# NECESSARY REALIGNMENTS FROM MENTAL ARGUMENTATION TO PROOF PRESENTATION

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*This paper deals with students' difficulties in transforming mental argumentation into proof presentation. A teaching / research tool is put forward, where the statement of a task is accompanied by a given written piece of argumentation suggesting a way to resolve the task intuitively. The student must convert this into an acceptable mathematical form. Three illustrative examples are given.*

**Key words**: mental argumentation; proof presentation; mathematical language; refinement of expression; transparency.

## INTRODUCTION

It has been noted in several papers (eg. Gusman, 2002; Moore, 1994) that in certain circumstances students can 'see' a proof but they cannot express their intuitive ideas in terms of mathematical language. The students use representations that are or have become over time divorced from the mathematical frameworks that allow explicit tools of exact analysis. Thus an impasse occurs.

On the other hand, the usual style of presentation of proof can seem 'monolithic'. It denies in most cases not only a history of aborted attempts, but also it does not communicate essential conceptual and cognitive input that supported the initial formation of the proof. In this respect, reading a proof has a facet that has to be deciphered. When assessing proofs we should not be only concerned in investigating the 'mechanics' that explain how a given proof succeeds in what it was meant to achieve. We also should be concerned with the creative processes involved in producing the 'mechanics' in the first place.

Hence, the circumstance where a student can discern an argument informally but cannot express it in a ratified mathematical format is exacerbated by the fact that past exposure to proof presentation hardly is supportive. A possible remedial measure might be to seek for a radical change in how proofs are written, to better reflect the cognitive input that otherwise would be repressed. However in the next section we will argue that there are compelling reasons to retain the traditional styling of proof presentation. Taking this in mind, if students are to develop the skills to convert mental argumentation into mathematical frameworks allowing deductive reasoning, channels have to be found to help the students to achieve this. In this paper, we put forward such a channel.

In particular, we consider the situation where a student is given not only a task, but also has an informal description how to deal with the task. The description can be self, peer, or teacher generated. The job of the teacher is to guide the student to transform the information that is provided into a strict proof. This is envisaged as a sustained teaching practice, which hopefully would encourage student emulation in their independent work. The education researcher also has a role. Beyond investigating which kinds of guidance given by the teacher will be the most effective, the researcher would be interested in identifying specific types of discrepancies that can occur between informal and formal reasoning, and their effect in cognitive terms.

The main body of this largely theoretical paper will comprise a discussion of three worked examples. These worked examples follow a certain format of design. We envisage that this format could be consistently adopted as a research tool for an educational program of a larger scale. For each example, its content will be carefully separated between the 'givens' and the 'material to be produced'. The 'givens' have two components; the first is a task or a proposition, the second is a mental argument that addresses it informally. The material to be produced will include a 'rigorous' solution or proof influenced by the given mental argument. In addition, in order to ease the transition to the proof, the material to be produced may further involve the formation of an enhanced version of the initial informal argument.

The examples are chosen to illustrate how the identification of structural properties in the informal argumentation can lead to an entry point into a mathematical framework, and ways that proof presentation may seem not to respect the informal line of thought. The approach taken here would be most pertinent to the upper-secondary and tertiary levels, as it is at these levels that the insistence of proof production becomes more poignant.

We acknowledge some points in our undertaking might deny some important aspects in combining intuitive and formal sources in the doing of mathematics. For example, ideally the students themselves could be constructing their own representations and mental argumentation. Representations and mental argumentation made by peers or the teacher may not be comprehended by the students. Further, often it is the case that mental argumentation and the thinking consonant to mathematical frameworks might evolve mutually. These points might suggest that what we are endeavouring to do in this paper has its limitations. However, we do believe that the direction we take constitutes an important device for analysing the learning and teaching of mathematical modelling, and the potential difficulties that are involved.

## MENTAL ARGUMENTATION AND PROOF; HOW DO THEY DIFFER?

It has often been observed both by mathematicians and educators that the proofs published in mathematical journals are far from being completely rigorous (e.g.,Thurston, 1995; Hanna & Jahnke, 1996). This has prompted some educators to view proof mostly in terms of conviction. However, in certain circumstances even a highly naive argument can be so compelling that any reasonable person would be 'convinced' of the proposed conclusion. The problem is that however 'obvious' or 'transparent' an intuitive argument is, there might not be a way to directly reduce it to fundamental principles. The point is not so much about conviction, but how we can clarify the bases of the reasoning employed. The notion of a 'mathematical warrant' (Rodd, 2000) addresses the issue of justifying the grounds that support students' belief in the truth of a mathematical proposition. Still, in how this term is employed suggests a certain primacy to 'embodied processes' over any mathematical setting demanding deductive argumentation.

