CERME 6 – WORKING GROUP 11

APPLICATIONS AND MODELLING

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APPLICATIONS AND MODELLING

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The red thread in the programme for working group 11, *Applications and Modelling*, was to identify and discuss different theoretical perspectives found in research on the teaching and learning of mathematical modelling. Particular emphasis was placed on the relation between research and development of practices of teaching. The presentation: *A survey of theoretical perspectives in research on teaching and learning of mathematical modelling* were given by Morten Blomhøj and Gabriele Kaiser to set the scene for the working group. And the work was ended with a closing panel discussion with Javier García, Gabriele Kaiser, and Hugh Burkhardt as panellists and Susana Carreira as moderator. In addition Hugh Burkhardt gave a historical perspective on the field by his presentation: *The challenge of integrating modelling in mathematics teaching practices – a historical view* by.

The presentation and discussion of the papers was structured according to five themes: (1) Teachers' professional development for teaching and assessing mathematical modelling, (2) The role of ICT in teaching and learning mathematical modelling, (3) Researching the teaching and learning of mathematical modelling within the Anthropological Theory of Didactics (ATD), (4) Researching the teaching and learning of mathematical modelling within the framework of Realistic Mathematics Education (RME), (5) Researching the teaching and learning of mathematical modelling under the Models and Modelling Perspective (MMP). Each theme was introduced shortly and rounded off with a general discussion. The proceedings is organised in accordance with the thematic structure of the programme and include the19 papers presented and discussed at the conference.

Theme 1 was introduced by Katja Maass and Geoff Wake and included six papers.

In the first paper, Rita Borromeo Ferri and Werner Blum present and discuss their experiences with modelling seminars as a way of integrating the teaching and learning of modelling in mathematics teacher education. As a basis for their design of the modelling seminars the authors have identified four main competencies related to the teaching of mathematical modelling: (1) Theoretical competency, (2) Task related competency, (3) Teaching competency, (4) Diagnostic competency. It is argued that mathematics teacher education should support the development of such competencies and include experiences with modelling activities in school practices.

Katja Maaß and Johannes Gurlitt write about the problem of how to evaluate teachers professional development in the teaching of mathematical modelling. Based on the authors experience from the international LEMA project the paper discusses the challenges related to the design and application of an evaluation questionnaire for teachers participating in a professional development project.

The paper by Barbara Schmidt is also related to the LEMA project. She analyses - also by means of questionnaires - the motives and obstacles experienced by the teachers for including realistic modelling activities in their teaching. According to the regulations of mathematics teaching it should include realistic modelling activities. However, different institutional and educational factors seem to form obstacles for this ambition. The findings suggest that it is possible to identify types of teachers that experience motives and obstacles for realistic modelling differently.

The paper by Jeroen Spandaw and Bert Zwaneveld reports on the development of a text book for secondary mathematics teacher education. One of its objectives is to further the coming teacher professional development for teaching mathematical modelling. The paper discusses issues such as the teachers' dispositions for modelling, educational goals for teaching modelling, design aspects, testing in modelling, the role of domain knowledge, and computer modelling. The paper also reflects on the relationship between mathematics, teaching of mathematics and modelling, and on the role of modelling in the Dutch mathematics curriculum.

The next paper is concerned with formative assessment in relation to mathematical modelling activities. Using a Cultural Historical Activity Theory perspective, Geoff Wake argues that modelling activities and related pedagogies and in particular the quest for formative assessment in relation to learners modelling processes have the potential to bring about a significant change in classroom activity for learners and teachers; and that such changes might support the learning of mathematics for more students and better prepare them to apply mathematics. This paper is also related to the LEMA project.

In the last paper of theme 1 Jonas Bergman Ärlebäck reports on a study on teachers' beliefs and affects about mathematical modelling. Five different domains of beliefs are identified as important for if and how teachers will include mathematical modelling in their teaching: (1) the nature of mathematics, (2) real world (reality), (3) problem solving, (4) school mathematics, (5) applying, and applications of, mathematics. Two teachers' beliefs are analysed according to these five domains.

Theme 2 was introduced by Morten Blomhøj and included four papers each presenting concrete cases of ICT supported modelling activities.

Maria Lucia Lo Cicero and Filippo Spagnolo in their paper report from an experimental project with three upper secondary classes that have been working with motion sensors and computers to produce graphs for different motion phenomena. From pre- and post-tests and analyses of the classroom interactions it is argued that students developed modelling competencies and that the modelling activities can enhance the students' mathematical and physical understanding of important concepts such as Cartesian graph, function, derivative, velocity and acceleration.

In the second paper Christina Roeckerath presents and analyses a simulation software package that can support the students' modelling and analyses of different types of biological interactions between species such as predator-prey, competition or parasitism. It is argued that such modelling activities can provide the students with an insight into the interdisciplinary relationship between mathematical modelling and theoretical population biology, and support their learning of biology.

Mária Lalinská and Janka Majherová discuss in their paper different aspects of visualization in relation to projectile motions modelled by secondary students using a spreadsheet and a graph drawing software. The motion of fireworks is used as a situational context to set the scene for the modelling activities and it is argued that ICT-supported modelling activities allow the students' to experience and understand better the mathematical and physical elements involved in the phenomena.

In the last paper of theme 2, Hans-Stefan Siller and Gilbert Greefrath present and analyse in detail modelling cycles in which technology is integrated by means of handheld or computer based software. The potentials in different types of software (CAS, DGS and SP) for supporting the students learning of modelling and mathematics are discussed and illustrated with the example of modelling "dangerous road intersections".

Theme 3 was introduced by Berta Barquero and Javier García and included five papers.

The first paper by Berta Barquero, Marianna Bosch and Josep Gascón introduces the metaphor of ecology and the notion of levels of didactic determination from ATD, and show theoretical constructs can be used to better understand the institutional constraints that hinder the large scale implementation of mathematical modelling activities. The theoretical ideas are exemplified through an analysis of "applicationism" - a notion used by authors to capture the set of beliefs that guides applications of mathematics in traditional mathematics teaching.

