DERIVATIVES AND APPLICATIONS; DEVELOPMENT OF ONE STUDENT'S UNDERSTANDING

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This paper reports on a longitudinal observation study characterising student's development in their understanding of derivatives. Through the Dutch context-based curriculum, students learn this concept in relation to applications. In our study, we assess student's understanding. We used a framework for data analysis, which focuses on representations and their connections as part of understanding derivatives, and it includes applications as well. We followed students from grade 10 to grade 12, and in these years we administered four task-based interviews. In this paper we report on the development of one 'average' student Otto. His growth consists of an increasing variety of relations, both between and within representations and also between a physical application and mathematical representations. We also find continuity in his preferences for and avoidances of certain relations.

Keywords: Derivative, applications, procedural and conceptual knowledge, process-object pairs, case study.

INTRODUCTION

In the Dutch mathematics curriculum for secondary schools, the role of applications increased over the past 15 years. When the concept of the derivative is taught in grades 10-12, most textbooks provide students with opportunities to learn the concept in different contexts. Often an introduction in grade 10 starts with contexts related to velocity, steepness of graphs and, for example, increasing or decreasing temperatures. Textbooks provide tasks on the average rate of change, average velocity and the slope of a secant. The step towards instantaneous rate of change is kept intuitive, as most textbooks avoid the use of the formal limit definition, or only mention it on one page without using the notation with a 'limit'. Also in the conceptual extension of the derivative in grades 11 and 12, most chapters contain applications.

During their school time, students construct their knowledge of different concepts. One of these concepts is the derivative, which is not only a multifaceted mathematical concept, it also has relations to other school subjects. Knowledge of the derivative may support the learning of physics and economics, but physics teachers complain that students cannot apply what they have learned in their mathematics classes (e.g. Basson, 2002). In our research, we investigate which aspects of the concept derivative are becoming available to students, and whether and how students can relate the concept between different subjects such as mathematics, physics and economics. Our aim is to describe and analyse the development of students' understanding of derivatives, not just as a mathematical concept in itself, but as a mathematical concept in relation to applications.

THEORETICAL BACKGROUND

Understanding the concept of the derivative

It is complex to determine to what extent a student understands the concept of derivative. Many publications on understanding concepts use words such as scheme, structure, connections and relations. Anderson and Krathwohl (2001) define conceptual knowledge as: the interrelationship between the basic elements within a larger structure that enables them to function together. Thus, they perceive it as more complex and organized forms of knowledge. Procedural knowledge is defined as: methods of inquiry and criteria for using skills, algorithms, techniques and methods. Hiebert and Carpenter (1992) describe understanding in terms of the way, in which information is represented and structured. The degree of understanding depends on the number and strengths of connections between facts, representations, procedures or ideas. Connections can have different characteristics. In our analysis of students' connections, we identify procedural and conceptual knowledge. To describe a student's understanding of the derivative in relation to applications, we describe the connections made by a student (Roorda, Vos & Goedhart, 2007), distinguishing:
(i) Connections between mathematical representations,
(ii) Connections within mathematical representations and
(iii) Connections between an application and mathematical representations.
We will explore these three types of connections further.

Connections between representations

Hähkiöniemi (2006) discusses different viewpoints on representations. According to him, the traditional view on representations is that a representation is conceived as something that stands for something else, and representations are divided into external and internal ones (cf. Janvier, 1987). In his study Hähkiöniemi defines a representation broader as:

".. a tool to think of something, which is constructed through the use of the tool; a representation had the potential to stand for something else but this is not necessary. A representation consists of external and internal sides which are equally important and do not necessarily stand for each other but are inseparable." (p. 39)

As such, a gesture by a hand in the air can be a representation of a tangent. Without ignoring the existence of internal representations, we will follow the more traditional view, because external representations can be observed and they can be considered as external indicators of someone's internal representations. In different research the following representations are distinguished: formula, graph, table, words, physical background, gestures (Asiala, Cotrill, Dubinsky & Swingendorf, 1997; Hähkiöniemi, 2006; Kendal & Stacey, 2003; Kindt, 1979; Zandieh, 2000). Kendal and Stacey (2003) look especially at three mathematical representations: formula, graph and table. Students can talk about derivatives from a formulae viewpoint (such as rate of change), from a graphical viewpoint (slope), or from a numerical viewpoint (such as average increase).