This primacy might be challenged by some. For example, the construction of a proof can be regarded as an activity to make argumentation more precise. From this viewpoint, proof refines any intuitively based argument. Perhaps a more balanced stance to take is that it is artificial to try to distinguish informal thinking from formal thinking. Thurston talks about a mathematical language (replacing the 'myth' of complete rigour). As in any language, there is ample space to express ideas in casual, incomplete, or inexact formulations. However mathematical language is strongly rooted to a vocabulary referring directly to defined mathematical entities, and its expression is conditioned by respecting previously ascertained properties. Drawing a sharp characterisation of this language might be a difficult undertaking, though preliminary remarks are made in Downs & Mamona -Downs (2005). Assertions made by Thurston are that it is very difficult for students to become fluent in the mathematical language, but ultimately it is in this medium that mathematical thought evolves.

In the introduction we employed the term 'mental argumentation'. What place does this have in our discussion above? From our perspective, mental argumentation rests on collating sources of intuitive knowledge. One character of intuitive knowledge is that, cognitively, it deals with self-evident statements. Unlike perception, intuitive knowledge exceeds the given facts (see Fischbein, 1987). Also, it is accumulative; it depends on past assimilation of conceptual matter. The collation involved in mental argumentation can be made either at the level of instinct or at the level of insight. Both rely on a certain degree of vagueness (see Rowland, 2000, for the importance of vagueness in the doing of mathematics). Mental argumentation should convince the practitioner but not necessarily others; the practitioner would be aware that someone else might demand a warrant. Mental argumentation can lie either inside or outside the mathematical language. Which of the two depends on whether the collation of intuitive knowledge is guided by mathematical insight rather than instinct. Indeed if the argument is based on instinct, there is a lack of self-awareness of the sources drawn on in making the reasoning, including mathematical backing.

Harel, Selden & Selden (2007) have put forward a framework for the production of proof by distinguishing a 'problem - oriented' part and a 'formal - rhetorical' part. (The word rhetorical here serves to point out that what is accepted as formal proof can include some standard linguistic devices beyond strict logic). We suggest that mental argumentation stresses the 'problem - oriented' part; the 'formal - rhetorical' part is as yet opaque, and it is drawn on only when it is required to bolster the intuitive line of thought. A 'naturalistic' proof is obtained by respecting the original problem solving aspects, but fills the 'gaps' in the reasoning by explicitly bringing in mathematical sources permitting tight deduction. A 'naturalistic' proof should be explanatory; Hanna & Jahnke (1996) suggest that proof that explains is preferable to proof that does not. However 'naturalistic' proofs are not always feasible; in the process of converting the original mental argumentation into a framework allowing deductive argument, certain mathematical constructs have to be made to accommodate the intuition, but in doing this there might well be clashes in cognition that cannot be side-stepped. Because of this, formal proof presentation often does not seem to communicate the thinking processes that first motivated its formulation. However, the formal presentation is not simply a contrived imposition, stipulating that your argument has to be validated by a vague standard of rigour. It is something that is encompassed in the mathematical language. In that context, the original thinking processes should be retrievable. Hence, we have a duality between the problem-solving element needed in forming a proof and that needed in reading a proof (see Mamona-Downs and Downs, 2005).

A teaching/research practise similar to that proposed in the introduction is forwarded by Zazkis (2000). It deals with relatively simple examples that only involve translation from mental argumentation to naturalistic proof.

## THREE ILLUSTRATING EXAMPLES

In this section we write down and discuss three tasks and proposed solutions. The purpose is to illustrate some cognitive issues concerning the conversion of mental argumentation into proof presentation. In considering just three tasks, our exposition will bring up only a sample of the points that potentially can be made; we believe that many other points and elaborations can be drawn in the future.

