SECONDARY-TERTIARY TRANSITION AND STUDENTS' DIFFICULTIES:

THE EXAMPLE OF DUALITY

Martine De Vleeschouwer

Unité de support didactique, University of Namur (FUNDP) – Belgium

Abstract : *We are presenting a study about duality and its learning in linear algebra. We have elaborated a device of follow-up of knowledge and difficulties of students enrolled in first-year university mathematics or physics programs, concerning this theme. We are presenting the results of this device categorizing students' difficulties. We present moreover a perspective on transition allowing us to interpret students' difficulties in duality in terms of transition.*

Key-words : *linear algebra, duality, tertiary level, institutional transition*

1. INTRODUCTION AND THEORETICAL FRAMEWORKS

The study presented here focuses on the teaching of duality at university. This work is thus naturally related with WG12 theme "Advanced mathematical thinking (AMT)" of CERME6, and is more precisely connected with the sub-theme "Effective instructional settings, teaching approaches and curriculum design at the advanced level".

Duality is taught in most countries only at tertiary level, and is even more 'advanced' than elementary linear algebra. One aspect of our contribution is to precise possible meanings of 'advanced', in order to enlighten students' difficulties, a necessary step before proposing a teaching design.

From an epistemological point of view, duality takes a central place in linear algebra. Indeed, the notion of rank, essential in linear algebra, has first emerged in what Dorier terms the *dual aspect*, meaning the smallest number of linearly independent equations (Dorier 1993, p. 159).

Even if since the mid-eighties didactical works are interested in linear algebra, they mostly concern elementary notions of this part of mathematics (Dorier 2000, Trigueros & Oktac 2005,…).

However, when the duality is studied as an object (Douady 1987) in a course of linear algebra in first year of university, we notice that the students are confronted with numerous difficulties. Our main objective is to understand the origin of these difficulties, and to be able, in a later work, to propose adapted teaching devices.

In our work, we try, in a first step, to identify different kinds of difficulties, according to mathematical content that can be problematic, and after to interpret these difficulties from an institutional point of view. So we try to answer the following questions :

- What are the difficulties tied to duality itself, those that are linked more generally to linear algebra, or also to other connected contents?
- How can we interpret these difficulties, which hypotheses can we do about their causes ?

Our work, beyond duality, also has for objective to enlighten the specific difficulties of novice university students. These difficulties have already been the object of numerous works (Artigue 2004, Gueudet 2008). Here we adopt an institutional point of view (Chevallard 2005). The difficulties don't only result from the fact that new knowledge is met. They can be caused by the fact that the same knowledge will be differently approached in the secondary school institution and in the undergraduate institution. So a same *type of tasks* can be associated with a new *technique, to solve the corresponding exercises* ; a same technique will be differently justified… So, in our research, we use the « praxeology » notion, also named « mathematical organization », introduced by Chevallard (2002). He defines a punctual mathematical organization as an union of two blocks [Π / Λ], each one containing two parts. The first block, Π = [T / τ], named « practico-technical » block, is made of a type of tasks T and a technique τ allowed to realize tasks related to type T. The second block, Λ = [θ / Θ], named « technologico-theoretical », is made of a *technology* θ, which is a discourse justifying the technique τ, and a *theory* Θ justifying the technology θ. A complete mathematical organization is then an organization that we can note [Π / Λ] or [T / τ / θ / Θ].

Let us illustrate these concepts by an example. Suppose we propose to a student to solve the following *exercice*: « Compute the dual basis of the canonical basis of $\mathbb{R}^4$ ». We can say that this exercise is related with the *type of task T* « given a n-(sub-)vector space E and one of its bases, to determine the dual basis of the given basis ». A *technic* τ associated with this type of tasks T consists in solving n systems (i = 1,…,n) of n equations in n unknowns (α_{ip}): $\begin{cases} \sum_{p=1}^{n} \alpha_{ip} x_{1p} = \delta_{i1} \\ \dots \\ \sum_{p=1}^{n} \alpha_{ip} x_{np} = \delta_{in} \end{cases}$ where x_{jp} are the coordinates of the j^{th} vector of the given basis. This technic τ is justified by a discourse, called *technology* θ : « To find the dual basis, firstly define the general expression of any linear form y in the given space : $\forall x \in E, y(x) = \sum_{p=1}^{n} \alpha_p x_p$ where x_p are the coordinates of a vector x in E. Then solve n systems of n equations in n unknowns : $\forall i, j = 1,\dots,n : y_i(x_j) = \delta_{ij}$ where x_j are the vectors of the basis given in the type of task ». This technology θ is justified by the *theory* 1 : « Given E an n-vector space, and $\{x_i\}_{i=1}^{n}$ a basis of E. Then there is a basis $\{y_i\}_{i=1}^{n}$ of the dual space E' so that

