EXPERIMENTAL AND MATHEMATICAL CONTROL IN MATHEMATICS

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This paper talk about a problem which can put students in the role of a mathematical researcher and so, let them work on mathematical thinking and problem solving. Especially, in this problem students have to validate by themselves their results and monitor their actions. The purpose is centred on how students validate their mathematical results. I also present the first results of my experimentations. So, this paper is related to learning processes associated with the development of advanced mathematical thinking and problem-solving, conjecturing, defining, proving and exemplifying.

BACKGROUND

The maths à modeler team (www.mathsamodeler.net) is developing a type of problem for the classroom called RSC [1] (Grenier & Payan, 1998, 2002 ; Godot, 2005 ; Ouvrier-Buffet, 2006). The aim of a RSC is to put students in the role of a mathematical researcher. Grenier and Payan (2002) define a RSC as a problem which is close to a research one and, often, only a partially solved problem. The statement is an easy understandable question which is situated on the outside of formal mathematics. Initial strategies exist, there are no specific pre-requisites. Necessary school knowledge is, as much as possible, the most elementary and reduced. But, many strategies to put forward the research and many developments are possible for the activity and for the mathematical notions. Furthermore, a solved question, very often, postponed to new questions.

A RSC seems very interesting for gifted students because it is a challenging problem where they can find new results and be confronted with uncertainly and doubt. However, a RSC was not developed to be used only by gifted students, a RSC is for all the students and the goal of a RSC is not only to challenge students but, firstly, to make them work on mathematical thinking and especially “transversal knowledges and skills” which means: Experimenting, Conjecturing, Modelling, Proving, Defining...

So, in a RSC, students are confronted with an open-field where they have to make their own investigations and validate by themselves their results and actions. They have also to manage their research, for example by trying to solve sub-problems or easier ones instead of the initial problem. Moreover, it can also be a way for students to develop their problem solving skills as it can be considered as a “non-routine” problem.

In French handbooks, it seems that problems do not give the responsibility of the validity of their results to the students. Whereas, it is important for students to be confronted with uncertainly and doubt in mathematical problems because first, they have to control their results to be sure that they are true. Second, they have to convince themselves and their colleagues that their results are true. So, even if they do not give a mathematical proof, they enter in a phase of argumentation which can let them give mathematical arguments like counter-examples. Third, they have to monitor more carefully their actions as they do not know a solution or a plan to solve the problem.

So, a RSC is a type of problem which can give responsibility to the students. But a RSC can also let students work on definition (Ouvrier-Buffet, 2006), modelling (Grenier & Payan, 1998), experimental approach (Giroud, 2007) and more generally on transversal knowledges and skills.

In this paper, I present a RSC, *the game of obstruction*, which is a discrete mathematics optimization problem. This problem is only partially solved. I propose this problem for 2 reasons: let students work on mathematical thinking and problem solving, and in his quality of very challenging problem.

I give a mathematical and didactic analyses of the problem. I also propose results of my experimentations that will be centred on how students control their mathematical results, especially with these types of control:

Different types of results control in mathematics

The experimental control: Dahan (2005) claims that there exists 2 types of experimentations in mathematics: generative experimentations, which are experimentations that we carry out to generate facts when we have no idea of the result ; and checking experimentations that we carry out to check an hypothesis [2] or a conjecture. So, the checking experimentation can be a way to control the results. But unfortunately, even if a result is experimentally checked as true a lot of time, it can be false. In mathematics, we need a proof. However, we can use the experimental validation before going to the proof stage to convince ourselves that the result is true.

For example, if we do not know whether the Goldblach conjecture: *all even number superior to 2 can be written as the sum of two prime numbers,* is true, we can control this proposition by carrying out checking experimentations on 2, 4, 6, 8, 1284... And as we seen that each times it works, it can convince us that the conjecture is true.

The mathematical control: the mathematical control is what we call proof. We can not have a "better" control.

It is essential to have a proof to name a fact theorem, for example the Goldblach conjecture is true for all even number higher than 2 and lower than $4*10^{14}$ (Richstein, 2000) but we can not call it theorem because we do not have a proof for all even numbers.

