A THEORETICAL MODEL FOR VISUAL-SPATIAL THINKING

Conceição Costa, José Manuel Matos, and Jaime Carvalho e Silva

Escola Superior de Educação de Coimbra, Universidade Nova de Lisboa, Universidade de Coimbra

This paper presents part of a study (Costa, 2005) intending to create, explore and refine a theoretical model for visual-spatial thinking that includes three visual-spatial thinking modes along with the thinking processes associated to them. This paper will focus on the final theoretical model.

Many researchers have emphasized the value of the visualization and the visual reasoning in the mathematics learning (Bishop, 1989; Presmeg, 1989, Zimmerman & Cunningham, 1991). In the literature we find terms such as visualization, visual thinking, visual reasoning, spatial reasoning, spatial thinking to name mental acts combining visual, spatial, and visual-spatial thinking. The visual reasoning often parallels visualization (Hershkowitz, Parzysz & Dormolen, 1996) and visualization itself has different definitions according to the context of mathematics education, mathematics, or psychology. The terms, spatial thinking or spatial reasoning appear frequently tied to spatial abilities (Clausen-May e Smith, 1998). Dreyfus (1991) included visualization as a component of representation crucial in AMT.

This paper presents part of a research (Costa, 2005) intending to create, explore and refine a theoretical model for visual-spatial thinking, thus deepening meaning of a thinking-centered perspective on AMT. This research was developed through a three-stage process. Firstly, an initial model for visual-spatial thinking, condensed from relevant literature, was developed; secondly, this initial model was confronted with data from an empirical study; finally, the initial model was refined. The methodology for the empirical study was qualitative, integrating video registrations of individual answers and tasks performed in classroom activity. These episodes were analyzed and a constant comparison approach was used to fine-tune the initial model. The refined version of the model was elaborated and evaluated according to the standards for judging theories, models and results proposed by Schoenfeld (2002).

This paper will focus on the final theoretical model. The theoretical framework took into account research in the areas of cognitive processes in mathematics education, embodiment in mathematics, a perspective on learning with emphasis on the social construction of knowledge and on semiotic mediation, theoretical perspectives on the teaching and learning of geometric concept.

A THEORETICAL VISUAL-SPATIAL THINKING MODEL

The final model for understanding the visual-spatial thinking differentiates four distinct modes of thinking: the visual-spatial thinking resulting from perception (VTP) — intellectual operations on sensory, perceptual and memory material —; the visual-spatial thinking resulting from mental manipulation of images (VTMI) —

intellectual operations related to the manipulation and the transformation of images —; the visual-spatial thinking resulting from the mental construction of relationships among images (VTR) — intellectual operations related to the mental construction of relationships among images, the comparison of ideas, concepts and model—; the visual-spatial thinking connected with transmission-communication and representation, that is to say, connected with the exteriorization of the thinking (VTE) — intellectual operations related to the representation, translation and communication of ideas, concepts and methods.

Visual-spatial thinking modes	Definition
Visual-spatial thinking resulting from perception (VTP).	Intellectual operations on sensory, perceptual and memory material.
Visual-spatial thinking resulting from mental manipulation of images (VTMI).	Intellectual operations related to the manipulation and the transformation of images.
Visual-spatial thinking resulting from the mental construction of relationships among images (VTR).	Intellectual operations related to the mental construction of relationships among images, the comparison of ideas, concepts and models.
Visual-spatial thinking resulting from the exteriorization of thinking (VTE).	Intellectual operations related to the representation, translation and communication of ideas, concepts and methods.

TABLE I. Visual-spatial thinking modes and respective definitions.

In the next sections, we will discuss each mode and characterize the associated mental processes.

VISUAL-SPATIAL THINKING RESULTING FROM PERCEPTION

The visual-spatial thinking mode resulting from perception (VTP) is the nearest to sensations, that is to say, to the electric impulses that arrive at the brain. Its intellectual operations occur on sensory, perceptual and memory material. It is constructed from sensory stimulus and takes advantage of information gained from experience. This thinking mode involves experiences of mental concentration, of control, and observation. The observation experiences involve perception and interpretation, depend on past experience, memory, motivation, emotions, attention, the individual neuronal mechanisms, previous knowledge, verbalizations, and cultural aspects and so, what we saw depends on our relationship to the situation. The sociocultural factors, from which the perception depends on, are not less importance and they regulate how the members of a culture see.