Connections between representations and the ability to switch between these are important features for solving tasks (Dreyfus, 1991; Hiebert & Carpenter, 1992). Hähkiöniemi (2006) states that conceptual knowledge often refers to the making of connections from one representation to another. However, we will show in this paper that a connection between two representations can also have a more procedural character.

Connections within representations

As mentioned above, not only connections between representations but also within one representation are important (Dreyfus, 1991). For the derivative, Kindt (1979) distinguishes four levels within each representation. For example, in the formulae representation the four levels are: function, difference quotient, differential quotient and derivative, in the graphical representation: graph, slope of a chord, slope of the tangent and graph of the derivative. Zandieh (2000) indicates the steps between these four levels as process-object pairs, since each level can be viewed both as dynamic process and as static object. To illustrate the idea of process-object pairs we look at the second level of the formulae representation, the difference quotient. A difference quotient $\Delta y : \Delta x$ is a division, which can be viewed as a process: divide a difference in y by a difference in x. The outcome of this division, denoted by $\frac{\Delta y}{\Delta x}$, is a value which can be seen as an object. Likewise, in the graphical representation: the division of two lengths is the process, which results in an object, the slope of a chord.

Zandieh (2000) explains why the differential quotient and the derivative function both also can be viewed as process-object pairs. In the difference quotient a limiting process is involved, and 'the derivative acts as a process of passing through (possibly) infinitely many input values and for each determining an output value given by the limit of the difference quotient at a point.'

When a student makes connections between levels within a representation, Hähkiöniemi claims this to be mostly procedural. However, these connections can also be conceptual, for example in a graphical explanation of the limiting process.

Connections between applications and mathematics

The mathematical concept 'derivative' has relations with different applications. Thurston (1994) describes different ways of understanding derivatives. One way is to understand derivatives in terms of the instantaneous speed of $f(t)$ when t is time. Also, derivatives are used in physics lessons for concepts such as velocity, acceleration or radioactive decay, and in economics lessons for calculating maximum profits of marginal costs and revenues. Zandieh (2000) included a column *physical* into her framework. She argued that the context of motion serves as a model for the derivative. This extension can be made to other applications of the derivative as well.

Our research question in terms of the described framework is: what are characteristics of a student's development with respect to connections made between and within representations, and between applications and mathematical representations?

METHODOLOGICAL DESIGN

To study the development of students' understanding, we designed a longitudinal multiple case study with twelve students. Between April 2006 and December 2007, approximately every six month a task-based interview was conducted, yielding four interviews of 75 minutes with each student. In the interviews, we used think-aloud and stimulated recall techniques. The interviews were videotaped and transcribed.

The first interview was held before students were introduced to the theory of derivatives. Between the second and the last interviews, derivatives were a re-occurring topic in mathematics lessons. For this paper, we report on interview 2 (I-2) in November 2006 and interview 4 (I-4) in November 2007, because these contained the same five tasks, enabling us to compare in time. We will report on the work of one student, Otto. By zooming in on the work of one student, we can look more precisely at the solution strategies and statements of this student. We selected an average student with a positive attitude.

All tasks in the test dealt with the concept of derivative, but this was not explicitly mentioned. The tasks were designed to give students many opportunities to show their understanding of derivatives in different representations and applications. We describe three exemplary tasks, named *Emptying a Barrel*, *Petrol* and *Ball*.

Barrel: A barrel is emptied through a hole in the bottom (Figure 1). For the volume of the liquid in the barrel, the formula $V = 10(2 - \frac{1}{60}t)^2$ and its graph are presented. The question is to calculate the out-flow velocity at $t = 40$.

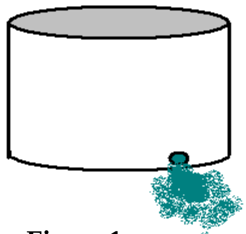

Figure 1

Petrol (Kaiser-Messmer, 1986): In a car an installation measures the petrol consumption related to the distance driven. The amount of petrol, used by a car, depends on the travelled distance. The task includes a graph and a table. $V(a)$ is the petrol consumption after a km. The question is to interpret $\dfrac{V(a+h)-V(a)}{h}$ (h is a value, which you can choose).

Ball: A ball falls from a height of 90 cm. A table, a graph and the formula for the height $h(t) = 0{,}9 - 4{,}9t^2$ are presented. The question is to calculate the velocity at a certain point.