We have also others types of control, for example if an analogue problem is known to be true.

But here, the 2 types of control that I will consider are the experimental and mathematical control.

These 2 kinds of control, mathematical and experimental, do not contradict each other. Considering Polya's distinction between plausible and demonstrative reasoning (1990), it appears that the experimental control is part of the plausible reasoning whereas the mathematical control is part of the demonstrative reasoning. And as Polya (1990) claimed:

> Let me observe that they do not contradict each other; on the contrary, they complete each other.

Indeed, in mathematics both are useful, we can use the experimental control to estimate the plausibility of a result and we need the mathematical control to be completely sure.

Now, I present the theoretical framework that I use to make my analysis.

THEORETICAL FRAMEWORK

I recall briefly what is a didactic variable. For Brousseau (2004), a didactic variable of a problem P is a variable which can change the solving strategies of P and which can be used by the teacher. So, by using the didactic variable the teacher can change the knowledge in game in P for the students.

I also use the notion of research variable (Grenier & Payan, 2002 ; Godot, 2005). A research variable of a problem P is a variable of P which is fixed by the students. The didactic choice for the teacher is to choose which variables of P will be used as research variables. This choice is made by considering the questions, conjectures, proofs that these variables could generate. In a RSC, there are research variables as it can let students manage their research.

The notion of didactic contract (Brousseau, 2004) is also used. The didactic contract corresponds with the implicit relations between the students and the teacher. An example in French classrooms is when students learn the factorization of polynomials, when the teacher asks a student to factorize $4X^2+4X+1$, the answer that the teacher wishes is $(2X+1)^2$ not a factorization like $4*(X^2+X+1/4)$ which is, even, a right factorization but not a factorization in irreducible polynomials which is implicitly asked.

And to analysis the experimentations, I use the framework developed by Schoenfeld (2006) to analysis mathematical problem solving behaviour:

> the key elements of the theory are:
>
> - knowledge;
> - goals;
> - beliefs;

– decision-Making.

The basic idea is that an individual enters any problem solving situation with particular knowledge, goals, and beliefs. The individual may be given a problem to solve – but [...] what happens is that the individual establishes a goal or set of goals – these being the problems the individual sets out to solve. The individual's beliefs serve both to shape the choice of goals and to activate the individual's knowledge – with some knowledge seeming more relevant, appropriate, or likely lead to success. The individual makes a plan and begins to implement it. As he or she does, the context changes: with progress some goals are met and other take their place. With the lack of progress, a review may suggest a re-examination of the plan and/or re-prioritization of goals. [...] This cycle continues until there is (perceived) success, or the problem solving attempt is abandoned or called to a halt.

THE GAME OF OBSTRUCTION

The situation was suggested by Sylvain Gravier. In order to present the problem we will need some useful definitions. A *(n, c)-card game* (or for short *card game*) is a set of cards having *n* lines, each of which contains a color in *{1, …, c}*.

Given a (n, c)-card game, the color of the ith line of a card C will be denoted by C_i. A *bad line* in a set of 3 cards C, C' and C" is a line i for which either ($C_i = C'_i = C"_i$) or ($C_i \neq C'_i \neq C"_i$ and $C_i \neq C"_i$).

1	3	3	2
2	3	2	2
1	3	1	3

Figure 1: A (3,3) card game

An *obstruction* is a set of 3 cards such that **all lines are bad**.

3	3	3	Plain
1	2	3	Strict multi-color
3	1	2	Strict multi-color

Figure 2: An obstruction

Now the problem can be stated as follows:

Given two integers n and c, find the largest (n, c)-card game which does not contain an obstruction. (P1)

Some examples:

2	2	2	3	3	3	4	4
1	1	2	4	1	4	1	4
2	3	2	1	2	2	1	1

Figure 3: A (3,4) card game containing an obstruction

1	1	1	3	3	3
1	1	3	1	2	3
2	4	2	2	2	4

Figure 4: An obstruction-free (3,4) card game

First, one can observe that: *one may consider a card game for which all the cards are distinct.* Indeed, given an obstruction-free card game of cardinality *m* for which all the cards are distinct, by duplicating each card, we obtain an obstruction free card game of cardinality 2*m*. Conversely, there are no 3 copies of the same card in an obstruction-free card game.