In the paper by Richard Cabassut it is argued that mathematical modelling activities can be analysed as a double didactical transposition within the framework of ATD. Real world problems and related techniques undergo a transposition when used in mathematics teaching similar to the transposition that mathematical concepts, techniques and theories undergo. This transposition process is analysed with respect to the modelling cycle, and examples of mathematisation tasks from the LEMA project are used to illustrate the elements in the transposition process.

García and Ruiz-Higueras in their paper illustrate how the ATD can be used as a theoretical framework for designing mathematical modelling activities for teaching. A design - also from the LEMA project - for 4-5 years children is presented and analysed to illustrate the theory based design process. The experiences from the implementation of this design show how even very young pupils can be involved in rich and meaningful mathematical modelling activities.

The paper by Vázquez reports about the ATD based design of modelling activities for engineering students. The processes of transposition of the praxeologies involved in a particular modelling task – the modelling of a motor – are analysed, and it is argued that in order to understand the technologies linked to the students techniques, it is necessary to take in account the different disciplines involved.

Serrano, Bosch and Gascón in their paper analyse from a ATD perspective the mathematisation process in the modelling cycling process. A modelling task for university students on forecasting the sale of a given product from an empirical time serie is used as an example. The experinces show that a modelling activity initiated with a real-situation can lead to mathematising that affects both the system and the model and that challenge the students' modelling competency and their learning of important mathematics.

Theme 4 was introduced by Mette Andresen and included two papers and a poster by Simon Zell and Astrid Beckmann.

In the first paper Mette Andresen presents a long-term research and developmental project concerning mathematical modelling in a multidisciplinary context in upper secondary teaching. A course of lessons based on the Vioxx case is used to illustrate the different levels of reflection in the students' modelling work in this context.

The paper by Roxana Grigoras deals with the modelling of real world phenomena where no numerical data are given. In the case studied, lower secondary students' are trying to make sense out of a picture of the surface of the planet Mars. In this very open modelling activity the students use a number of fundamental mathematical ideas. The activity is analysed using RME as a theoretical framework.

Theme 5 was introduced by Nicos Mousoulides and included only one paper. Here N. Mousoulides, M. Chrysostomou, M. Pittalis and C. Christou present and discuss a case where a class of 11-years students worked with the fresh water shortage problem in Cyprus. It is a real life problem, the students' used relevant technology (Google Earth and spreadsheet) and they were in fact able to compare, judge and reflect on the different models developed. The activity was design and analysed within the framework of MMP.

MATHEMATICAL MODELLING IN TEACHER EDUCATION – EXPERIENCES FROM A MODELLING SEMINAR

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Mathematical Modelling has recently become a compulsory part of the mathematics curriculum in Germany. Hence future teachers must have a strong background about different aspects of modelling and also about appropriate methods how modelling can be taught. That means that the content and the methodology of university courses on modelling have to include all these aspects. In our paper, we will report on university seminars on modelling for students in their fourth year of study. Among other things, the students had to write a "learning diary" over the whole semester. The results give interesting insights in students' learning processes of modelling, their progress and their problems during the semester and their considerations about teaching modelling.

INTRODUCTION

Although mathematical modelling is now a compulsory part of the mathematics curriculum in Germany and one of the main competencies within the national Educational Standards, it is not at all guaranteed that pupils will be taught by teachers who have a sound knowledge of modelling. One reason for this is the fact that modelling has normally not been taught in teacher training courses at University, because modelling is not contained explicitly in the curriculum for future math teachers in Germany. However, there is no doubt (see, e.g., Krauss et al. 2008) that teachers have to be experts in modelling themselves in order to be able to teach students effectively and that their thinking has to be shaped towards creating rich classroom environments that enable students to be actively involved in modelling (Chapman 2007).

In the last few years, a lot of empirical studies have dealt with the question of how modelling can be taught in school (see, e.g., Maaß 2007 or Blum & Leiß 2007) or how students at University can be sensitized for modelling through complex modelling tasks (see Lingefjaerd & Holmquist 2007, Blomhoj & Kjeldsen 2007 or Schwarz & Kaiser 2007). The results of these studies opened new ways of thinking about modelling and the way it can be integrated in school mathematics in a profitable way. However, the question of how these aspects can be integrated in teacher education still remains open. Two of the main questions are:

1. How can future teachers be prepared in university courses for teaching modelling at school, which contents and which methods are appropriate?

2. How do students' processes of learning and understanding develop during such courses, what are their main difficulties and problems, and how can progress be observed?

In this paper, we will report on such a modelling seminar which has been taught at the University of Hamburg by the first author and with similar features at the University of Kassel by the second author, and with which we have tried to tackle these two questions. Our guiding principle for the conception of this seminar was: If we want our students to teach modelling in an appropriate way (with a correspondence between content and method, cognitive activation of pupils, reflection on learning and integration of summative assessment) we as lecturers have to conceive our own teaching in exactly the same way (correspondence between content and method, cognitive activation, reflection, summative assessment).

CONCEPTION, GOALS, CONTENT AND STRUCTURE OF THE SEMINAR

The main basis for our data collection was a modelling seminar for students in their fourth year of study at the University of Hamburg. In this course altogether 25 future teachers from all school levels were participating, including teachers for students with special needs. (The authors' experiences from other modelling seminars showed that this kind of mixture builds a good basis for discussions and is important for arguing that modelling is suitable for all kinds of school levels and types.) The course took place once a week for 90 minutes over one semester that means 14 lessons altogether. According to the meaning of a "university seminar", the students were expected to be actively involved in all activities and to cover a major part of the course by their own presentations. In the following we will describe more precisely the conception of this seminar and the way the students were observed over the semester. Mathematical Modelling as a subject in teacher education may, of course, be structured in many different ways because it is a vast field and contains a lot of important aspects (see Blum et al. 2007). In our considerations for planning and structuring a modelling seminar in a new way, the content and the methods should fit to each other. This is also a challenge for the lecturer. Concerning content, we regard the following competencies concerning modelling as particularly important:

(1) Theoretical competency (knowledge about modelling cycles, about goals/perspectives for modelling and about types of modelling tasks)

(2) Task related competency (ability to solve, analyse and create modelling tasks)

(3) Teaching competency (ability to plan and perform modelling lessons and knowledge of appropriate interventions during pupils' modelling processes)

(4) Diagnostic competency (ability to identify phases in pupils' modelling processes and to diagnose pupils' difficulties during such processes)

We did not include an "Assessment competency" (that is the ability to construct and mark tests appropriate for modelling). This competency is, of course, very important for in-service teachers but can, in our view, not be expected from students who have not enough experience in assessment.