$\forall i, j = 1,...,n : y_i(x_j) = \delta_{ij}$. The defined basis $\{y_i\}_{i=1}^n$ is also called the dual basis associated with a basis of the primal space E ».

We also use a framework proposed by Winsløw (2008), especially focused on "concrete-abstract" transition issues, and drawing on praxeologies. Winsløw considers that when a student arrives in an undergraduate institution, he/she is confronted with two types of transition. The first type of transition origins in the secondary school's teaching, where almost only the block « practico-technic » intervenes. The first transition that a student meets changing institution, is that at university, the « technologico-theoric » block is also present, completing the mathematical organizations. But a second transition appears when the recently introduced elements of « technologico-theoretical » block also become objects that the students have to manipulate, constituting then the « practico-technic » block of new mathematical organizations. We will explain why the learning of duality in linear algebra at university depends of this second type of transition.

In this article we present the analysis of responses to a survey that has been proposed to students enrolled in first year university mathematics or physics programs in the University of Namur (Belgium) concerning duality. In a first step (part 2), we describe the survey. Then in part 3 we present the analysis of the survey's results.

2. DESCRIPTION OF THE SURVEY

In (DeVleeschouwer 2008), we describe how the teaching of the duality in linear algebra is structured, focusing on the concepts of dual (as vector space), linear form, dual basis, annihilator and transposed transformation. Through the analysis of various textbooks (books and course notes), we have analysed the duality as an object (Douady 1987) of teaching in the university institution. We also studied the different aims of the tool function of the duality : we distinguished the analogy-tool, the resolution-tool, the illustration-tool, the definition-tool and the demonstration-tool for duality.

Thanks to these analyses we have designed a survey addressed to students enrolled in first year of university, meeting the teaching of duality in linear algebra. This survey, which focuses on the duality in its 'object' aspect (Douady 1987), is based on the elements identified in the analysis of textbooks, and will enable us to precise the difficulties faced by the students.

This survey contains two parts :

- The first one is constituted of a *questionnaire*. 37 students enrolled in the first year of mathematics or physics programs at the University of Namur answered to this (written) questionnaire (February 2008). The students had two hours to answer it. Some interviews allowed to highlight the answers brought to the questionnaire for 16 of these students (May 2008).

- The second part of the survey is a *group work*. 23 students enrolled in the first year of mathematics programs took part of this group work. The students, divided in four groups of 5 or 6, had 5 weeks to return a written report about the asked work. It was recommended then to consult an assistant during the two first weeks of their work ; and an interview (varying from 30 to 90 minutes) was mandatory when giving the written report (March 2008).

Before the survey, the students have already seen, in the theoretical course and in the exercises, the vector spaces (algebraic structures, linear dependence and dimension, sub-vector spaces) ; the linear applications, the associated matrices; the linear forms, and also the dual space (and bases) and the reflexivity; the linear and transpose transformations. The theoretical course had already approached determinants (without exercises).

We have to precise that in the secondary school Belgian pupils have only approached the vector's notion at the *geometric level* (Hillel 2000, p.193). The notion of transpose was only presented to the pupils of the secondary school who specialize in mathematics, principally when approaching the definition of the inverse matrix (using the transpose of the cofactors *matrix*).

2.1. THE QUESTIONNAIRE

The questionnaire (appendix 1) comprises two parts, each one composed by the same questions but contextualized in different frames. The two chosen frames are the vector space $\mathbb{R}^4$; and the frame of matrices with real coefficients, particularized to 2 by 2 matrices ($\mathcal{M}_{2x2}$).