According to that, we will now only consider card games for which all the cards are distinct. The cardinality of a largest (n, c)-card game with no duplicated cards will be denoted by *Max(n, c)*.

Mathematical analysis

It is worth noticing that (P1) is still an unsolved problem so before trying to solve it one may study a weaker version: *(P2) How can we build a set without obstruction ?*(P2) problem suggests determining an efficient method (algorithm) to check if a given set of cards contains an obstruction. I will denote this problem by (P3).

Another way of simplification will be to fix *n* and/or *c*. To work on optimization problems, we need to consider the following problem: *(P4) How can an upper bound be found?*

(P2) and (P4) split (P1) into the two aspects of an optimization problem: lower and upper bounds.

Unfortunately, since (P1) is still not solved, we do not have yet a general strategy to solve (P4) efficiently. Mainly, a strategy (SP4) to answer (P4) is based on *enumerating all possible obstruction-free card games*. For a low value of n, an easy enumerating argument shows that theorem:

Theorem 1: *For any integer c ≥ 2, we have Max(1, c) = 2 and Max(2, c) = 4.*

Now, I present some strategies to solve our problems. First, concerning (P3), a “naïve” way would be to check all sets of 3 cards among a given card game. Nevertheless this strategy fails when the number of cards *m* is large since it requires $O(m^3)$ cases to be explored. Nevertheless, a strategy based on the structure of the given card game exists. For i in {1, …, c}, the *i-block* of a card game G is the subset $C^1, \ldots, C^t$ of G such that $C^1_1 = \ldots = C^t_1 = i$.

(SP3) First check that each block does not contain an obstruction (you can apply this strategy recursively). Secondly, search obstructions that have at most one card per block.

In general, this strategy is no more efficient than the “naïve” way. Nevertheless, it appears that for large obstruction free card game G, the colours are recursively and equitably distributed on each block, therefore (SP3) checks in $O(\mathrm{Log}_c\,(m)^3)$ steps that G has no obstruction.

Another interest for using (SP3) is that it allows first results on Max(n, c) to be obtained. Indeed, consider an obstruction-free (n, c)-card game, then each block is at most Max(n-1, c) in size. Therefore Max(n, c) ≤ c.Max(n-1, c), which gives an answer to (P4).

Moreover, from an obstruction free (n-1, c)-card game G of cardinality t, one can build an obstruction free (n, c)-card game of cardinality 2t. Indeed, for i=1, 2 , consider the obstruction free (n, c)-card games G_i obtained from G by adding a line to each card and assigning color i to this new line. The set G' = $G_1 \cup G_2$ is an obstruction-free (n, c)-card game of cardinality 2t, which gives an answer to (P2).

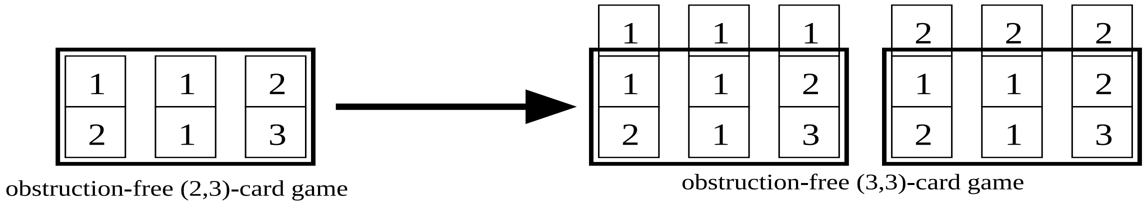

Figure 5: An example of the inductive construction based on SP3

These two remarks lead to:

Theorem 2: *Given integers n and c ≥ 2, we have that:*

$$2.Max(n\text{-}1, c) \leq Max(n, c) \leq c.Max(n\text{-}1, c).$$

Observe that for c = 2, we get: $Max(n, 2)=2^n$. Notice that this result can be proof without theorem 2 by giving an inductive proof.