These four competencies were the basis for the structure of the seminar. The seminar was subdivided in the following five parts, also in order to have an appropriate balance between more theoretical and more practical phases:

Part 1 (Theory): Theoretical background about modelling (3 lessons)
Part 2 (Practice): Solving and creating modelling problems (3 lessons)
Part 3 (Theory and Practice): (1) Students analyse transcripts of pupils' work on modelling problems; (2) What are modelling competencies;* (3) Types of teacher interventions while modelling; (4) Methods how to teach modelling in school (4 lessons)
Part 4 (Presentations): Groups of students present their own modelling tasks and how pupils in school solved these tasks. (3 lessons)
Part 5: Last lesson – reflection of the whole work over the semester

*At the end of this part there was an intermediate evaluation of the seminar on the basis of a questionnaire.

One important goal of the seminar was that students do not only solve or construct modelling tasks but also learn methods how they can teach modelling. For us as lecturers it seemed important not to merely say which methods could be useful, but to integrate them directly into the work in the seminar. We decided to use teaching strategies from the field of "Cooperative Learning" (see e.g. Johnson & Johnson 1999, Kagan 1990), also because the first author had good experiences using this while teaching modelling at school. We think that Modelling as the content and Cooperative Learning as a teaching strategy fit together very well also at university seminars. Research has shown (see Johnson & Johnson 1995) that cooperative learning techniques promote pupils' learning and academic achievement, increase pupils' retention, enhance pupils' satisfaction with their learning experience, help pupils develop skills in oral communication, develop pupils' social skills, and promote pupils' self-esteem. Several studies on modelling made clear that modelling is better done as a group activity (Ikeda, Stephens & Matsuzaki 2007), also because this supports discussions about mathematics or extra-mathematical aspects, trains argumentations and gives the chance to profit from group synergy. That is why in the first lesson of the seminar the students had to build so-called "basis-groups" of five people who were supposed to work together over the whole semester; altogether there were six such groups. However, working in groups is only under certain conditions more productive than competitive and individualistic efforts. Those conditions are (Kagan 1990): Positive Interdependence, Face-to-Face-Interaction, Individual-&Group-Accountability, Interpersonal-&Small-Group Skills and Group Processing. So we had to take care that all group activities fulfil these conditions. We combined these activities with the content-parts of the seminar:

Part 1: Students had to know about different *directions* in the discussion on modelling and different *modelling cycles* (see e.g. Kaiser, Sriraman & Blomhoj 2006 and Borromeo Ferri 2006 as literature which was given to the students). They learned this content with the activity "Jigsaw": Each group member is assigned some particular material to learn and later on to teach to his group members (in this case each student had one direction of modelling as his particular topic, e.g. realistic modelling, and in the second round one version of the modelling cycle). Students with the same topic worked together in "expert-groups", so the basis-groups were divided. After working in these expert-groups, the original basis groups reformed and students taught each other. So at the end of this part the students had learned this content mostly on their own. It was, of course, important for the students that they also could ask all kinds of questions, especially in the last lesson of this part, and that we reflected both the theory and the activity Jigsaw.

Part 2 started with the question "What is a good *modelling task*?" For that we used the activity "Think-pair-share". This involves a three step cooperative structure. During the first step, individuals thought silently about a question posed by the instructor. Individuals paired up during the second step and exchanged their thoughts. In the third step, the pairs shared their responses with the entire group. After that the basis-groups solved the modelling task "Filling Up" ("Tanken", see Blum/Leiß 2007). For a better understanding we showed the students a possible solution process by means of the seven-step modelling cycle that we ourselves use in our work (Blum & Leiß 2007, Borromeo Ferri 2007), in order to help them to understand which part of their solution can be regarded as a real model or a mathematical model and so on. The six groups had then time for sharing ideas for their own modelling tasks which they had to construct and to test in school. For that "creating part" we used the method "RoundRobin Brainstorming"**:** One person of each group was appointed to be the recorder. A question or an idea was posed with many answers, and students were given time to think about the answers. After the "think time," members of the team shared responses. The recorder wrote down the various answers of the group members. The person next to the recorder started and one person after the other in the group gave an answer until time was called. At the end of this part, the groups had finished creating their modelling tasks and in addition they had learned how they could do a subject-matter analysis of the problem. Similar to the first part, we discussed questions and reflected the used methods for potential uses in school.

Part 3 contained a lot of interesting aspects of modelling. So we started each aspect with a short theoretical input and the students then had an activity on their own. Concerning aspect (1), the basis-groups worked on the transcripts of *pupils' solution processes* to the modelling task "Lighthouse", and we had a discussion afterwards especially about the distinction of the phases while modelling. Before we started with our input for aspect (2), we used the method "silent writing conversation". Every group got a big sheet of paper. In the middle of the paper they were to write "*modelling competencies*". The students had to do a brainstorming about what modelling competencies could be, however without saying a word. So they had to comment the products of the other group members also in a written way. After that we gave information on modelling competencies and had then a discussion in the plenum, mainly about how teachers can support modelling competencies and how they can assess these in school. Like before, we started aspect (3) with an activity, this time "Inside-Outside-Circle" before we gave a theoretical input about the meaning of "*intervention*" and "*self-regulated learning*". The activity "Inside-Outside Circle" follows the principle that all students are integrated in the learning process. So the students form an inner and an outer circle. Those in the inner circle look outside, those in the outer circle look inside. Then the whole group was asked: "What do you think a teacher has to know when teaching modelling so as to be able to intervene appropriately in case of students' difficulties?" The students stood opposite to each other and discussed this question in pairs. After five minutes, the outside circle moved on and students in new pairs exchanged their thoughts. The same was done with the second question, thus addressing aspect (4): "What do you think are good or bad methods for teaching modelling?" We closed this lesson with a discussion and a reflection about the five activities of cooperative learning we had so far during the seminar and how they fit to the contents of the seminar. Simultaneously this was meant to be a meta-reflection on different levels: 1. the students had to think about each method and about teaching them in school in connection with modelling; 2. we as lecturers had to reflect whether the chosen activities were useful to teach the contents of the seminar.