This mode uses concrete images and memory images (Brown & Presmeg, 1993). *Concrete images* may be thought of as “a picture in the mind”, and are not the same for all persons; *memory images* are produced when images of experience are brought up again. These are representations of visual information connected to the perception of movement, for example, the images remaining immediately after we visually check for in-coming vehicles, before crossing the street.

Mental processes of this mode

Thinking processes involved in this visual-spatial thinking mode are: primary intuitions; intuitive inference; visual construction; representation again and image evaluation; visual recognition; objects and models identifications, formation of a “gestalt”, global apprehension of a geometrical configuration; perceptual abstraction and abstraction connected with recognition; and generation of concepts.

The first mental processes associated with the VTP mode are intuitions. Using the terminology of Fischbein (1987), we include in this mode the primary intuitions, — cognitive acquisitions that develop in individuals independently of any systematic instruction as an effect of personal experience. The primary intuitions are connected, for instance, with space representation related to body movement, and to images as models. Images may inject properties and relationships in the process of concepts construction that do not belong to the conceptual structure (points as spots, lines as bands). It also includes intuitive inferences, which are shown, for example, when a child sees a ball, runs after it according to the ball’s position and adapts his reactions to the ball’s movements. The child not only sees the ball moving, but also expects that it goes on moving, existing and preserving its shape and properties.

Visual construction is a mental process, which is present in this mode and may be illustrated, for instance, when alterations of distance or size “are seen” in optic illusions (even though the mind knows the perception is illusory), or when we perceive the fluctuations of the figure-ground in ambiguous designs.

The mental process of evaluating an image consists in representing again the image and this act of re-presentation is complex and subtle (Wheatley, 1998). These re-presented images are not immutable, because they may undergo change over time. In many cases the re-presented image may have been modified or it might be a prototype, which is then transformed, based on the demands of the context. The nature of the re-presentation is greatly influenced by the intentions of the individual and in many cases the re-presented image may come again more elaborated.

The information that comes through our eyes is involved in visual perception containing two phases (Gal & Linchevski, 2002), the visual information processing phase which consists in registering the sensory information, and the visual pattern recognition phase, which involves the interpretation of the identified shapes and objects. In the first stage of visual perception, shapes and objects are extracted from the visual scene. To form the object we need to know “what goes with what” and they are organized into groups similar to the gestalt principles. In the second phase of visual perception, shapes and objects are recognised. Recognition is the result of feature analysis, in which the object is segmented into a set of sub-objects, as the output of early visual processing of the first phase. Each sub-object is classified, and when the pieces out of which the object is composed and their configuration are determined, the object is recognized as a pattern composed of these pieces. The cognitive processes designated by visual recognitions, objects and models identifications, formation of a *gestalt*, global apprehension of a geometrical configuration belong to the second phase of visual perception while the remainder are included in the first phase of visual perception.

Although abstraction is more developed in the others thinking modes, it shows in VTP as a basic perceptual procedure — when we isolate (identify) something from the visual scene —, or in the recognition of a familiar structure in a given situation. Generation of concepts is done when the recognition of relations and idea emerge.

VISUAL-SPATIAL THINKING RESULTING FROM MENTAL MANIPULATION OF IMAGES

Visual-spatial thinking mode resulting from mental manipulation of images (VTMI) embraces different levels of imagery processing, mainly to foresee the result of transforming an image and envision the trajectory of that same transformation. We will include in this thinking mode the dynamic imagery and the pattern imagery proposed by Brown and Presmeg (1993). Dynamic imagery involves the ability to move or to transform a concrete visual image and pattern imagery is a highly abstract form of imagery where concrete details are rejected and pure relationships are depicted in a visual-spatial scheme. Owens (without date) using the conceptual frame of Presmeg, showed a kindergarten child extending a square using pieces of bread to make a “skinny” rectangle. This child also used dynamic imagery foreseeing (mentally) the result of the transformation a square into a rectangle before executing (physically) this same transformation. According to Owens (1994) the dynamic imagery was the means by which the child was linking her images for the concepts of squares and rectangles. Another child, for instance, makes the medium triangle with the small triangles in the tangram puzzle (Owens, without date). This child also used a patterned imagery because she can see a certain configuration, structure (triangle) as a composition of other structures.

The VTMI mode incorporates the transformational reasoning referring to the foresight and mental transformations of objects, postulated by Simon (1996). Simon assumes, more than the inductive and deductive reasoning used in the comprehension and validation of mathematics ideas, a third type of reasoning, transformational reasoning, is defined as

> “The mental or physical enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and

> the sets of results of these operations. Central to transformational reasoning is the ability to consider, not a static state, but a dynamic process by which a new state or a continuum of states are generated" (p. 201).

