaching experience shows that, nevertheless, at the initial stage it is expedient to acquaint the students only with the concept of an isomorphism, as it is easier to be interpreted as the "similarity" of groups in some sense (for example, the similarity of the multiplication tables of finite groups); it is easier and more natural also to consider various examples of isomorphisms than those of homomorphisms.

d) Analysis from the point of view of possible applications.

The concept of a group since several decades became rather popular part of the cultural property of mankind. For example, the psychologist J.Piaget tried to use this concept for theoretical study of the psychological theory; the experts in the quantum mechanics believed that the group theory can be used for solving any problem. The group theory turned out to be extremely useful in the search of elementary particles and in the study of the structure of chemical molecules. Of great interest are the consideration of symmetry groups of geometrical figures and the use of groups for the research of patterns. Good examples of the applications of the group theory are the investigation of the "Fifteen puzzle" and graceful group-theoretical proofs of number-numerical theorems of L.Euler and P. Fermat.

**2) Analysis from the point of view of the arrangement of a subject matter, of the opportunities of use of various means of representation of objects, concepts and ideas and of the influence on students.**

Using results of the genetic elaboration, it is possible to offer the following version of the arrangement of a subject matter and of the use of means of influence.

As the theory of groups has grown out of generalizations of diverse ideas and constructions, we offer also to use some lines leading to group-theoretical concepts from the different perspectives: numbers, cosets, bijective transformations and permutations.

In accordance with the official abstract algebra syllabus, we devote to the study of groups several (four) stages at different places of curriculum, and such arrangement allows to effectively use elements required by principles of concentrism and multiple effect. As a result, students cumulatively acquire the necessary knowledge and skills, not losing their interest and motivation to the learning from the beginning to the end of the study of group theory.

The first stage: already at the introductory lecture it is possible to suggest to the students to consider systems of integers under the addition and non-zero rational numbers under the multiplication, to recollect properties of these arithmetic actions. It is expedient to help the students to reveal the properties of associativity, of the existence of neutral and inverse elements in the system of integers,  and the students will be able to reveal independently by analogy the same properties in the system of non--zero rational numbers. Further it is necessary to try to lead the students to the idea that it would be useful to study properties of arithmetic actions based on the revealed fundamental properties and abstracting from the concrete number systems considered above. Here is “the moment of truth” (Safuanov, 2005) where axioms of group should be formulated. Note that the moment of truth is similar to the act of reflective abstraction (as the interior co-ordination of operations of the subject in a scheme) in the theory of Piaget (Dubinsky, 1991), and also to a moment of reification (Sfard, 1991). Such organization of teaching may be difficult and not always completely possible. Therefore, sometimes the appropriate help of the teacher may be useful.

In the ideal case, students should do it independently. Nevertheless, most likely, on this stage the teacher will have to formulate axioms of group himself or to offer the students to find the definition in a textbook.

At this first acquaintance the concept of a group will not be quite strict, as it will be based only on students’ intuitive representations about binary algebraic operations (“actions on elements of sets”), and the possibility of non-commutativity of an operation is not emphasized at all. In effect, this preliminary concept serves only as the anticipation of more detailed acquaintance at the following stages.

The second stage: after the consideration of the addition of cosets and the addition tables for small modules (for example, 2, 3, 4), it is possible to raise the question about the performance of  addition in a set of cosets modulo arbitrary $n>1$. Properties will be similar to properties of the addition of numbers. The students can guess the fulfillment of laws of associativity and commutativity, the existence of neutral and inverse elements, and even in some extent to participate in proving these properties. After that it is possible to introduce a stricter definition of a group, beginning with the definition of ordered pairs and binary algebraic operations (as the rules putting in correspondence to every ordered pair of elements of a given set a certain element of the same set - at this stage students are not yet familiar with the concept of a direct product of sets). Here it should be underlined that the considered groups of cosets under the addition, as well as groups of integers under the addition, are Abelian (commutative), though there are also examples of non-commutative groups.

The third stage: preliminary, but already quite strict statement of elements of the theory of groups after the consideration of elements of the theory of sets, direct products, mappings, including bijective ones, and permutations. At this stage all formal definitions of concepts necessary for the strict introduction of group-theoretical concepts are available as well as sufficient amount of motivating and illustrating properties and examples. At this stage, after the introduction of the formal definition of a group and proof of the elementary properties, it is expedient to consider symmetry groups of geometrical figures. It is useful also for the maintenance of interest to the theory of groups and for the accumulation of the necessary amount of interesting and useful examples for the illustration of further constructions. Just at this stage the examples of non-commutative groups (symmetry groups and groups of permutations) are considered.

At this stage the concepts of a subgroup and isomorphism of groups should be strictly introduced, but in detail they should not be studied yet: they only anticipate systematic study of group-theoretical concepts and constructions at later stages, after studying linear algebra.

