DEFINING, PROVING AND MODELLING: A BACKGROUND FOR THE ADVANCED MATHEMATICAL THINKING

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This paper is a part of the large study that explores what 16-18 year old students have learnt with respect to defining, proving and modelling, considered as metaconcepts that constitute a background to the advanced mathematical thinking. In particular, we focus on the characterization of students' justifications and its persistence (or not) when making decisions related to tasks that involve those metaconcepts. Through the study, we have identified different types of considerations that underlie students' justifications. Our results have shown how students that maintain different types of considerations do not react in the same way to the same mathematical situations.

Key words: students' understanding, students' justifications, defining, proving, modelling

INTRODUCTION

The mathematical background of first year university students is an issue of concern and debate in our country. Throughout the last years, university mathematics teachers have been observing in the first year students a lack of understanding of basic mathematical ideas, which affects in a significant way the access to the mathematical advanced thinking. In order to improve this situation, some Spanish universities are offering courses of basic mathematics to students who want to access scientific and technological degrees. In this context, the highest grade (16-18 year-old students) of Secondary Education in Spain requires special interest. This grade is a non-compulsory level and its duration is two academic years. Among their aims is its importance as preparatory stage, which should guarantee the bases for tertiary studies.

Our study seeks to explore the understanding of students of the 16-18 level with respect to three metaconcepts that we consider fundamental in mathematics and didactics of mathematics: defining, proving and modelling. We consider them metaconcepts, due to their complex, multidimensional and universal configuration, admitting that each of them includes several aspects of very different complexity. In addition, we assume that they are key elements in the construction of the mathematical knowledge, and we decide to approach them jointly, since they contribute in different and interrelated ways to the above mentioned construction, and therefore to the students' learning process.

We want to emphasize that, at least in Spain, those metaconcepts are not explicitly mentioned in the school curriculum, but students approach them in an indirect way, through other mathematics curricular topics.

CONCEPTUAL FRAMEWORK

We think that the acquisition of intellectual skills is closely linked to sociocultural context (Brown, Collins & Duguid, 1989; Lave & Wenger, 1991). From this basic assumption, we approach students' understanding related to metaconcepts through:

- the use they make of the metaconcepts when they solve tasks in which the mathematical objects are those metaconcepts (metaconcepts are involved), and
- the justifications that they provide about their decision-making.

From a theoretical point of view, we needed to select some elements that allowed us accessing to that 'use' and those justifications.

With respect to the use, in an initial phase of our research we selected some elements that were considered the 'variables' of our study:

- identification variables, considered the *characteristics* that allow for a clear identification of metaconcept, and

- differentiation variables: *role*, representing different facets of the metaconcepts, and *type*, establishing differences inside them, including different systems of representation.

We think these variables are 'aspects' that can represent or describe in some way the metaconcepts and, furthermore, the relationship between the student and those aspects can inform us about his/her understanding of those metaconcepts.

These variables were specified for each metaconcept.

The variables in the case of defining. We considered "defining", among other characteristics, as prescribing the meaning of a word or phrase in a very specific form in terms of a list of properties that have to be all real ones. This prescription had characteristics that could be imperative (not contradictory, not ambiguous, and invariant under the change of representation, hierarchic nature) or optional (for example, minimality) (van Dormolen & Zaslavsky, 2003; Zaslavsky & Shir, 2005).

With respect to the differentiation variables, we selected the four roles mentioned by Zaslavsky & Shir (2005), which included: introducing the objects of a theory and capturing the essence of a concept by conveying its characterizing properties, constituting fundamental components for concept formation, establishing the foundation for proofs and creating uniformity in the meaning of concepts. In addition, we contemplated two types of definitions. Procedural type refers to what different authors consider definitions for genesis (Borasi, 1991; Pimm, 1993), which included what has to be done to obtain the mathematical defined object. Structural type referred to a common property of the object that is defined, or of the elements that constitute the object.

The variables in the case of proving. The contributions of different authors (Balacheff, 1987; Moore 1994; Hanna, 2000; Healy & Hoyles, 2000; Knuth, 2002; Weber, 2002) led us to include among the characteristics of proving the existence of both a premise / terms of reference / proposition and a sequence of logical inferences, which are accepted as valid characteristics by the mathematical community in the sense of 'not erroneous'.