This transformational reasoning is supported by transformational reproductive images or by antecipatory images. Reproductive images evoke objects or events already known and anticipatory images represent, through figural imagination, events (movements or transformations, for example) that have not previously been perceived. In either case, someone is able to visualize the transformation resulting from an operation; however, transformational reasoning is not restricted to mental imaging of transformations. A physical enactment may be used to examine the results of a transformation. For example, a student who is exploring the validity of the statement, "If you know the perimeter of a rectangle, you know its area", might work with a loop of string observing what happens to the area as she makes the rectangle longer and thinner. But in order for the student to model this problem it is required a mental anticipation, that is, he must know, before handling the string, how to model the rectangles and use the string to observe the results of the operation (Simon, 1996). In both transformational reasoning and VTMI mode, mental operations or transformations on objects may be made and mentally envisioned as well their results.

Mental processes of this mode

The following mental processes are associated with this visual-spatial thinking mode: secondary intuitions and anticipatory intuitions; unitizing; mental transformations; reflective abstraction, constructive generalization; synthesizing; spatial structure; coordination; and visual construction.

The intuitions associated to VTMI, following the Fischbein´s terminology, are of two types: secondary intuitions and anticipatory intuitions. The secondary intuitions are affirmative intuitions that represent a stable cognitive attitude with regard to a more general, common, situation. The secondary intuitions are developed as the result of a systematic intellectual formation and they are interpretations of various facts taken as assured. Integration into dynamic and perceptively rich situations, as for instance, the use of a microworld, seems to enrich the acquisition of intuitions. Particularly secondary intuitions may be acquired (Fischbein, 1987).

Anticipatory intuitions also characterize this visual-spatial thinking mode. These intuitions do not simply establish a (apparently) given fact. They appear as a discovery, a preliminary solution to a problem, and the sudden resolution of a previous endeavour. Moreover, one may assume that anticipatory intuitions are inspired, directed, stimulated or blocked by existing affirmative intuitions. The anticipatory intuitions may be the effect of a creative activity in mathematics, of a constructive process in which inductive procedures, analogies and plausible guesses play a fundamental role (Fischbein, 1987).

Unitizing, which consists in the mental operation of constructing, creating and coordinating abstracts mathematical units, identified as a base for much mathematical activity in both geometric and numeral settings, are present in VTMI.

The term mental transformation is used to refer a type of process which involve the change of a mental representation in one of two aspects or in a composition of the two: to dislocate, that is to say, to change the position and to transform, where there is only a change of shape. These two aspects are related to each other and there is only a difference of complexity between displacements and transformations. In particular, to change the shape of an object may consist in dislocating the parts. Reciprocally, when we dislocate an object without changing its shape, this may dislocate en reference to another and changing the configuration of the whole.

Gusev and Safuanov revealed three types of operating with images (in order of their increasing complexity): transformations resulting in the change of a spatial position of an image (1st type); transformations changing the structure of an image (2nd type); long and repeated performance of transformations of first two types (3rd type).

This thinking mode is characterized by a particular type of abstraction, the reflective abstraction — essentially the construction by the subject of mental objects and of mental actions on these objects. The subject, in order to understand, deal with, organize, or make sense out of a perceived problem situation or to know a mathematic concept, uses schemes that invoke a more or less coherent collection of objects and processes. Understanding the trajectory as a coordination of successive displacements to form a continuous whole is an example of reflective abstraction in children thinking (Dubinsky, 1991). The pseudo-empirical abstraction (in the Piaget sense) as a sub-variety of the reflective abstraction is present in this visual-spatial thinking mode, focused on children actions and the properties of the actions and it appears from their successive coordinations.

Constructive generalization creates new forms, new contents, that is to say, a new structural organization. The mental process synthesizing that means to combine or compose parts in such way that they form a whole, an entity, is a basic prerequisite to the abstraction. The spatial structuring is the mental act of constructing an organization or form for an object or set of objects. It determines an object's nature or shape by identifying its spatial components, combining components into spatial composites and establishing interrelationships between and among components and composites (Battista, 2003).

A fundamental cognitive process to the understanding of the reasoning in this thinking mode VTMI is the coordination which involves diverse aspects, one of them is that indicated by Battista (2003, p. 79) "it arranges abstracted items in proper position relative to each other and relative to the wholes to which they belong". Another aspect of the coordination is related with the ability of using structures (references systems) as a way to organize the thinking. So, for instance, a student adopts structures of references to codify the spatial positions of the objects that may come to be defined: references systems centred in himself, references systems centred in the objects or in external structures which are or provided by the spatial structure or they are imposed mentally by the space (environment).