Our analytic framework (presented in Roorda et al. 2007) contains elements of earlier frameworks of Zandieh (2000), Kindt (1979) and Kendal & Stacey (2003) In one dimension we have three mathematical representations: (a) formulae, (b) graphical; (c) numerical. In the other dimension we have the three object-process layers as connections between the four levels. See Table 1.

Table 1: Representations and levels of the concept derivative

	Formulae	**Graphical**	**Numerical**
Level 1	F1: f : function	G1: graph	N1: table
Level 2	F2: $\dfrac{\Delta f}{\Delta x}$ difference quotient	G2: average slope	N2:average increase
Level 3	F3: $\dfrac{\mathrm{d}f}{\mathrm{d}x}$ differential quotient	G3: slope of a tangent	N3:instantaneous rate of change
Level 4	F4 : f' derivative	G4: graph of derivative	N4: table with rates of change

To solve an application problem, students can choose which mathematical representation can be helpful. In this way, they make a connection between an application and a mathematical representation. In the table below, different non-mathematical representations are displayed, matching the format of the table above.

Table 2: Different applications

	General application	**Economics**	**Physics: velocity**	**Physics: acceleration**
Level 1	S1: $A(p)$: A depends on p	E1: TK total costs	Pa1: $s(t)$ displacement	Pb1: $v(t)$ velocity
Level 2	S2: $\dfrac{\Delta A}{\Delta p}$ average change of A	E2: $\dfrac{\Delta[TC]}{\Delta q}$ average increase of costs	Pa2: $\dfrac{\Delta s}{\Delta t}$ average velocity	Pb2: $\dfrac{\Delta v}{\Delta t}$ average acceleration
Level 3	S3: $\dfrac{dA}{dp}$ instantaneous rate of change	E3: $\dfrac{d[TC]}{dq}$ marginal costs	Pa3: $\dfrac{ds}{dt}$ instantaneous velocity	Pb3: $\dfrac{dv}{dt}$ for $t=a$ instantaneous acc.
Level 4	S4: $A'(p)$ derivative	E4: MC marginal costs	Pa4: $v(t)$ velocity	Pb4: $a(t)$ acceleration

The difference with earlier frameworks is that we operationalise understanding of the concept of the derivative through the *connections* between representations, within representations and between representations and applications. In our analysis, we use arrows (as connectors) to visualize the connections in the scheme above. During the problem solving process a student may switch, for example, from a function (F1) to the derivative function (F4), yielding the code F1→F4. Another difference is the role of applications: these are not only viewed of as a support for understanding mathematics, but also as a part of other school subjects. When, for example in an economic problem, a student focused on the graph, drew a tangent line, and calculated the slope, without economic interpretation, we will denote this as: E1→G1→G3. However, when a student solves a problem by calculating marginal costs, without mentioning relations with functions, graphs or table, we will denote this as E1→E4→E3.

RESULTS

In this section, the analysis and coding of students' strategies in terms of our framework is illustrated by looking at the task *Barrel*. In Table 3 we summarise Otto's work on this task during I-2 and I-4.

Table 3: Otto's typical statements and activities; Associated codes for Otto's connections; task *Barrel*

Interview 2 (I-2)	Interview 4 (I-4)
Otto: *I have to calculate the velocity at that point* [plots the graph and uses the option 'Tangent' of his graphing calculator. In the window of the calculator the tangent appears and the formula $y = -0{,}4428191485x+35{,}49$..] Otto goes on to say: *I think I have to differentiate, I get the formula of the tangent by differentiating*. He calculates the derivative, without using the chain rule, fills in $t = 40$, makes a calculation error, writes down $V'(40) = -493{,}333$. To check his answer, Otto tries to calculate the average out-flow velocity of the tank over the whole period, by a self-made rule: $\frac{begin + end}{2}$	Otto calculates the derivative with some errors: $V'(40) = 59{,}8$. He discovers a miscalculation, corrects his answer into −555,56 litre per minute. To check his answer, Otto draws a tangent into the graph of the task and calculates $\frac{\Delta y}{\Delta x} = \frac{35-0}{80}$ = 437,5 l/m. He says: *This is a bit imprecise. I think it is possible. […] you can check with a graphical calculator by drawing a tangent.* Otto plots the graph and the tangent: [O writes down: GR→ tangent(40)→-0,444x+35,56] He writes down 444,4 l/min. He thinks he made a miscalculation in the derivative.
Connections interview 2 S1→F1→G1→G3: use of formula; plots the graph; plots tangent S1→F1→F4→F3→S3:derivative (with error); derivative at t = 40; back to application	**Connections interview 4** S1→F1→F4→F3→S3: formula; derivative (with error); fills in t = 40; back to application S1→G1→G3→S3: graph; tangent; application F2→G2 slope of tangent with $\frac{\Delta y}{\Delta x}$ F1→G1→G3→S3 graph, tangent; application