The different types of tasks (Chevallard 2005) associated with the exercises proposed in the questionnaire are described in (De Vleeschouwer 2008). We only propose here a short description of types of tasks present in the questionnaire :

- « Example of linear form », noted **T_Exemp_FL** : given a (sub-)vector space, give an example or counter-example of a linear form.

- « General expression of a linear form »,noted **T_ExpGen_FL** : given a (sub-)vector space, describe a general expression of a linear form defined on the studied space.

- « Primal and dual basis », noted **T_Base_P&D** : given a *n*-(sub-)vector space and a set of *n* vectors of the considered vector space, determine if this set is a basis of the vector space and if it is, to find the dual basis.

For the rest of or study, we had to subdivide the type of tasks T_Base_P&D into sub-types of tasks :

- « Primal basis », noted **ST_Base_P** : given a *n*-(sub-)vector space and a set of *n* vectors of the considered vector space, determine if this set is a basis of the vector space .
- « Dual basis », noted **ST_Base_D** : given a *n*-(sub-)vector space and a set of *n* vectors of the considered vector space, determine its dual basis.

- « Coordinates functions », noted **T_FctCoor** : given a basis and its dual basis, determining the coordinates of a vector from the primal vector space.

- « Definition of the transpose transformation», noted **T_Def_TTransp** : given a linear transformation defined on a (sub-)vector space, to define its transpose transformation.

2.2. THE GROUP WORK

The group work (GW) is composed of several parts, that we will not present in details in this article. The two first parts of the GW are corresponding to the questionnaire. What follows complements then the questionnaire, notably :

- asking for the relation between the two parts of the questionnaire ;

- taking the same plan that the two parts of the questionnaire, but in the algebraic theoretical frame because « il s'agit de proposer des apprentissages qui portent sur divers cadres à propos d'une même connaissance »[1] [Robert 1998, p.155]. Knowing that « ce n'est pas toujours le travail dans un cadre général, formel, qui est le plus difficile »[2] [Robert 1998, p.151], we adapt the common plan of the two parts of the questionnaire notably with bringing new types of tasks for the algebraic theoretical frame. For example, concerning the transpose :

- « Representation of the transpose », noted **T_Repr_TTransp** : explain, choosing one or several semiotic representation registers, what represents the transpose transformation. We want to know if the students think that the transpose transformation is defined on the dual space, or if they feel that the transpose transformation applied to a linear form is in fact the compound of the linear form and the initial transformation.
- « Properties of the transpose », noted **T_Prop_TTrans** : establish or prove transpose's properties. Especially, we ask the students if it is possible to claim that $(f^t)^t = f$. They have then to justify their answer. That question allows us to investigate the students' perception about the relation between the bidual and the primal and more especially about the canonic isomorphism between these two finite-dimensional spaces.

3. RESULTS OF THE SURVEY

The first observations of the analysis of the student's answers to the survey lead us to perceive different natures of students' difficulties when learning duality. Drawing on this analysis, and on our analysis of the way duality is structured in textbooks, and articulated with linear algebra (DeVleeschouwer 2008), we have chosen to classify the appeared difficulties in three main categories: difficulties tied to an insufficient mastery of elementary concepts of linear algebra, difficulties common to the

[1] " We have to propose learnings which concern diverse frames about the same knowledge."
[2] " It is not still the work in the general, formal frame, that is most difficult."

elementary linear algebra and duality, and finally difficulties specific to duality. Naturally, intersections between these categories are possible.

Some difficulties, obviously, are even more general : for example, we observed a confusion between a function *f* and the value of the function in an element of the departure's space : *f(x,y,z,t)*. Another well-known fact is that mathematical writing is not mastered by the students yet (obstacle of formalism, Dorier 2000). We don't detail here these types of difficulties, preferring to focus on linear algebra.

All the listed difficulties can be analyzed from an institutional point of view (the same object is differently considered in different institutions). In particular, we shall show (section 3.2) that the difficulties listed in the third category can be interpreted in term of second type of transition (Winsløw 2008).

3.1. OBSERVATION AND CLASSIFICATION OF DIFFICULTIES IN DUALITY

3.1.1. Insufficient mastery of elementary concepts of linear algebra

By elementary concepts of linear algebra, we mean concepts considered as elementary *with regard to* the notion of duality which we study.