The visual construction process included in this visual-spatial thinking mode is related with making or modifying a spatial structure in such way that it meets certain predetermined geometric criteria. The visual construction comprises abilities such as the anticipation and the logic organization.

VISUAL-SPATIAL THINKING RESULTING FROM THE MENTAL CONSTRUCTION OF RELATIONSHIPS BETWEEN IMAGES

The intellectual operations of the visual-spatial thinking mode resulting from the mental construction of relationships between images (VTR) are related to the mental construction of relationships between images, the comparison of ideas, concepts and models.

Mental processes of this mode

We consider that the visual-spatial thinking resulting from the mental construction of relationships between images, mode VTR, may be associated to the following thinking processes: anticipatory intuitions; discovery of relationships between images, properties and facts; comparisons; synthesis; reflective abstraction; metacognition. The metacognition process is fundamentally understood as a regulation of cognition which includes the planning before beginning to solve the problem and the continuous evaluation while solving the problem.

VISUAL-SPATIAL THINKING RESULTING FROM THE EXTERIORIZATION OF THINKING

The visual-spatial thinking mode resulting from the exteriorization of thinking (VTE) is connected to the process by which mental representations are materialized, to the communication and the dissemination of ideas, to the construction of argumentation, to the description of the mental dynamics and to the support of conceptualizing abstract entities. The VTE mode has a nature different from the other thinking modes because is like the conveyor of those thinking modes. The VTE mode is a cognitive space of action, representation, construction and communication and as a whole may integrate components such the body, the physic world and the culture. This mode allows us to infer the imagery and the mental dynamics of students and to understand how they perform mathematical tasks.

For communicating their mental representations, the students may construct patterns, drawings, figures, and graphics, musical and rhythmic productions, to use gestures (corporal language, facial expression), actions, verbal descriptions (spoken or written), mathematic representations, etc. The VTE thinking mode relies fundamentally on verbal and gestured, visual language and it requires the use of concrete, memory, dynamic, pattern images and also kinaesthetic images (Brown &

Presmeg, 1993) which involve muscular activity of some type (the muscular activity may be limited to the use of hands and fingers).

Mental processes of this mode

The mental processes associated to the visual-spatial thinking mode resulting from the exteriorization of thinking are: representations; translation; description of the mental dynamics through verbalization and gestures; construction of argumentation, of conjectures; and the use of analogies. The concept of representation is essential to understanding constructive processes in learning and doing of mathematics and, roughly speaking, an external representation is a configuration of some kind that represents something in a special manner. For instance a word may represent an object of the real life, a numeral may represent the cardinal of a set, or even the same numeral may represent a position in a numeric line. The representations do not occur in isolation and usually they belong to highly structured systems, either personal and idiosyncratic or cultural and conventional (Goldin & Kaput, 1996). Among the external representations we find external physic embodiments, structured external physical situations or a set of situations which may be mathematically described or seen as embodying a mathematic idea; linguistic expressions, verbal or syntactic and formal mathematic constructions.

The representation of visual-spatial information used by the student is going to depend on the context where the problem is posed. The same task may require from the student different spatial abilities or different levels of abstraction. This representation may be a concrete image or a diagram or a concept representation: the reflection around a line, or the pattern construction or a tessellation.

Translation is a process that is intimately related to the conversion among representations. For example, the conversion of what is given of symbolic form in information given by figures or passing a problem from natural language or graphic form to some other form.

The description of the mental dynamic designates mental images evidenced in oral language, actions or gestures and in metaphoric expressions. Gesture is used to refer to any of a variety of movements, we want to identify mainly movements of hands, non-conventional gestures (gesticulations and language-like gestures) that accompany the speech with which they form an integrated whole. The description of the mental dynamic is going to be designated by factual if the objects of description are geometric objects and by analytical if the objects of description are geometric properties.

Analogies or metaphoric expressions are appealing modes of externalizing visual-spatial thinking, particularly ways of mathematic communication and of building of meaning. Two objects, two systems are said to be analogical if, on the basis of a certain partial similarity, one feels entitled to assume that the respective entities are similar in other respects as well. The difference between analogy and trivial similarity is that analogy justifies plausible inferences. So analogies imply similarity of structure (Fischbein, 1987). The visual-spatial thinking mode VTR may involve the use of analogies, which may conduct to new images, to new models or to draw comparisons, transformations and discoveries of relationships between images. Gusev and Safuanov (2003) say that the new images processed under the influence of some associations and analogies emerge frequently with unexpected qualities, creative imagination and they are the result basically operating the second and third type of transformations (behind explained). The visual-spatial thinking mode VTE is the conductor of those analogies, is linked to the externalization through the language, actions and gestures or through a distributed blend of perceptual sources coming from the screen and the gestures, if the student has not yet a language to describe and to theorize the events, appropriately.