Each example will be divided into three parts. The 'givens' is the material that would be given to the student if a fieldwork were undertaken. The 'material to be produced' always includes a form of a suitable proof presentation, but might also involve a middle step enhancing the original mental argumentation. The 'material to be produced' is made in a putative spirit rather than regarding it as a 'model solution'. Finally, the 'comments' relate the cognitive factors extracted from the examples.

Example 1

**Givens**

Task: Two persons, A and B, start a walk at the same time and place along a particular path of length d. Person A walks at speed $v_1$ for half of the time that A takes to complete the walk; after he walks at speed $v_2$, where $v_2 < v_1$. Person B walks at $v_1$ for half of the distance, and after walks at $v_2$. Who finishes the walk first?

Mental argumentation: Person A covers more distance in the first half of the time when walking at $v_1$ than the distance achieved in the second half of the time walking at $v_2$ (as $v_1 > v_2$). Thus A walks further than the half point in distance, i.e. d/2, at the faster speed $v_1$, whereas person B walks only the half- distance at $v_1$; hence A arrives first.

## Material to produce

Proof presentation: Let $d_1$ be the distance at which A changes speed. Let $t_1$, $t_2$ be the time for A, B to complete the walk respectively. Then

$$\begin{array}{l} d_1 = \frac{1}{2} t_1 v_1 \\ d - d_1 = \frac{1}{2} t_1 v_2 \end{array} \Rightarrow d - d_1 < d_1 \quad (as\ v_1 > v_2) \Rightarrow d_1 > \frac{d}{2}$$

$$t_2 = \frac{\frac{d}{2}}{v_1} + \frac{\frac{d}{2}}{v_2} = \frac{\frac{d}{2}}{v_1} + \frac{(d_1 - \frac{d}{2})}{v_2} + \frac{(d - d_1)}{v_2} > \frac{\frac{d}{2}}{v_1} + \frac{(d_1 - \frac{d}{2})}{v_1} + \frac{(d - d_1)}{v_2} \quad (as\ v_1 > v_2)$$
$$= \frac{1}{2} t_1 + \frac{1}{2} t_1 = t_1$$

## Comments

This example constitutes a relatively smooth transition from the mental argumentation to the proof presentation. Even so, we envisage that many students might have problems in executing it. Even the required assignation of symbols ($d_1$, $t_1$, $t_2$) has a modest constructive element that should not be assumed easy for the students to adopt. The thrust of the proof lies in the transformation of d/2 into ($d_1$ - d/2)+(d- $d_1$). The motivation in doing this is ($d_1$ - d/2) represents the distance that A walks at the highest speed $v_1$ beyond B does; (d – $d_1$) represents the distance for which both A and B walk at the lower speed $v_2$. Hence one term pinpoints where the behaviour of A and B is different, the other where their behaviour is the same. This 'move' might be difficult to make unless you have the support of the mental argumentation, so the student would have to have a tight grasp of how the intuitive reasoning is guiding the algebra.

This task appears in Leikin & Levav-Waynberg (2007) in the context of connecting tasks. Another approach different to the one above would be to take the strategy: explicitly determine the time that A and B take separately and then argue which time is the shorter. However, there is not a sense here that a mental argumentation is playing a role; the task is immediately modelled into an algebraic context, and the argumentation is accomplished completely at this level. This latter approach certainly provides more explicit information (beyond what was demanded), but lacks the transparency that the first provides.

Example 2

**Givens**

Task: Suppose that the real sequence ($a_n$) is convergent, and there is an infinite subset M of the set of natural numbers N and a real number t such that $a_n$=t whenever $n\in M$. Prove that the limit of ($a_n$) is t.

Mental Argumentation: There is an 'infinite number of terms' that take the value t, so however far the sequence has progressed there must still be a term having the value t not reached as yet. At the limit, the terms must be tending to the limiting value, but as far progressed the sequence is, t 'occurs', so the limiting value must be t.

**Material to produce**

Enhanced mental argumentation: Suppose that in fact it is not true that the limiting value is t. Then the value must be a number $l\neq t$. There is an explicit number expressing the distance between l and t. However progressed is the sequence, the value t 'occurs' and so there will always be terms that have a certain fixed distance from the limiting value. This contradicts the idea that the sequence is tending to the limiting value. Thus it cannot be true that l and t are different.