Let us consider for example the notion of linear application or linear form. Indeed, only 62% of the students who answered to the questionnaire give a correct example of linear form within the frame of $\mathbb{R}^4$. This rate decreases to 27% in the matrix frame.

The students also have difficulties to build examples of vector spaces. They propose for example the set of polynomials of degree 3; or still the set of polynomials of degree superior or equal to 3. Asking the students to design for examples, is frequent at the university, and hardly present at secondary school; it is thus difficult for novice students (Praslon 2000).

We can also notice that generally speaking, the students prefer to work within the frame of $\mathbb{R}^4$ rather than within the frame of matrices. The exercises corresponding to the various types of tasks are also better solved there. The vector space of the 2x2 matrices is not familiar to the students. In the University institution, it is necessary to consider objects recently defined in linear algebra as familiar objects on which and from which we are going to work. For example the fact that the object *matrix* can be considered as an element of a vector space, that's to say a vector. We can thus consider the coordinates of a matrix, or define linear applications acting on matrices.

Being able to change frames is important for the learning of a notion. In the case of duality this requires in particular the knowledge of several vector spaces.

3.1.2. Difficulties common to linear algebra and duality

We also observe difficulties common to elementary linear algebra and to duality, for example the confusion between a vector and its coordinates. This confusion, well known in linear algebra (Dorier 1997), becomes crucial when learning duality.

Within the framework of 4-tuples, we could say that the confusion between vectors and coordinates is natural or unnoticed. We can think that it is one of the reasons for which the students privilege this frame in the questionnaire. We notice that the students tend to work with the coordinates of objects (vectors, matrices, linear forms) and not with objects in themselves. So, it is frequent to see appearing in the answers the equality between the i^{th} linear form of dual basis (often noted y_i by the students) and the 4-tuple taking back its coordinates in the canonic base (that the students nevertheless learnt to note $[y_i]^{e'}$).

Another problem that we identified is the fact that the students prefer to present the solution of an exercise as an element of the vector space being of use as frame to the task ($\mathbb{R}^4$ or $\mathcal{M}_{2x2}$). So, during the resolution of exercises corresponding to the type of task T_FctCoor, concerning the computation of the coordinates of an element (quadruplet or matrix) of the considered vector space, it is frequent to see students presenting calculated or deducted coordinates (in the second part of the questionnaire by analogy with regard to the first part) as a 4-tuple or as a matrix.

So, the only student having correctly solved the exercise corresponding to the type of task T_TTransp within the framework of 4-tuples ends then his answer by identifying $f^t(y)$ with a 4-uplet containing his coordinates in the canonic dual basis, without mentioning however these are coordinates in this basis. In the matrix frame, this student presents the transpose in the form of matrix.

3.1.3. Difficulties directly related with duality

We can also classify difficulties directly related with duality, often connected with the very abstract character of the involved objects. It will lead us naturally to the following section dealing with the “concrete-abstract” transition (Winsløw 2008).

The definition of the transpose transformation can illustrate our comments because it is about a transformation defined on a vector space which elements are linear forms.

So, during the resolution of an exercise corresponding to the type of tasks T_Def_TTransp, within the frame of 4-uplets, three students mix up the transpose transformation with the inverse transformation. They have a general idea of a “reverse” process, associated both with inverse and with transpose. We also can notice, within the frame of $\mathbb{R}^4$, that some students don’t even try to work with the given transformation : they only give the theoretical definition of the transpose or another explanation onto what they think the transpose should be, without trying however to resolve effectively the proposed task. For these students, the transpose is only a part of the abstract world, and they don’t manage to mobilize it in a contextualised frame.

Within the frame of the 2x2 matrices, we find almost the same proportion of students working with the given transformation among the students trying to solve the question corresponding to the type of tasks T_Def_TTransp. But in this frame, the answers are more varied because the students associate the proposed type of task with a notion approached on the institution secondary school in Belgium : the transpose matrix. For example, to resolve an exercise depending from the type of tasks T_Def_TTransp in the matrix frame, some students simply take back the matrix which is given to them in the statement and transpose it. The notion of matrix dominates on the notion of application when the term “transpose” is used.