Some observations: Otto used in I-2 and I-4 similar solution methods, such as differentiating the formula and plotting the tangent. Differences are also visible, for example in I-4 Otto checked his solution additionally by drawing the tangent on paper. Also, the connection between applications and mathematics G3→S3 was added, because Otto interpreted the slope of the tangent in terms of the application. In table 4 the same overview is given for the tasks *Ball* and *Petrol*. We will analyse the data of these three tasks by examining the connections between representations, within representations and between application and mathematical representations.

Connections between representations

In I-2 the connection F1→G1 is frequently observed. In the tasks, Otto used the given formula as a starting point to plot a graph on his graphical calculator. Only one time we saw Otto make a table with his graphing calculator. Throughout I-2, Otto made a connection between derivative and tangent (F3/F4 →G3), but he could not explain this relation precisely. He said, for example: *When you differentiate you get the formula of the tangent* (see Table 3) and: *to approximate the tangent, you use the formula* $\frac{V(a+h)-V(a)}{h}$ (see Table 4).

Table 4: Otto's typical statements and activities; Associated codes; tasks *Ball* and *Petrol*

Interview 2	Interview 4
Otto reads the task ***Ball*** and says: *I think I have to use a derivative.* He calculates the derivative but he fills in t = 2,4 instead of t = 0,24. Then he says: *When you differentiate you get the formula for the tangent, and that corresponds to the velocity, I think.* On his graphing calculator he plots a graph and a tangent but after a long silence he states: *I don't get any wiser from this.* **Connections:** Pa1→F1→F4→F3 formula; derivative; fills in a wrong value for *t*. F1→G1 →G3 graph; tangent	Otto thinks he can calculate the velocity of the ***ball*** by the formula $v = \Delta x / t$. He calculates the average velocity over de first 0,24 seconds. This is followed by some confusion because Otto thinks the ball also moves horizontally. When de interviewer asks him to check his answer, Otto calculates the derivative. This answer is better, according to him, because in it he recognizes the derivative 9,8 as the gravity acceleration. He also says: *I could draw a tangent and calculate the slope of it.* At last Otto mentions a method with kinetic energy, but for that he needs the mass of the ball. **Connections:**Pa1→F1→F4→F3→Pa3: formula; derivative; fills in a value for *t*; velocity G1→G3 slope of tangent
Statements of Otto in the task ***Petrol*** *It's the oil consumption at that point.* *On a small interval it becomes precise.* *On a small part you can approximate the tangent.* *Differentiating is for the formula of the tangent.* *It is a specific value for the tangent* *How many liters per kilometer he uses* (F2→S2)	Statements of Otto in the task ***Petrol*** *It is the approximation on a certain point;* *It is a certain slope, when you take a small h you calculate exactly the slope at a certain point* (F3→G3; F2→F3); *You get the consumption very precisely;* *When h is larger it is the average consumption over a certain distance.* (F2→S2); *It is a formula to calculate the consumption over a certain period of time.*

Compared to I-2, in I-4 we observed more relations between representations, also at different levels of the concept. Otto more often used the given graph to solve the task. In I-4 Otto stated that the value of the derivative equals the *slope* of the tangent. He also made a connection between the formula of the difference quotient and the slope of a secant. He never used the numerical representation.

Connections within representations

Both in I-2 and I-4, we often coded the connection between levels F1→F4→F3 and G1→G3. These two connection strings (calculating a derivative and plotting a tangent) were standard procedures for Otto, displaying a strong procedural understanding, but in I-2 Otto cannot yet explain this relation accurately.