Moreover, we took into account the five roles proposed by Knuth (2002). This author, on the basis of several roles identified by previous authors and proposed in terms of the discipline of mathematics, which he considered to be useful for thinking about proof in school mathematics, suggested the following roles:

> "to verify that a statement is true, to explain why a statement is true, to communicate mathematical knowledge, to discover or create new mathematics, or to systematize statements into an axiomatic system" (Knuth, 2002, p.63).

In addition, we identified three types: pragmatic proof, intellectual proof and formal proof. Pragmatic proof is restricted by the singularity of the event. That is, it fails in accepting the generic character and, in occasions, it depends on a contingent material that can be imprecise or depending on local particularities. Intellectual proof requires the linguistic expression of mathematical objects that intervene and of their mutual relationships. Lastly, formal proof makes use of some rules and conventions, universally accepted as valid by the mathematical community (Balacheff, 1987; García & Llinares, 2001).

The variables in the case of modelling. Mathematical modelling was characterized as a translation of a real-world problem into mathematics, working the mathematics, and translating the results back into the real-world context (Gravemeijer, 2004). Among the different roles, we included solving word problems and engaging in applied problem solving, posing and solving open-ended questions, creating refining and validating models, designing and conducting simulations, and mathematising situations. We selected two types: 'model of' and 'model for'. 'Model of' deals with a model of specific situations. 'Model for', deals with a model for situations of the same type (Cobb, 2002; Lesh & Doerr, 2003; Lesh & Harel, 2003).

With respect to the students' justifications, they have been considered in mathematics education from very different context and points of view (Yackel, 2001; Harel & Sowder, 1998). In particular, in our case they were analyzed according to the two main types of considerations identified by Zaslavsky and colleagues (Shir & Zaslavsky, 2002; Zaslavsky & Shir, 2005). Mathematical considerations included principally arguments in which mathematical concepts and relationships are involved. Communicative considerations were mainly based on ideas as clarity and comprehensibility, among others.

The part of the large study reported here focuses on the characterization of students' justifications and its persistence (or not) when making decisions related to tasks that involve the different metaconcepts.

METHOD

Participants

Ninety-eight students (aged 16-18 years) participated in this part of the study. They belonged to three different Secondary schools (A, T and C in the text) of three different towns, with no special characteristics in relation to their socio-cultural context. The role of teachers and schools was not considered in the part of research reported here.

Data collection

Our data source included questionnaires and semi-structured interviews for teachers and students. Considering the aims of this part of research, we focus on the results of students' questionnaire, we will detail only this research instrument.

The questionnaire consisted of an initial presentation followed by three parts (one for each metaconcept). These parts had in general lines the same structure. They included two types of statements to access to different aspects related to the way in which the students had constructed the different metaconcepts, so that they allowed gathering a variety of points of view (Healy & Hoyles, 2000).

In the first type of statements, students were asked to provide descriptions on every metaconcept, expressing in their own words the associated meaning, and including an example that they were considering more suitable.

The second type of statements presented different possibilities for each metaconcept according to the type and role (differentiation variables). These statements were related to two mathematical topics. They included three correct/incorrect expressions for each topic. The mathematical topics belonged to different mathematical domains (Algebra, Analysis and Geometry), and were practically extracted from the textbooks used at school. For example, with respect to the metaconcept defining, we selected three definitions of perpendicular bisector (mediatrix) and three of the greatest common divisor (they are not included due to the limitation in extension of this paper). The students had to indicate whether or not these definitions were correct, which one they preferred and which one they thought their teacher would prefer, giving reasons for each of their answers.

The initial version of the questionnaire thus obtained was then sent to five expert secondary teachers, who were asked to comment on the general structure of the set of statements, and to give comments and suggestions about specific items. Their comments were used to modify the formulation of almost every statement.