Nevertheless, when c ≥ 3, one can find obstruction-free card game of larger cardinality than 2.Max(n-1, c). To find such obstruction-free card game one can apply "greedy" strategies:

(S1P2) Start from an obstruction free card game G (it can be empty) and add a card C such that G∪C is still obstruction-free until there is no such card.

(S2P2) Start from a card game G and while there is an obstruction in G, remove a card from this obstruction.

Observe that these two strategies give Max(n, 2) since there is no obstruction in a (n, 2)-card game. In general, an obstruction-free maximal card game G is built (i.e. for every card C not in G, G∪C contains an obstruction). It is worth noticing that (SP3) produces also obstruction-free maximal card game G, but this requires additional arguments. If one chooses an appropriate order for eliminating cards one can find an optimum of (P1) using (S1P2) or (S2P2). Of course, finding such an order remains an open problem. Nevertheless, when n is 'large', one may use a suitable order which ensures that one considers all possible cards ; for instance the lexicographic ordering. Unfortunately, even when n=3, the lexicographic ordering gives a maximal obstruction free (3, 3)-card game of cardinality 8. However, by applying (S1P2) or (S2P2) with other orderings, one can find an obstruction free (3, 3)-card game of cardinality 9 (> 2.Max(2, 3)). Similarly, one can exhibit an obstruction free (4, 3)-card game of cardinality 20.

Moreover, by applying a (SP4) strategy one can prove:

Theorem 3: *Max(3, 3)=9 and Max(4,3)=20.*

Didactic analysis

I decided to use *n* the number of lines and *c* the number of colours as research variables (Grenier & Payan, 2002 ; Godot, 2005). Since they can lead to new questions like: what is the link between a n-line game and a n+1-line game ? Trying to solve this question would provide an inductive construction of obstruction free card games which can be seen as an inductive proof. Moreover, it can let students generalize some results, especially with 2-colours. So, students can use these variables to manage their research.

There exists a more general problem than (P1), in which the size of an obstruction is a variable of the problem, but here, I decided to use it as a didactic variable by fixing its value to 3. I choose a size of 3 because for 1 or 2, the situation is very easy. It becomes sufficiently complex from 3.

Through mathematical analysis one can determine the following knowledge involved in solving (P1):

- The definition of an obstruction requires the understanding of **logic quantifiers**.
- (S1P2) and (S2P2) suggest using an **algorithmic** approach to solving (P2) using **eliminating ordering** (for example **lexicographic ordering).** Moreover, since these strategies build a maximal obstruction-free card game, one can discuss **local /global maximum.** Therefore**,** these strategies will produce solutions which can be conjectured as optimal.
- (SP3) allows a card game to be **modelled** which can be reinvested to (partially) solve (P2) and (P4) as shown in proof of Theorem 3. Moreover, (SP3) applied on (P2) gives an **inductive construction** of obstruction-free (n, c)-card game based on two copies of an obstruction-free (n-1, c)-card game.
- (SP4) is an **enumerating** approach for solving (P4). To reduce the number of cases to be considered it will be convenient to use **variables** for the enumerating.
- The distinction between problems (P2) and (P4) is related to **lower and upper bounds** on an optimization problem (P1) which is closely related to **necessary and sufficient conditions**.
- Solving (P1) with c = 2 provides all possible 2^n cards in a card game on n lines to be **counted**.

OUR EXPERIMENTATIONS

Two experiments were carried out, one with a “seconde” (tenth grade) class, E1, and another with a “première scientifique” (eleventh grade) class, E2. Pupils worked in groups of 3-4. In each class, we let them search for 2 hours. The E1 experiment was carried out before the E2 one. We filmed one group in each experiment.

The problem was presented orally with examples on the blackboard. We gave to them some material with which they can experiment. In E1, we gave plain circles of 4 different colours and in E2, we added n-line cards with no colours and n=1, 2, 3, 4.

But in both experiments the problem is posed generally as (P1), we did not ask students to only use the number of colours or the number of lines that is given materially.

Results of experimentations

First, my analyses are focused on how one group of the tenth grade class tried to solve (P3), that is to say, how they control the presence of an obstruction in a card game.