In *part 4,* all groups presented their *modelling tasks* and their *experiences* they had, in the meantime, gathered in school with these. Because of the participation of future teachers for all school levels, also the presented experiences were from primary to upper secondary school. The final *part 5* rounded off the seminar with a *summary* of all aspects.

Taking into account the rather elaborate conception of the seminar, we liked to know if the students felt sufficiently well-equipped to teach modelling at school. Furthermore we were interested in students' individual learning and understanding processes and how these develop during such a course as well as in their main problems and difficulties.

METHODS OF ANALYSING LEARNING PROCESSES

Reflection was a major issue for the students in the seminar, because thinking over one's own actions generally deepens the understanding a lot. To get insight into the thinking and learning processes, the students had to write a "learning diary" (see e.g. Gallin & Ruf 1990). One important goal of a learning diary is to write down one's individual learning story. It also helps stabilising the competencies related with the contents. For the goals of our explorative study, the "learning diary" was the adequate instrument to stimulate reflections on students' own learning processes over a long time. Interviews could have been an alternative, but not with a whole seminar. The organization of a learning diary looks mostly as follows: write down the date, the topic of the lesson and the activity; write down why you had to do the activity; look back and think about where you are in the learning process. The students in the seminar had to do this in a similar way concerning their learning of the topic of modelling. In the last five minutes of each lesson, the students had time for writing their reflections into their learning diary. At the end of the semester, all diaries were collected in order to analyse them with respect to understandings and problems referring to a) the different parts on the content, b) the methods used, c) the way how the seminar was taught, and d) the students' own reflections on teaching and learning modelling in school. So we coded (Strauss/Corbin 1990) and categorized statements of the students according to these four aspects to get an overview and to find patterns. In addition, we analysed each diary with regard to hints concerning the learning process of the individuals.

RESULTS

Most of the students knew from their first semester course a little bit about modelling and what it means, but that was only a small part of the lecture. So 18 from 21 students wrote in their reflections after the first lesson that they had not known that modelling is such a big field.

"In this lesson I got a first insight in the theme "modelling". There it became apparent for me that this theme is very wide and does not only exist of the modelling cycle I know from my first semester course." (Katrin, 2nd of April 2008)

Not unexpectedly, dealing with part 1 was not easy for the students. To distinguish between different directions and then again between different modelling cycles was a high demand for them, what the reflections clearly show. But the method Jigsaw was a helpful strategy for the students to help each other and to become more clarity about the content. Anyhow the students felt that this strong theoretical part was helpful to get appropriate background knowledge.

"Sometimes it was not easy to understand one direction of modelling in the expert-groups, because of the shortness of the text. But this method [Jigsaw] is perfect! Everyone of the group has to explain something and so we discussed till I understood it better." (Swetlana, 9th of April 2008)

A progress in the learning process of the students could be reconstructed in Part 2. All students reflected that they understood the seven-step modelling cycle finally through the modelling task "Tanken" which we presented to them in a detailed manner after they solved this problem. Furthermore they felt that now the background from part 1 will help them to create an own modelling task, so for them theory and practice linked together here.

“It was good, that we went through the modelling cycle with an exemplary task. Thereby one became aware how complex a modelling task can be [...]. Now it will be easier for us to create our own modelling task.” (Sarah, 16th of April 2008)

“Slowly I understand the modelling cycle better. Working with the “Tanken”-Task helped me to distinguish several steps of the modelling cycle.” (Alexander, 16th of April 2008)

When analyzing the reflections on part 2 it became very clear that creating modelling tasks is as important for learning and understanding modelling as solving modelling problems. The students had to think over the school level in which they wanted to test the problem, how complex the task should be, how much time the pupils would probably need, and so on. Helpful for them was the method used in this context.

“It was good that we were to create our own modelling task in our basis-group. However we recognized that this will be a difficult undertaking. But the method RoundRobin was exactly adequate to get helpful suggestions from other basis-groups.” (Anna, 23rd of April 2008)

Thus the three lessons of part 2 were once again a linking between theory and practice for the students, and a progress in their process of understanding could be reconstructed especially concerning the modelling cycle. Furthermore they had to deal with the question of authenticity and complexity while creating their own modelling task. The students were confronted with a lot of aspects of modelling in part 3 as described above. We have no space to go more into detail here, but we try to summarize the important points. Analyzing transcripts of pupils’ modelling processes in aspect (1) was helpful for the students to distinguish several modelling phases.

“The transcripts of the pupils helped me in some part to distinguish several modelling steps.” (Heidy, 7th of May 2008)

Modelling competencies and beliefs were interesting for the students. Most of them liked the question of how modelling competencies could be supported. They commented that one lesson was not enough for this content and that they would like to know more about this topic.

“The silent-writing-conversation was very fruitful at the beginning concerning the meaning of modelling competencies. Of high interest for me was the question of how modelling competencies can be supported. This is especially for a teacher an important question.” (Jan, 21st of May 2008)

Starting aspect (3) with the method Inside-Outside-Circle was for all students a good start for the topic of teacher interventions. Most of the students started to reflect more about themselves as a teacher personality and also liked to have more time for this topic.

“After the discussion in the Inside-Outside-Circle I think that a teacher must be well prepared when he has a modelling problem for his lesson, because he has to analyse and to diagnose his pupils quickly to help them.” (Carolin 21th of May 2008)

“Today I learned a lot about different kinds of teacher interventions, firstly theoretically and then practically through group work with a case study of a teacher. But I take much more out of this lesson today: The case study showed me how invasive a teacher can intervene, so that this intervention is restricted only to the content. But I will look to myself how I intervene to correct my interventions.” (Andreas 21th of May 2008)

The reflection of the methods (aspect 4) was very constructive, because the students learned the methods on their own through the seminar. So they were able to decide about advantages and disadvantages. All of the students agreed also that these methods can be integrated while teaching modelling, but they have to be practiced.