Next, the revised version of the questionnaire was piloted. For this purpose, a sample of 26 secondary students was chosen. These students belonged to one of the secondary schools that participated in our study, but they were not included in the final sample. According to the analysis of their answers, some items were subsequently deleted from the questionnaire, because the original formulation was ambiguous or unclear, or not provided important information. The final version of the questionnaire was administered to the 98 students.

Data analysis

The data in this part of the study consisted of individual students' written responses to the different items of the questionnaire. From a qualitative / interpretive approach, in a first step we followed an inductive and iterative process in which every response was divided in units of analysis. In a second step, these units were categorized depending on the type of considerations (mathematical or communicative) identified in the justifications. We exclusively considered the questionnaires belonging to students that had answered all the items. Because of that, only 67 were selected.

RESULTS

This section reports and discusses the results of the study and is organized around the two aforementioned research questions: the characterization of students' justifications and its persistence (or not) when they make decisions related to tasks that involve the different metaconcepts.

In the j*ustifications* provided by our students, we have found the two main types of considerations identified for Zaslavsky and colleagues (Shir & Zaslavsky, 2002; Zaslavsky & Shir, 2005). In addition, we have found some considerations on the basis on institutional-cultural aspects. This type of considerations was based in the context provided by schools that includes teachers, curriculum, principals and so on. The students identified as A217 and T17 (the first letter identifies the school, the following number the course (1 or 2) and, finally, the last numbers indicate the student) were representatives of this type of considerations:

Student A217: [I chose this…] because teachers explained it this way and this is how they taught me this topic

Student T17: Because that is how we were taught this topic at primary school and I have got used to it …..

With respect to the *persistence* of the students' justifications through the different metaconcepts, we have been able to identify:

- seventeen students that always followed considerations communicative or mathematical, independently of the considered metaconcept;

- six students that always combined mathematical and communicative (mathematical/commnicative) considerations, independently of the considered metaconcept;

- thirty-one students varied their considerations depending on the metaconcept. These considerations could be mathematical, communicative, institutional/cultural or they combined these types the considerations;

and

- thirteen students that used different considerations depending on the different statements in each metaconcept; in this case, we were not able to identify the type of consideration and they were not considered here.

In relation to the 17 students that maintained a common consideration, we show in the Table 1 the types of considerations identified and the corresponding students:

Types of considerations	***Students***
Communicative	A15,A16,A28,A213,A216,C16,C19,C120,C135
Mathematical	A25,T13,T14,T113,T114,T21,T25,C127

Table 1: Students that maintained communicative or mathematical considerations

The nine students situated in a communicative perspective considered their own person as the 'centre' of the considerations. The following excerpt is representative of this:

Student A16: I like statement 1 because it seems to be the easiest one for me

In general, communicative students' decisions were related with ideas as clarity, comprehensibility and so on. They saw *mathematics* and *teacher* (considered as a vehicle of communication between student/mathematics) from a very personal point of view.

In the case of the eight students situated in a mathematical perspective, their considerations were related to the use of mathematical expressions, lack of accuracy and so on. The following excerpt exemplifies this aspect:

Student A25: Statement 1 is not correct because it tells you what normally happens … in the majority of cases is the greatest number… but it doesn't not always have to be this way … it is incomplete ….

These students were able to consider separately the mathematical aspects from the personal aspects.

In addition, communicative students made a weak distinction of the identification variables (characteristics that allow the identification of a metaconcept). In relation to students situated in mathematical considerations, we can say that the majority of these students identified the incorrect expressions of the three metaconcepts, although they showed different degrees of accuracy in their mathematical arguments for justifying their decisions. The percentage of communicative students that were able to decide whether or not a statement on the different metaconcepts was correct was less than 40% in all cases. This percentage increased up to a 90% in the case of students that adopt mathematical considerations.

In particular, in the case of defining, 7 out of 9 communicative students chose both for teacher and students the same definition of mediatrix and the greatest common divisor, independently of characteristics, role and type and representation system. The communicative students did not see these characteristics as relevant because the centre was his/her own person. This result was also found in proving, with a slight difference between topics (7 of 9 and 6 of 9 in each case), and in modelling. This result differed in the case of mathematical students, who did not show a clear coincidence.