Proof Presentation: Suppose that lim $a_n$=l and $l\neq t$. Let $\varepsilon = (|l - t|)/2$. Then there is a natural number N such that for all n>N, $a_n\in(l-\varepsilon, l+\varepsilon)$ and we have chosen ε such that $t\notin(l-\varepsilon, l+\varepsilon)$. Now there are only a finite number of $n\in N$ such that $a_n\notin(l-\varepsilon, l+\varepsilon)$. This means that only a finite number of $n\in N$ satisfy $a_n$=t. This is a contradiction.

**Comments**

The first mental argument could persuade some students on reading it, but the basis of its acceptance rests on a degree of personal instinct that likely would not be shared by others. An enhanced mental argument might arise as an attempt to remedy some of the shortcomings of the first; if the argument lacks concreteness when it is used to justify a proposal, you might be forced to consider the consequences if the proposal was not true. These consequences might run contrary to the specifications of the task environment. In this way, we believe that logical devices such as proof by contradiction can, up to a point, be naturally handled in the confines of mental argumentation.

There remains a point of vagueness shared by both mental arguments, i.e. the claim 'however the sequence has progressed there must still be a term having the value t not reached as yet'. Likely the acceptance of this would depend much on the student having a suitable mental image of what an infinite sequence is. Without this, a student might be doubtful about how the claim could be justified.

For a justification, one has to refer to the mathematical definitions providing the means to decide on issues dealing with limits. Much research has reported clashes of intuitive images with the dictates of the definition of the limit. With this in mind, it is not surprising that some switches of focus have to be made to transform the mental argumentation into a proof presentation, Mamona-Downs (2001). What the definition provides is an 'ε-strip' around l that stipulates that however small ε is, there is a 'stage' of the sequence beyond which the values taken must be trapped in the strip. (This makes use of imagery that is usually made available in the teaching process.) By choosing ε small enough, we can arrange the ε-strip to 'avoid' the value of t if t≠l. Then there are only a finite number of terms 'at the start of the sequence' that can possibly take the value of t, and we reach a contradiction.

The switch then is that instead of employing the fact that there are infinitely many terms taking the value of t as a basis for argument, one employs the definition of the limit of a sequence as a basis for finding contrary evidence. The character of the contradiction here is somehow different from the one found in the enhanced mental argument. The difference could be expressed by comparing "if the result was not correct, then a condition is transgressed" with "a perceived property (tending to the limit) is contravened".

Note that the negotiation of what direction the proof should follow is itself couched in casual terms. This illustrates how mental argument can be a part of the mathematical language. Even though the supporting mental argument guides the structure of the proof, the proof presentation does not acknowledge its role. Particularly stark is the setting, almost as a fiat, of the value of ε. However, from our strategy making, the choice of ε is pre-motivated, and it could take any value in the interval $(0, |l-t|)$. A reader of the proof might not appreciate this. Another feature of the proof presentation is the compression involved in the statement ' we have chosen ε such that $t\notin(l-\varepsilon, l+\varepsilon)$ '. Set theoretically, a justification of it would take several lines. But because the value of ε was picked especially to satisfy the property involved, these details can be safely suppressed. In general, the transition from one line to another in a proof presentation often goes beyond deductive implication; it often 'hides' input from mental argumentation. The skeletal form of the proof presentation has an advantage in that the 'gaps' that appear can be filled through insight, but if this fails one can always resort to the mathematical tools available to complete the minutiae synthetically. This discussion throws a light on the respective roles of mental argumentation and proof presentation in the mathematical language.

Example 3

**Givens**

Task: Let n be a natural number. Suppose that $r_n$ is the highest power of two dividing the factorial of $2^n$. Find $r_n$.

Mental argumentation: (Student produced)

"We know that from the numbers 1, 2, 3, …, $2^n$, there are $2^{n-1}$ numbers which are divisible by 2. We note that from the numbers

1, 2, 3, …, $2^{n-1}$, there are $2^{n-2}$ numbers that are divisible by 2. We note that from the numbers 1, 2, 3, …, $2^{n-2}$, there are $2^{n-3}$ numbers that are divisible by 2. Continuing to the end we have that $2^n! = 1.2.3…2^n$ is divisible by 2 raised to the power

$2^{n-1} + 2^{n-2} + 2^{n-3} + … + 2^2 + 2 + 1$.