“It is good that we are learning not only modelling as a subject in this seminar, but also the methods how we can teach this at school!” (Katja 28th of May 2008)

Testing the modelling task at school and then presenting the results in part 4 was particularly important for the learning processes of the students. Whereas the processes of understanding of the students concerning modelling partly stumbled in part 3 because of the diversity of the aspects, part 4 stood for their progressives. The reflections indicated that they learned and understood more about what modelling means on a theoretical level and also how to teach it.

“Today my group and I had our presentation. I think it was good! […] Overall the testing was helpful for me as a teacher, because I could see where pupils had problems while modelling. Also to get the self-awareness to walk between the small level of intervention and reservation was important for me. Furthermore it showed me that the task should be phrased precisely and to allow enough extra time.” (Benjamin 4th of June 2008)

“Testing the modelling task in grade five was important and helpful for my understanding of modelling and the practical transformation in school. […] It was good to have a chance testing modelling problems at school.” (Birgit 18th of June 2008)

Summary of the results

We summarize our results concerning the two questions at the beginning. First we asked how teachers can be prepared in university courses for teaching modelling at school, which contents and which methods are appropriate. On the basis of our experiences, we are sure that in general a balance between theory and practice must be given. Both should be connected by means of an appropriate teaching strategy, which must be reflected in the seminar. Of course, the contents of such a seminar may vary, but according to our experiences, the following contents are well suited for such a seminar (see the competencies referred to at the beginning): (1) Knowledge about modelling cycles, goals/perspectives and types of tasks; (2) Solving, creating and analysing modelling tasks; (3) Planning and practising modelling lessons; (4) Diagnosing actual modelling processes of pupils.

Second we asked how do students’ processes of learning and understanding develop during such courses, what are the main problems and how can progress be observed. We decided that students had to write a learning diary to help us to answer that question, also in combination with the evaluation of the seminar. These were the main problems of the students: to understand several directions of modelling and the distinctions between modelling cycles in the literature; to distinguish phases of the modelling cycle in general and also analysing transcripts of pupils’ modelling processes; subject-matter analyses of modelling problems; and finally dealing with the question of authenticity while creating a problem. Progress of the students concerning these difficulties could be reconstructed mostly when, pragmatically speaking, they linked theory with practice. Reflecting these developments during the seminar helped the students, undoubtedly, on their way to become competent teachers of mathematics.

In conclusion, we would like to emphasize once again the necessity that university students who are to become mathematics teachers must have vast opportunities to deal with mathematical modelling both on a theoretical and on a practical level, including experiences with modelling at school. This will not only contribute to preparing them to be competent teachers for mathematical modelling but will also contribute to further develop their understanding of mathematical subject matter and of mathematics as a discipline (Lingefjaerd 2007).

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DESIGNING A TEACHER QUESTIONNAIRE TO EVALUATE PROFESSIONAL DEVELOPMENT IN MODELLING

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LEMA is an international project to design a professional development course for modelling. In order to measure the effects of the course, an evaluation questionnaire was developed and pilot tested. Based on theoretical background about modelling, this paper outlines the challenges of the design process, presents reliability and validation data, and exemplifies each scale with a few sample items.

Keywords: Modelling, professional development, international approach, design of an evaluation questionnaire, empirical study

INTRODUCTION

Researchers, practitioners and policy makers in mathematics education agree that educationist aim should be to enable students to apply mathematics to their everyday lives (PISA, OECD, 2002) and contribute to the development of active citizenship. However, modelling is still rare in day-to-day teaching around Europe. LEMA (Learning and Education in and through Modelling and Applications) is a transnational European Project[1] that tackles the problem at teacher level by developing a common course of professional development in mathematical modelling. The aim of this paper is to provide an approach to the evaluation of professional modelling development in different national contexts and settings that is theory-based and driven by analysis of needs.

Teachers' knowledge and beliefs about the nature of the subject, their views on how to educate the subject and their self-efficacy beliefs about teaching the specific subject influence how they design or select tasks, plan, implement and evaluate their lessons (e.g. Brickhouse, 1990). Thus, to successfully implement mathematical modelling in their actual classroom practice, teachers need to (amongst others) (1) know the key concepts of mathematical modelling, (2) change their beliefs about the nature of mathematics education (if not appropriate for modelling), and (3) increase their awareness of their own competency to implement mathematical modelling in their actual classroom practice (self-efficacy).

THEORETICAL BACKGROUND

Mathematical modelling means applying mathematics to realistic, open problems. There are many descriptions of modelling processes (Blum & Niss, 1991; Kaiser-

[1] Partners of LEMA: Katja Maass (Coordinator) & Barbara Schmidt, University of Education Freiburg, Geoff Wake, University of Manchester, Fco. Javier Garcia Garcia, University of Jaen, Nicholas Mousoulides, University of Cyprus, Ödon Vancso & Gabriella Ambrus, University of Budapest, Anke Wagner, University of Education Ludwigsburg, Richard Cabassut, IUFM Strasbourg. The project is founded by the EU.

Messmer, 1986). They vary according to the described modelling cycle, the relevance given to the context and the justifications seen for modelling in mathematics lessons (Kaiser & Shiraman, 2006). In this study, we follow the description of the modelling process in PISA (Prenzel et al., 2004), albeit restricting it to context-related problems.

Modelling competency is the ability to carry out modelling processes independently. It comprises the competencies to carry out the single steps of the modelling process as well as competencies in reasoning mathematically and metacognitive modelling competencies (Maaß, 2006). Similar distinctions have already been made by Kaiser and Blum (1997) and indirectly by Money and Stephens (1993), Haines and Izard (1995) und Ikeda and Stephens (1998) in setting up assessment grids.