With respect to the thirty-one students who adopted different justifications depending on the metaconcept, the three main types of considerations (communicative, mathematical and institutional-cultural) were combined in some cases. We were able to identify several types of mixed considerations (communicative/ institutional-cultural, communicative /mathematical, mathematical/ institutional-cultural). We show in the Table 2 the students that were situated in each consideration.

	Proving	***Defining***	***Modelling***
Communicative considerations/mathematical considerations in each metaconcept			
Communicative	A210	A215, A217	A210, A215
	T11, T15	T112	T17, T112
	C12, C116, C119 C122, C123, C132 C138, C139		C119, C122, C123 C134, C138
Mathematical		A211	A29, A211
	T18, T19, T112 T22,T23,T29, T210	T12, T28, T119	T11, T18, T19 T115, T119 T22,T23,T28,T29
	C137	C116, C139	C129
Mixed considerations in each metaconcept			
Communicative /Institutional-cultural	A217		A217
	T17		
	C129	C119, C123,C134	
Communicative /Mathematical	A29, A211, A215	A29, A210	
	T12, T115, T119 T28	T11, T17, T18, T19, T115 T22, T29, T210	T12 T210
	C134	C12, C122, C129, C132, C137, C138	C12, C116, C132, C137, C139
Mathematical/ Institutional-cultural	T118	T15, T23, T118	T15, T118

Table 2: Students that varied their considerations depending on the metaconcept

As we can see in the Table 2, globally considered there were not significant differences between the number of communicative or mathematical considerations (23 and 26 respectively). The communicative/mathematical considerations (C/M) prevailed, being the most common in the three metaconcepts. Communicative considerations had a significant presence in proving and modelling with respect to defining.

In addition, 6 students (A14, A19, A21, C110, C126, and C130) maintained communicative/mathematical considerations in all metaconcepts. These students used communicative considerations when the focus of their justification was the relationship between the metaconcept and themselves; when the relationship was between metaconcepts and the teacher, the type of consideration was mathematical. We can say that in these cases those considerations were associated with the 'character' (student or teacher).

It is worth to point out to the great number of students that belong to the Secondary School T and who were situated in mathematical considerations. Although the reasons provided by the teachers in the large research have been very useful in explaining, from their point of view, some of the differences between the different Secondary Schools, as we mentioned above this is not the aim of the part the research reported here.

CONCLUSIONS

Our study examines three metaconcepts that we consider basic in the construction of students' mathematical knowledge. The findings suggest that the type of research instrument we designed has proven to be a valuable research tool in the identification of students' justifications.

Students' communicative and mathematical considerations proposed by authors as Shir & Zaslavsky (2002) for defining have been enlarged in the case of other metaconcepts as proving and modelling. In addition, the presence of institutional-cultural considerations showed in the other kind of justifications, which indicate the importance of the aspects linked to school context, that are considered as a 'source' for the justifications. Moreover, we were able to see the presence of mixed considerations (Communicative/institutional-cultural, communicative/ mathematical, and so on).

Our results have shown the students that justify their decisions on the basis of mathematical or communicative considerations do not react in the same way to the same mathematical situations. In particular, we have been able to see the difficulties communicative students have in making decisions both on distinguishing the characteristics of metaconcepts and on differentiating between the teacher and themselves, showing that their decisions are related to personal aspects. For mathematics teachers this fact implies the importance of considering the existence of students whose analytical tools are based on communicative aspects and the difficulties that means in helping them to construct other types of reasoning.

With respect to the findings related to the students that varied their type of considerations depending on the metaconcepts, they inform us about the necessity of going deep into the relationships among the motives that students have to link a specific type of considerations to a specific metaconcept. In some way, these relationships could inform us about some characteristics of students' understanding.

Finally, although it has not been considered in this paper, the differences among secondary schools that we have identified in our findings lead us to the need to incorporate in the design of future research some instruments that allow us to answer the following question: up to which point is the adoption of any determined consideration influenced by the specific education (training) of a secondary school and particularly by secondary school teachers? As researchers, we need to deepen the characteristics of the relationships between students and teachers in a specific secondary school that might encourage a determinate type of considerations.

NOTES

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