3.2. « CONCRETE-ABSTRACT » TRANSITION

The difficulties directly related to duality presented in the previous section can be interpreted in terms of "concrete-abstract" transition (Winsløw 2008), which corresponds to the second type of transition described in the section 1. According to Winsløw, in the secondary school institution, it is essentially the "practico-technical" block of the mathematical organizations that is worked. This coincides with what we can notice when we analyze the answers of the students who were asked to say, in the work group, if there is, according to them, a link between the first two parts ($\mathbb{R}^4$ frame and matrix frame). The students concentrate themselves on the practico-technical part of mathematical organizations described by Chevallard (2005), and generally let down the technologico-theoretical block. Indeed, students answer that “both exercises represent the same transformations in two very similar vector spaces” and that “the question 2 is exactly the same than the question 1, there is only their representation which changes”. By using the term “similar”, the students do not identify the vector spaces, but indeed elements constituting the vectors of each of these two spaces. The students notice that only the “representation changes”. We can suppose that by writing it, the students think of applying identical techniques (computation of dual basis,…) to the various proposed statements. Always concerning the link between both parts of the questionnaire, the other students say, in the end, that "we find the same solutions". They fall again into the practico-technical block : according to them, the numerical values appearing in the solution are the most important. They do not mention the isomorphism used to justify this practice.

In the University institution, the technologico-theoretical block takes more importance. It is a first transition. Some students already adapted to this evolution. To illustrate our comments, let us turn to the exercises corresponding to the types of tasks T_Exemp_FL and T_ExpGen_FL. Even if these exercises did not a priori require any justification, a student justifies explicitly the fact that the supplied example is a form and also that the linearity is verified.

A second transition appears when elements constituting the technologico-theoretical block of a mathematical organization become elements on which calculations will be made and in which techniques are going to be applied. These elements constitute then the practico-technical block of new mathematical organizations. It is what happens when we work with the duality as an object : linear forms are considered as vectors because the set of linear forms is a vector space. The theories developed on the dual justify techniques applied to the linear forms. But when we consider the transpose transformation, the dual shifts from the technologico-theoretical block of a previous mathematical organization to integrate the practico-technical block of a new mathematical organization, because the dual is then considered as the departure space of the transpose transformation. According to Winsløw, this second transition is even more difficult than the first one. Indeed, concerning the type of tasks T_Def_TTransp for example, we observe that the students have difficulties to define correctly the departure space of the transpose transformation.

However, when we ask the students, in the group work, if we can assert that $(f^t)^t = f$, we notice that the question is very well answered by all groups. To solve a task of the type T_Prop_TTrans presented in an algebraic theoretical frame, the students choose, rightly, the technologico-theoretical block. For the transpose of the transpose, the students agree spontaneously to look for the solution in the theory. Sometimes, to make the link between the theory and the examples is more difficult than to stay in the theory.

4. CONCLUSIONS, DISCUSSION AND PERSPECTIVES

We classified the difficulties observed in the students' answers in three principal categories: the difficulties tied to an insufficient mastery of elementary concepts of linear algebra, those common to the elementary linear algebra and duality, and finally those specific to duality. We have seen, particularly, that the movement from elementary linear algebra to duality can be interpreted as a transition, according to Winsløw's meaning (2008). This confirms that transitions exist beyond the precise moment of the university's entry.

So, proposing a teaching device which searches to improve the learning of duality, asks to sit solid bases of linear algebra, and to devote specific attention to very abstract concepts as the transpose; but also to think about transition between elementary linear algebra and duality.

We will use these facts to propose an experimental teaching of duality in first year of university, in a further stage of our work.

5. REFERENCES

Artigue M. (2004). *Le défi de la transition secondaire-supérieur. Que peuvent nous apporter les recherches en didactique des mathématiques ?* Actes du premier congrès franco-canadien de sciences mathématiques, Toulouse.

Chevallard, Y. (2005). *Steps towards a new epistemology in mathematics education*. Opening conference at CERME4, Sant Feliu de Guixols (Spain), february 2005.

DeVleeschouwer, M. (2008). *Contribution to the study of the duality in linear algebra. First approach.* Proceedings of ICME11, Monterrey (Mexico), July 2008 (to be published).