Modelling lessons**:** A trans-national project, which aims at developing a common, research-based professional development intervention for Europe, faces the challenge presented by partners having different theoretical backgrounds related to the teaching of modelling situated in their different national contexts. Thus, we sought to identify where common ground and indeed differences might be used to enrich such a project. For example, the English partner adopts a socio-cultural approach – drawing particularly on ideas of the development of learner identity and using Cultural-Historical Activity Theory (CHAT) (Engestrom & Cole, 1997). Research of the Spanish team, on the other hand, relied upon the Anthropological Theory of Didactics (Chevallard, 1999). Finally, the German partner's position is located in the international discussion on modelling and focuses on authentic tasks showing the usefulness of mathematics.

Drawing upon all of these approaches, a theory-based design emerged as follows: The German partner provided descriptions of the modelling cycle and focused on authenticity, both aspects being used to design tasks and support didactical development in the classroom. With relation to the Spanish partner, authenticity can also be seen in the search for questions that are crucial for students as social individuals. The English partner's perspective that engaging with mathematics can be considered a social activity provides teachers and researchers with a range of new learning and pedagogical models.

In short, the theoretical approach – related to the teaching of modelling – used in this study is a synthesis of a variety of theoretical backgrounds. This allows for and ensures that combined expertise guided this trans-national project.

Professional development of teachers: When considering teachers' competencies in teaching, we follow Krauss et al. (2004) and Shulmann (1986) by distinguishing professional knowledge (content knowledge, pedagogical content knowledge, pedagogical knowledge), beliefs, motivational orientation and competencies in reflexion and self-efficacy.

Empirical studies of teachers' professional development (e. g. Tirosh & Gerber, 2003; Wilson & Cooney, 2002) show that professional development interventions lead to changes if the courses are long-term, with embedded phases of teaching and reflexion, and if further factors which might have an impact on teachers' possibilities to teach modelling (such as the framework, the head of the school, parents, teacher's own beliefs) are taken into account.

Teachers' beliefs about the nature of mathematics and its education are believed to have a major impact on if and how a teacher employs innovation in everyday teaching. According to Pehkonen and Törner (1996), beliefs must be understood here as being composed of a relatively lasting subjective knowledge of certain objects as well as the attitudes linked to that knowledge. Beliefs can be conscious or unconscious, whereas the latter are often distinguished by an affective character.

Kaiser (2006) showed that innovations required by the curriculum are interpreted by the teacher in such a way that they fit into his or her existing belief system. Grigutsch, Raatz & Törner (1998) classified beliefs about mathematics into various aspects: the aspect of scheme (fixed set of rules); the aspect of process (problems are solved); the aspect of formalism (logical, deductive science); and the aspect of application (important for life and society).

Teachers' self-efficacy beliefs in this context can be described as teachers' beliefs in their capabilities to organize and execute mathematical modelling activities in their planning and classroom practice (see Bandura, 1997, Bandura, 2006). Self-efficacy is a future-oriented belief about the level of competence a person expects he or she will experience in a given situation. Self-efficacy beliefs influence thought patterns and emotions that enable actions and effort for reaching goals and persist in the face of adversity. "The self-assurance with which people approach and manage difficult tasks determines whether they make good or poor use of their capabilities. Insidious self-doubts can easily overrule the best of skills'' (Bandura, 1997, p. 35).

Considering pedagogical content knowledge about modeling as an external measure of learning success, both beliefs about the nature of mathematics and the education of mathematics and self-efficacy beliefs touch the individual's own perception and motivational aspects potentially relevant for application in the actual classroom setting.

DEVELOPMENT OF THE COURSE

Analysis of needs: In order to design a suitable professional development course for teachers, we first conducted a needs analysis to assess teachers' mathematical beliefs, their use of tasks, and their attitude towards given modelling tasks. Altogether, N = 563 teachers from the partner nations participated in the needs analysis. The measurement instrument included items about beliefs, which were rated on a 4-point rating scale ranging from 1 (strongly disagree) to 4 (strongly agree). Additionally, items related to the tasks teachers use (e.g. tasks practising basic skills vs. problem solving tasks) and three concrete modelling tasks were given to the teachers (all related to the same context, but differing in their task openness). Teachers were asked how likely they were to use each task and to justify their response.

Results revealed that teachers gave high rating scores for belief items in the process dimension (e.g. mathematics allows you to solve problems: M = 3.49) and the application dimension (e.g. mathematics is useful in everyday life: M = 3.5). Conversely, they gave relatively lower scores for the formalism and scheme dimension (e.g. mathematics is a fixed body of knowledge: $M = 2.44$). However, when asked which tasks they would most likely use in their lessons, the majority selected “tasks that practised basic skills” ($M = 3.48$) as opposed, for example, to “problems with other than one solution” ($M = 2.38$). Accordingly, when asked if they would use any of the given modelling tasks, the closed tasks proved to be very popular, while the more open versions drew less enthusiasm. Commonly cited detractors for the open tasks were perceived difficulty and class time constraints.

In terms of designing an evaluation questionnaire, the analysis of needs also made clear that the more related the questions are to day-to-day teaching (e.g. related to concrete tasks), the more objections become evident.

The course of professional development: Based upon the findings of the needs analysis, the following considerations were given particular attention: First, we addressed the teachers’ concerns and difficulties in using modelling tasks by providing further information about the benefits inherent to each modelling task. We also addressed different ways to assess and support students in their development of modelling competencies.

Based on the needs analysis and the synthesis of various theoretical backgrounds, we developed the professional development course into five key aspects (modules):

(1) *Modelling*: To implement modelling in lessons, teachers need background information about this concept (sub-modules: What is modelling? Why use it?) (Blum & Niss, 1991).

(2) *Tasks*: When it comes to planning lessons, teachers need to learn how to select appropriate tasks for their students and anticipate the modelling outcomes. An emphasis was placed on authentic tasks (sub-modules: Exploring, Design, Classification and Variation) (see e.g. Maaß, 2007, Burkhardt, 1989, Galbraith & Stillman, 2001, Kaiser-Meßmer, 1986)

(3) *Lessons*: Research has shown that group work, discussion and working independently all support the development of modelling competencies (sub-modules: Methods, Using ICT, Supporting the Development of Modelling Competencies, Exercising Mathematical Content Through Modelling), (see e.g. Tanner & Jones, 1995, Maaß, 2007, Ikeda & Stephens, 2001)

(4) *Assessment*: If modelling is implemented in lessons, it also has to be evaluated. Assessment should be used not only for grading but also for supporting learning through feedback (Williams & Black, 1998) (sub-modules: Formative Assessment, Summative Assessment, Feedback).