This means that $r_n$ equals $2^n - 1$."

**Material to produce**

Proof production: Here there is a choice. One tack that can be taken is to conjecture that the result obtained is correct and then use induction. This is fairly easy to do, and it will be left to the reader. The other tack is to produce a proof not assuming the result. Such a proof might follow the lines as below:

For each i = 1,…, n, let

$A_i = \{s \in \mathbf{N}: s \leq 2^n \text{ and is a multiple of } 2^i\}$

$B_i := \{t \in \mathbf{N}: 2^i \text{ divides } t \text{ and } t/2^i \text{ is odd}\}$

$a_i := |A_i|, \quad b_i := |B_i|$

By construction,

$$r_n = \sum_{i=1}^{n} i\, b_i, \quad a_i = b_i + b_{i+1} + \ldots + b_n \quad and \quad a_i = 2^{n-i}$$

*Hence, for* $i \neq n$

$$a_i = b_i + a_{i+1} \Rightarrow b_i = a_i - a_{i+1}$$

$$\Rightarrow r_n = n + \sum_{i=1}^{n-1} i(a_i - a_{i+1}) = n - (n-1) + \left(\sum_{i=2}^{n-1}(i-(i-1))a_i\right) + a_1$$

$$= 1 + \sum_{i=2}^{n-1} a_i + 2^{n-1} \quad = \quad \sum_{i=1}^{n-1} 2^i \quad = \quad 2^n - 1$$

**Comments**

In this example, contrary to the previous two, the mental argumentation was produced by two students (working together) whilst doing project work, and this constituted their final answer. In a subsequent interview, it became clear that they did not consider their response to constitute a proof, however the terse manner of their exposition seems to be influenced by an image of a proof being minimally expressed. In the interview the students were able to explain the origin of the stated lists of numbers, but only in informal terms. It is significant that the students did not spot the induction option, as in other work they showed themselves adept in identifying and applying this general proof technique. The impression was that they wanted a proof that reflects and respects the procedure for which they invested a lot to obtain the answer, rather than building up an argument employing the answer as a working conjecture. Quite likely, if their presentation were shown to other students to refine, those students would be more inclined to take the induction method. This proposition illustrates that we should expect some differences in student behaviour when they are reacting to their own mental argumentation rather than that provided by others.

The proof stated was achieved by the students with guidance of one of the authors during the follow-up interview. The degree of guidance will not be described here; in accordance with the other two examples, the proof will be discussed hypothetically in terms of cognitive demands in producing it from the existing mental argumentation. First, notice that the proof involves the construction of families of sets. Although the importance of sets (and functions) to the foundations of mathematics is usually emphasized in teaching at the tertiary level, generally students tend to be poorly equipped to design sets for specific purposes. Returning to the example, the family of sets $A_i$ reflects the process that is implied in the mental argumentation; had the two students based their argumentation on these sets, the exposition of the solving algorithm would have been clarified. The family of sets $B_i$ had the role to model the situation given by the task environment. The $B_i$' s give the grounding, the $A_i$' s the calculating power. Thus the $B_i$ 's appear from theoretical considerations, and are related (in the form of their orders) to the $A_i$' s to realize the numeric expression sought. In this way, the translation from the mental argumentation to a proof presentation needed the construction of sets together with a strategic understanding how these sets would avail what was desired. We see then that proof production can involve significant problem solving aspects, as noted before.

## CONCLUSIONS

There is plenty of evidence that students experience severe difficulties in the production of mathematical proofs. A particularly frustrating circumstance for a student is when he/she can 'see' a reason why a mathematical proposition is true, but lacks the means to express it as an explicit argument in one form or another. One problem is that students feel that the 'reason' has to be immediately couched in 'rigorous' mathematical terms. In fact, there is no harm in trying to write informal descriptions, which can be a first step in developing mental argumentation ultimately giving access to 'mathematization'. The paper proposes a teaching / research tool designed to give students support in this process. This tool provides, beyond the stated aim of the task, an informal account how the aim might be achieved. This format has several advantages. One is that it should help students to regard mental argumentation as being legitimate. Second, mental argument comprises an environment that allows refinement of expression. Third, mental argumentation is not just a way of negotiating an entry into established mathematical systems, but even the writing of proof presentation is highly dependent on it, though its influence is usually left implicit.

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