In the tasks *Barrel* and *Ball,* Otto never mentioned the difference quotient at a small interval or slope of a secant; the tasks obviously did not activate his potential knowledge of the limiting process of the derivative (connections within level 2 and 3) although the task *Petrol* gave ample opportunities to reason about the impact of a larger or smaller *h*. In both interviews, Otto was unable to explain the formula precisely, but in I-4 Otto made more correct statements than in I-2 (see table 4). As we see in I-4, Otto tried to explain the limiting process, but even in I-4 his formulations are not very accurate.

Connections between applications and representations

In I-2 Otto connected derivative, tangent and velocity, when saying: *"When you differentiate you get the formula of the tangent, and that corresponds to the velocity, I think."* Nevertheless, Otto did not accurately put these concepts together. In I-4 Otto mentioned and used more relations between formula/graph and applications. He interpreted the tangent-formula correctly to find the velocity of the ball, and in the *Petrol*-task the link between the mathematical notation and the application is correctly described by Otto.

In I-2 Otto did not connect mathematical and physical methods (such as using the formula $v = a \cdot t$). A year later, in I-4 Otto made a few remarks, in which he connected mathematics and physics. For example, Otto noticed that in the derivative $h'(t) = -9{,}8t$ the value 9,8 is the acceleration of gravity, and he mentioned a calculation method using kinetic energy. In I-4 Otto stated (in another task): *"the derivative is the formula for the velocity, and the second derivative is for* ***distance moved*** *[..] Once, my math teacher* ***gave this as notes****."* This is an incorrect formulation, because Otto meant 'acceleration' instead of 'distance moved'.

CONCLUSIONS AND DISCUSSION

This study uses a case study methodology, the focus of the data analysis is on the student as an individual. From individual results we can not prove any generalizations, which is clearly a limitation of this paper, but we can find counterexamples and existence proofs.

In this paper, we reported on Otto's development in understanding the derivative. Compared with I-2, we measured in I-4 an increased number of connections, both between and within representations. Connections made in I-2 reoccurred in I-4. Otto's preference for the graphical and the formulae representation was continued in I-4 and also his avoidances of the numerical representation. The preference for graphical representation corresponds to research by Zandieh (2000), who observed that six out of nine students prefer the graphical representation in tasks and explanations about derivatives. In the case of Otto, we saw that this preference prevailed throughout the learning process.

In I-2 at several occasions, Otto equalled the derivative to the tangent, instead of 'the slope of the tangent'. This was not a slip of the tongue, because Otto repeatedly displayed an incorrect idea about the connection between 'tangent' and 'derivative'. This phenomenon is also reported by Asiala et al.(1997) and Zandieh (2006). In addition to the research of Zandieh, we see that Otto's misstatements hinder him during problemsolving. A year later in I-4, Otto knows that the derivative yields the *slope* of the tangent, so his understanding of the formula of a tangent is corrected.

Basson (2002) reported that physics teachers frequently complain that students cannot use what they have learned in their mathematics classes. In the case of an average student such as Otto, we observe indeed difficulties to connect mathematics and physics correctly. Although there is some progress in the accuracy of statements, for example in recognizing the gravity acceleration, the use of the rule 'derivative is velocity', his understanding of these connections stays weak.

Otto improved his procedural knowledge. Although he often uses the same procedures, especially plotting the graph (F1→G1), plotting a tangent (G1→G3), or calculating a derivative at a point (F1→F4→F3), he seems to be more certain of his work and he is more sure about the connections between the different procedures. On the other hand, a recurring feature with Otto was that he sometimes chose an incorrect method, for example in the task *Ball*, in which he calculates in I-4 an average velocity instead of an instantaneous velocity, without any corrections on his work.

Between I-2 and I-4, his conceptual knowledge increased. In I-4 Otto could explain relations between mathematics and physics to a certain extent, the connection between tangents and the derivative function improved and he connected more frequently to the levels 2 and 3 of the derivative. On the other hand, the connections made were not verbally well explained and some possible connections were not mentioned. So his conceptual knowledge increased, but nevertheless remained weak.

We have used a framework for analysing students' understanding of the derivative in application problems. The resulting arrow-schemes describe students' strategies in a structured way by indicating patterns between cells of the table (see table 1). This facilitates the interpretation of students' statements and operations. Our framework also gives a clear description of transitions between applications and mathematical representations, which students make during problem solving. We added notes on procedural and conceptual knowledge displayed by the students. A challenge remains to use students' misstatements, which are presently not described although these can be indicators of students' understanding.

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