REFERENCES

Battista, M.T. (2003). Levels of sophistication in elementary students´ reasoning about length. In N. A. Pateman, B. J. Dougherty & J. T. Zilliox (Eds.), *Proceedings of 27th PME conference* (Vol. 2, pp. 73-80). Honolulu: CRDG, College of Education, University of Hawai.

Bishop, A.(1989). Review of research on visualization in mathematics education. *Focus on Learning Problems in Mathematics, 11*(1), 7-15.

Brown, D. & Presmeg, N. (1993). Types of imagery used by elementary and secondary shool students in a mathematical reasoning. In I. Hirabayashi, N. Nohda, K. Shigematsu & F. L. Lin (Eds.), *Proceedings of the 17th International Conference for the Psychology of Mathematics Education* (Vol. 2, pp. 137-144). Ibaraki, Japan: University of Tsukuba.

Clausen-May, T. & Smith, P. (Eds.). (1998). *Spatial ability: A handbook for teachers*. Berkshire, UK: National Foundation for Educational Research.

Costa, M. C. M. (2005). *Modelo do pensamento visual-espacial: transformações geométricas no início da escolaridade* [A model for visual-spatial thinking: geometric transformations in the beginning of the school]. Unpublished doctoral dissertation, Universidade Nova de Lisboa.

Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 25-41). Dordrecht, Holland: Kluwer.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 95-123). Cambridge: Kluwer.

Fischbein, E. (1987). *Intuition in science and mathematics: An educational approach*. Dordrecht, Holand: D. Reidel.

Gal, H. & Linchevski, L. (2002). Analysing geometry problematic learning situations by theory of perception. In A. D. Cockburn & E. Nardi (Eds.), *Proceedings of 26th PME Conference* (Vol 2, pp. 400-407). Norwick, UK: University of East Anglia.

Goldin, G. e Kaput, J. (1996). A joint perspective on the idea of representation in learning and doing mathematics. In L. P. Steffe, P. Nesher, P. Cobb, G. Goldin & B. Greer (Eds.), *Theories of mathematical learning* (pp. 397-430). New Jersey: Lawrence Erlbaum.

Gusev, V. & Safuanov, I. (2003). Thinking in images and its role in learning mathematics. In N. A. Pateman, B. J. Dougherty e J. T. Zilliox (Eds.), *Proceedings of 27th PME Conference* (Vol 4, pp. 87-94). Honolulu: CRDG, College of Education, University of Hawai.

Hershkowitz, R., Parzysz, B. & Dormolen, J. (1996). Space and shape. In A. Bishop, K. Clements, C. Keitel, J. Kilpatrick & C. Laborde, *International handbook of mathematics education* (pp. 161-204). Dordrecht: Kluwer.

Owens, K. (1994). *Spatial problem solving in mid-primary school*. Unpublished manuscript, University of Western in Sydney.

Owens, K. (without date). *When qualitative and quantitative analysis and complementary: an example on the use of visual imagery by primary school children*. Unpublished manuscript, University of Western in Sydney.

Presmeg, N. (1992). Prototypes, metaphors, metonymies and imaginative racionality in high school mathematics. *Educational Studies in Mathematics, 23*, 595 -610.

Presmeg, N. (1989). Visualization in multicultural mathematics classroom. *Focus on learning problems in mathematics, 11*(1), 17-24.

Schoenfeld, A. (2002). Research methods in (mathematics) education. In L. D. English (Ed.), *Handbook of International Research in Mathematics Education* (pp. 435-487). London: Lawrence Erlbaum.

Simon, M. (1996). Beyond inductive and deductive reasoning: the search for a sense of knowing. *Educational Studies in Mathematics, 30*, 197-210.

Wheatley, G. (1998). Imagery and mathematics learning. *Focus on Learning Problems in Mathematics, 20*(2 e 3), 65-77.

Zimmermann, W. & Cunningham, S. (1991). Editors´ introduction: What is mathematical visualization? In W. Zimmermann & S. Cunningham (Eds.), *Visualization in teaching and learning mathematics* (pp. 1-7). Washington, USA: Mathematics Association of America.