Dorier, J.-L. (1993). L'émergence du concept de rang dans l'étude des systèmes d'équations linéaires, *Cahier du séminaire d'histoire des mathématiques* (2ème série) 3, 159-190.

Dorier, J.-L. & al (2000). *On the teaching of linear algebra*. Mathematics Education Library, Kluwer Academic Publishers.

Douady, R. (1987). *Jeux de cadres et dialectique outil/objet*, Recherches en didactique des mathématiques, 7/2, 5-32.

Gueudet, G. (2008). Investigating the secondary-tertiary transition, *Educational Studies in Mathematics, 67/3, 237-254.*

Hillel, J. (2000). Modes of description and the problem of representation in linear algebra, in Dorier, J.-L. (ed.) *On the teaching of linear algebra* (pp.191-207). Mathematics Education Library, Kluwer Academic Publishers.

Praslon, F. (2000). *Continuités et ruptures dans la transition terminale S/DEUG Sciences en analyse. Le cas de la notion de dérivée et son environnement*. Thèse de doctorat, Université de Paris VII Denis Diderot.

Robert, A. (1998). Outils d'analyse des contenus mathématiques à enseigner au lycée et à l'université, *Recherches en didactiques des mathématiques, 18/2*, 139-190.

Trigueros, M. & Oktaç, A. (2005). La théorie APOS et l'enseignement de l'algèbre linéaire. *Annales de Didactique et de Sciences Cognitives, Vol. X*, 157-176.

Winsløw, C. (2008). Transformer la théorie en tâches : la transition du concret à l'abstrait en analyse réelle, in Rouchier A. et Bloch, I. (ed.) *Perspectives en didactique des mathématiques. Cours de la XIIIième école d'été de didactique des mathématiques*, La Pensée Sauvage, Grenoble.

APPENDIX 1 : Questionnaire

To answer the questions below, you may use as you prefer, the formal mathematical language, the French language, graphics or drawings,…

1. Consider the vector space, built on the field of reals.
   a. Give an example on a linear form defined on $\mathbb{R}^4$.
   b. Give the general expression of a linear form defined on $\mathbb{R}^4$.
   c. Given $x_1 = (1,2,0,4)$, $x_2 = (2,0,-1,2)$, $x_3 = (1,0,0,-1)$, $x_4 = (2,0,0,3)$;
   given $X = \{x_1, x_2, x_3, x_4\}$. Is the set X a base of $\mathbb{R}^4$?
   If yes, determine its dual basis.
   d. If the set $X = \{x_1, x_2, x_3, x_4\}$ defined above is a basis and if you were able to compute its dual basis, what could be the coordinates of the vector $(15,8,10,5)$ in the basis X ? Please explain your solution.
   e. Given the linear transformation $f : \mathbb{R}^4 \to \mathbb{R}^4$ so that $f(x,y,z,t) = (2x-t, 2y-z, x-y-t, -3z)$.
   How will you define the transpose transformation ?
2. Consider the vector space $\mathcal{M}_{2x2}$, the vector space of 2 lines, 2 columns matrices, with real coefficients, built on the field of reals.
   a. Give an example of linear form defined on $\mathcal{M}_{2x2}$.
   b. Give the general expression of a linear form defined on $\mathcal{M}_{2x2}$.
   c. Given $M_1 = \begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix}$, $M_2 = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$, $M_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $M_4 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$;
   given $X = \{M_1, \ M_2, \ M_3, \ M_4\}$.
   Is the set X a basis of $\mathcal{M}_{2x2}$? If yes, determine its dual basis.
   d. If the set $X = \{M_1, \ M_2, \ M_3, \ M_4\}$ defined above is a base and you had computed the dual base, what could be the coordinates of the matrix $\begin{pmatrix} 30 & 20 \\ 16 & 10 \end{pmatrix}$ into the base X ? Please explain your solution.
   e. Given the linear transformation $f : \mathcal{M}_{2x2} \to \mathcal{M}_{2x2}$ so that $f\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 2a-d & a-b-d \\ 2b-c & -3c \end{pmatrix}$.
   How will you define the transpose transformation ?