The group-theoretical knowledge acquired at the third stage, is used at the construction of concepts of rings, fields (in particular, of the field of complex numbers) and vector spaces.

The fourth stage: systematic study of elements of the theory of groups (including generalized associativity, cosets, normal subgroups, Lagrange's and homomorphism theorems). This knowledge already is sufficient for further study of quotient rings, Galois theory etc.

As to means of influence on students, in the teaching of elements of the theory of groups it is possible to use various evident ways of representation of a subject matter, considering, for example, permutations, symmetry of geometrical figures, geometrical transformations. Among ways of representation of groups it is possible to employ, in case of finite groups, lists of elements, multiplication tables etc. Among other means of influence one can mention the contrast (examples of groups versus semigroups which are not groups, normal subgroups versus subgroups that are not normal), variation (Abelian and non-Abelian groups, additive and multiplicative ones etc.).

## 3. DESIGN OF THE PROCESS OF STUDY OF GROUP-THEORETICAL CONCEPTS.

In the designing process of teaching we take into account all the results of the preliminary analysis, and thus the task of designing becomes considerably facilitated. Note that after designing and checking the intended system of study of a theme in practice, using a feedback, results of the control and assessment, it is necessary to bring in corrective amendments, sometimes essential, to the designed system. So, for the study of the theory of groups we at the third stage (after studying permutations) at first intended to prove the generalized associativity. However, the experience has shown that this rather short inductive proof nevertheless requires from students the well-developed logic reasoning and inordinately large efforts for mastering. Therefore, we have transferred this proof to the last, fourth stage devoted to systematic study of algebraic systems.

### 1) Construction of a problem situation.

As is already shown, for the successful construction of a problem situation it is necessary to organize it (including new questions, naturally arising from it) so that in a certain time there would occur the “moment of truth” when the students independently or with the minimal help of the teacher would open for the new concept for themselves.

For the first time such moment of truth arises already during the introductory lecture, when the preliminary version of the concept of a group arises as a generalization of properties of arithmetic actions in sets of integers (addition) and non-zero rational numbers (multiplication). At further stages this preliminary version of the definition forms the basis for the motivation of the consideration of the concept of a group, basis for its stricter study. So, for example, studying properties of the addition of cosets or multiplication of bijections of a set, permutations of a finite set, symmetries of a geometrical figure, the students already can find out that each time they deal with groups – and thus new moments of truth arise.

### 2) Statement of new naturally arising questions.

For example, when constructing a problem situation at the third stage (when passing to types and elementary properties of groups), one can use questions of the following kind: whether are groups under consideration commutative? Whether there exists an infinite non-commutative group? Is the neutral element of a group unique? For a given element of a group, is an inverse element unique? Is it possible to solve equations in groups? At the fourth stage (systematic study of more complicated group-theoretical concepts) the questions are pertinent: do the right and left cosets coincide? Do cosets of a normal subgroup form a group under multiplication? etc.

**3) Conceptual and structural analysis and logical organization of educational material.**

Conceptual and structural analysis and logical organization of group-theoretical concepts is rather complicated, as is seen, e.g., from the genetic decomposition of the homomorphism theorem (Safuanov, 2005. p. 260). This process is not straightforward, but rather long and, moreover, often occurs in several stages divided in time. From group axioms the properties of groups are deduced, and at final stages of study of groups a number of rather difficult theorems is proved.

**4) Development of applications and algorithms.**

Despite the importance of the theory of groups, its applications are too non-trivial: so in an obligatory course it is problematic to consider such major applications, as the Galois theory or, say, geometrical applications, which are more appropriate for considering in detail in a geometry course. Nevertheless, it is important to consider such simple and interesting examples of applications as the fifteen puzzle, group-theoretical proofs of number-theoretical theorems of L.Euler and P.Fermat, symmetry groups of geometrical figures etc.

The students also should learn such procedures as construction of the multiplication table of a finite group, finding cosets of a normal subgroup (i.e. construction of a quotient group) etc.

Concerning the development of cognitive strategies note that, according to the genetic approach, it is important to teach the students to construct analytical proofs, i. e. such ones that start from the statement that must be proved, and include the search of the facts necessary for the proof of the final statement. Then one searches how to find these necessary facts etc. It resembles going from the end of the proof to the beginning (in computer science such approach is referred to as “backtracking”) (see Goodman&Hidetniemi, 1977). The theory of groups gives such opportunities.

## 4. IMPLEMENTATION.

This system of teahing was successfully implemented in practical teaching at the pedagogical universities of Ufa and Naberezhnye Chelny for two decades. The students studying abstract algebra course by this system constantly show much better achievements and, most important, more positive attitude and interest to the subject than students studying the discipline by traditional deductive and “definition – theorem – example – exercise” approach.

Of course, the genetic approach can be applied for teaching other mathematical topics and mathematical disciplines.

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