They started by building an obstruction free card game with 3 lines and 4 colours with the additive strategy (*S1P2).* They built a card game G1 of cardinality 4 and then they added a card C. Then they searched obstructions in G1∪C by trying to check "randomly" all triple of cards. They did not find any obstructions but they were not sure to have tested all triple. Here, the knowledge of how to find all triple is missing. Then, they formulated this question (P3a): *How can we know if all triples of cards were checked ?* They tried to answer (P3a) during one minute but they did not find a solution. After that, they concluded that they checked all triples of G1∪C although they did not. Thus, they decided to give (P1) a higher priority than (P5). Seeing that they could not solve (P5) quickly and believing that their experimental control based on *"checked all triples"* is sufficiently efficient, they decided to rely on the experimental control.

During all the session they relied on the experimental validation for the obstruction's property although, I showed them obstructions in their card games. They did not decide to re-examine their plan by searching an other strategy to solve (P3) than "*check all triples"*. Despite that, they observed that this strategy is *too difficult to do* and that the experimental control based on this strategy was not efficient.

So, it seems they gave (P3) a lower priority than (P1). It joins Schoenfeld (1992) observations that students are more concerned about the initial problem than to sub-problems, although sub-problems can be key elements. And here, (P3) is key element to make progress on (P1). The group said 11 times that a card game was obstruction-free and it was true only once.

In the two experimentations, none of the group seemed to search an efficient method to answer (P3), they only used strategies based on *"checked all triples"*, although many of them were confronted to (P3). So it seems that students decided to rely on the experimental validation and not on the mathematical validation for the obstruction free property. An interpretation could be that students did not find a solution so they decided to rely on the experimental control to progress in (P1). However, for the group above, it seems, as they only search for one minute, that they decided to not spend too much time on (P5). So, they did not recognize the role of (P5) and (P3) for solving (P1).

Summarize of the experimentations:

It appears that the use of material during experiments E1 and E2 led pupils to carry out their own experiments in mathematics. Students started to manipulate and carry out experimentations to solve (P3) and (P2). Even if (P3) was identified, they stayed in the experimental control. Consequently, there were some group which did not obtain results on 3 lines. But, they made hypotheses or conjectures that they checked with experiments like "*this card game is maximum*", "*by using this strategy, we build an obstruction free card game*" or "*with only 2 colours on each card, there are no obstructions*", which allowed them to find counter-examples. Here, students are responsible of deciding the validity of their propositions. But for one group, it was not the case, they made an experimental control of the obstruction free property of their card game and after called us to validate their results. They did not take the responsibility of the result's validity. There was a problem in the didactic contract.

They proved Max(n, 2) for n=1, 2 and 3. But only one group generalized this result and this group made the 2 experimentations.

They used at most 4 colours and did not try to generalize with more. Moreover, they tried to use all the colours. Here, we can see a consequence of the didactic contract: use all that is given and not more. So, the didactic contract has to be changed to let students manage their research.

The concept of variable useful in an enumerating strategy like (SP4) was not discussed. Similarly no good eliminating ordering was proposed by the pupils ; they remained in a 'naïve' strategy.

BRIEF CONCLUSION

This situation was experimented with "ordinary" students and show that this problem can let students take the role of a mathematical researcher. Although they did not use the variables of the problem to try to solve easier sub-problems, they carried out experiments to try to answer their own questions, formulated conjectures and made proofs. Moreover, it seems, as in Schoenfeld (1992) studies, that contrary to an expert they have some difficulties to identify one of the key element to solve (P1) ; although they identified (P3), they relied on the experimental control.

Students did not work on all knowledges identified in the didactic analysis, especially the concept of variable which is a powerful abstract concept. We tested this situation on a longer time (18 sessions during one year). In this context, strategies (SP3) and (SP4) were developed and their corresponding results were obtained.

NOTES

1. RSC: Research Situation for the Classroom.

2. Here the definition of hypothesis used is: a proposition that we enunciate without opinion. It is not the same as the usual definition of a mathematical hypothesis,

3. In France, seconde corresponds at a tenth grade class, it is a general section. Première scientifique corresponds to a eleventh grade class and it is the scientific section.

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