(5) *Reflexion*: As outlined above, reflexion about implementation in lessons and dealing with challenges is crucial for the success of professional development courses (sub-modules: Implementation, Challenges).

Piloting: This course was piloted and evaluated in all 6 partnership countries. Piloting took place in 2008 and comprised 5 days. Implementation, however, was quite different. For example, in France the training was given as a one-block course in January 2008, addressing teachers teaching students aged 6-8 years. In Spain, the course contained two blocks in April and May. Finally, in Germany the course consisted of 5 separate days from January to November and addressed primary and secondary teachers at the same time.

The main question was how such a course, which was piloted under different conditions and in different national contexts, could be evaluated. We did not consider students for evaluation because this seemed to be almost impossible given the huge variety of students concerned (age 6 to 16) and the given national contexts. Focusing on teachers, we used questionnaires and interviews and exemplary videos. Questionnaires and interviews give insight into teachers' point of view and so provide information about teachers' intentions and thus about necessary preconditions of the change of day-to-day teaching. Here, we will focus on the teachers' questionnaire.

DESIGN OF THE QUESTIONNAIRE

Instrument Development and Field Testing**:** The questionnaire was prepared to assess all teachers taking part in the course (6 countries, 10-40 teachers per country). To measure possible knowledge gains and belief changes we implemented a pre-post-control-group design. The development and testing of the instrument took place in 5 steps.

Step 1: Establishing rationales guiding the design: *First*, items should mirror the theoretical background and key-aspects of the modules of the professional development. Thus, the questionnaire sought information about the pedagogical content knowledge, beliefs, and self-efficacy about mathematical modelling as well as beliefs about mathematics and its education. Items covered these categories and all five modules. *Second*, considering the target group and their understandable preference for a short questionnaire, our aim was to find a balance between a reasonable length and what would still provide a reliable assessment. *Third*, careful guidelines were developed to improve compliance in filling out the questionnaire and the quality of the implementation of the questionnaire (i.e. we provided information regarding the necessity of an evaluation for further improvement and emphasized that it was the course and not the teachers that was being evaluated) .

Step 2: Procedure and materials preparation: Considering the first rationale of Step 1, the scale construction was based on established scales wherever possible. The scale of belief items about the nature of mathematics and its education was based on Grigutsch, Raatz and Törner (1998). Items were related to four aspects of beliefs (see above). Learners rated their beliefs on a 5-point scale, ranging from strongly disagree to strongly agree.

Based on Bandura's method for measuring self-efficacy beliefs (Bandura, 2006), we designed a self-efficacy scale assessing efficacy beliefs related to modelling on a

100-point scale, ranging in 10-unit intervals from 0% ("cannot do at all"), to 100% ("highly certain can do").

For the assessment of the pedagogical content knowledge, we decided to use questions in an open format, the main consideration here being sensitivity for measuring knowledge provided in the course. Teachers were supposed to show their active knowledge, as this is probably the knowledge which they use in teaching. The open questions used for this knowledge assessment were rated by two independent, trained raters considering the amount and quality of arguments based on an expert solution.

Step 3: First Item Refinement – A Small Tryout: First, we conducted a small pilot study. The tryout instrument was administered to a group of 7 teachers and 3 teacher trainers. In addition to filling out the questionnaire, participants were asked to comment on the items they found misleading or difficult to understand. Consequently, items that were mentioned as being misleading were adapted. Items where the answers of teachers showed a lack of focus were reformulated. Considering the target group, the initial questionnaire was too long (time needed > 60 min). Thus, we analyzed the questionnaire for time-efficiency and possibilities to omit certain items. For example, the first questionnaire contained an open item referring to beliefs about the different areas where mathematics can be useful. As this was not directly linked to the content of the course, we omitted this item due to time vs. diagnostic value considerations. The first question "What is modelling?" was moved to the end, because some teachers were unable to answer it and therefore became discouraged right from the beginning.

Step 4: Second Item Refinement – Expert Questioning: To improve the content validity of the items, the questionnaire was submitted to 10 modelling experts, each with more than 5 years' teacher education. They were asked to evaluate whether the item statements would adequately deliver information about the proposed modelling curriculum.

As a result, certain questions were reworded, for example the rather general "What is modelling?" became "Name as many characteristics about modelling tasks as possible". In addition, it was moved back to its original location at the beginning of the questionnaire, so that examples of modelling tasks given in other parts of the questionnaire would not influence one's response to this question. To address possible feelings of discouragement among participants, we decided to provide the following introduction: *Whether or not you have already heard of or know anything about mathematical modelling and modelling tasks, it does not matter here. We simply want to know the starting point for the teacher training course.* This introduction also served the purpose of informing participants that they were not going to be tested. Another useful lesson gained from the modelling experts was to clarify the intention of the items related to beliefs by focusing on the beliefs about mathematics education and to omit items related to beliefs about mathematics itself. Again, the questionnaire was shortened. For example, the original questionnaire included three suitability ratings of different tasks, that took almost 20 minutes to answer but only comprised three single Likert-type scales accompanied by short comments.

Step 5: Testing and Item Selection: After conducting the above-mentioned revisions, we conducted a pilot study with prospective teachers, including 24 experts in modelling and 23 novices in modelling, to simulate pre- and post-testing. This testing targeted the following research questions:

1. How reliable is the pedagogical content knowledge-scale, the beliefs about modelling scale and the self-efficacy scale?
2. How good is the interrater-agreement between two independent raters scoring the open format knowledge questions?
3. Is the developed scale able to differentiate between novices (without experience in modelling and experts (in modelling)?

The first two questions about psychometric properties of the scales can be answered as follows. The reliability of the aggregated pedagogical content knowledge score was good (Cronbach's $\alpha = .83$). Forty percent of the open format questions were co-rated by a second rater (for 18 of the 47 participants), and the interrater agreement, shown by the intraclass correlation coefficient (ICC) was good ($ICC2,2 = .91$). Thus, only one rater coded the rest of the protocols. The reliability of the aggregated beliefs about modelling scale was good (Cronbach's $\alpha = .87$). The reliability of the aggregated self-efficacy belief score was high (Cronbach's $\alpha = .96$).

To answer the third question of whether the scales were able to differentiate between novices and experts we used a one-factorial ANOVA to analyze the data. An alpha level of .05 was used for all statistical tests. As an effect size measure, we used partial η^2, qualifying values <.06 as small effects, values in the range between .06 and .13 as medium effects, and values >.13 as large effects (see Cohen, 1988). Results of the analysis of variance showed that the experts had significantly higher knowledge scores about modelling than the novices, $F(1.41) = 23.22$, $p < .001$, $\eta^2 = .36$ (large effect). The analysis also showed that the experts had significantly higher scores related to beliefs than the novices, $F(1.34) = 13.97$, $p < .001$, $\eta^2 = .29$ (large effect). Last, the analysis revealed that the experts had significantly higher self-efficacy beliefs about modelling than the novices, $F(1,35) = 6.68$, $p < .014$, $\eta^2 = .16$ (large effect).

These findings provide evidence that lead to the conclusion that also from a quantitative point of view, the developed questionnaire shows good reliability and construct validity. We also surveyed 8 practicing teachers with the questionnaire and found that descriptively they scored close to the novices concerning modelling skills.

In order to address concerns about further shortening the questionnaire, items that did not seem absolutely necessary for measuring the pedagogical content knowledge, beliefs about mathematical modelling or self-efficacy were screened for discriminatory power and difficulty. In other words, if the items were too general or too easy, they would not be able to measure improvement. The final questionnaire contained the following sections**:** biography, beliefs about mathematics lessons, pedagogical content knowledge and beliefs related to modelling, and self-efficacy. The following examples provide a closer look at some of these sections:

Beliefs: We used items based on Grigutsch, Raatz and Törner (1998) but with a focus on school mathematics, for example – each of them with a 5-point Likert scale.

	School mathematics in my lessons from my point of view as a teacher	Strongly dis-sagree				Strongly agree
5.1.1	School mathematics is a collection of procedures and rules which determine precisely how a task is solved.					
5.1.2	School mathematics is very important for students later in life.					
5.1.3	Central aspects of school mathematics are flawless formalism and formal logic.					

Pedagogical content knowledge: Within this section we addressed the following aspects: modelling, reasons for and against modelling, tasks, methods and assessment. All items were related to a corresponding modelling task because, as the needs analysis clearly showed, being as concrete as possible was paramount to getting valid results. Most of the questions had an open format. For example:

Imagine you are teaching children whom you regard the right age for this task. The following 5 questions are all related to the task below and all connected with each other.

It is the start of the summer holidays and there are many traffic jams. Chris has been stuck in a 20-km traffic jam for 6 hours. It is hot and she is longing for a drink. How long will the Red Cross need to provide everyone with water?

Imagine you are teaching children whom you regard the right age for this task.	not very likely				very likely
6.1 Please x one to show how likely you are to use this type of task	○	○	○	○	○

6.2 Give as many reasons as possible (pros and/or cons) and mark them as such (+/-).

To evaluate the concept of assessment, a student's solution was given to the teachers and they were asked to provide written feedback. Further, teachers were asked also about methods they would use in a specific situation, all in relation to the task given.

(7) Self-efficacy: The scale was based on Bandura (2006) and included the following sample items that had to be rated on a 100-point scale, ranging in 10-unit intervals from 0% ("cannot do at all"), to 100% ("highly certain can do"):

I feel able to teach mathematical content using a modelling approach.

I feel able to develop detailed criteria (related to the modelling process) for assessing and grading students' solutions to modelling tasks.

FINAL NOTES

This paper exemplified the development process of designing a questionnaire evaluating the success of a professional development course on mathematical modelling. The greatest challenge was accommodating the participants' preference for a short questionnaire and evaluating the multifaceted aspects of the course as accurately as possible.

The results of the evaluation will be finalized in September 2009. The final evaluation questionnaire is available on request from the first author.

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MODELING IN THE CLASSROOM – MOTIVES AND OBSTACLES FROM THE TEACHER'S PERSPECTIVE

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Modelling is not only written into educational standards throughout Germany; other European countries also stipulate the integration of reality-based, problem-solving tasks into mathematics at school. In reality, however, things look quite different: in many places maths lessons are still dominated by exercises in simple calculation. So why? What is stopping teachers from introducing modelling? What would motivate them? In order to explore this issue in depth, a supplementary empirical study was conducted as part of the EU Project LEMA[1]. This paper intends to introduce the project, the development of the questionnaire and the survey design. Finally, first results will be presented.

THE LEMA PROJECT TEACHER TRAINING PROGRAMME

Within the framework of LEMA (Learning and Education in and through Modelling and Applications), a concept for a further training course for teachers on the theme of modelling and reality-based teaching was developed, piloted and evaluated. The aim was for teachers to become familiar with contemporary didactic and methodical concepts. They should acquire a basic knowledge of mathematical modelling and reality-based tasks in the school context, and after the training, they should be aware of why modelling should be learnt in maths lessons and how their pupils can learn it. In other words, they should know which subject matter, teaching forms and methods are most suitable for supporting pupils in their learning, at which point in the lesson modelling can be introduced and how a basic knowledge can be secured. In addition, practical concepts for putting together and evaluating and grading tasks for class tests should be acquired. A further aim was to be able to analyse, modify and describe the learning potential inherent in modelling tasks, and to be able to develop tasks which take into consideration the heterogeneity of school classes[2].

The course content was designed for about five days of further training. The modular structure of the course allows for a choice of content and is flexible in terms of the length of the training. Furthermore, it is conceived in such a way that teachers from all types of schools and of all academic abilities can take part. In Germany, two parallel training courses were to take place on five days spread out over the year (Jan. 08 – Nov. 08). There should be about two months

[1] LEMA = **L**earning and **E**ducation in and through **M**odelling and **A**pplications. Coordinator: Katja Maaß Pädagogische Hochschule Freiburg. Participant countries: DE, EN, FR, ES, HU, CY

[2] www.lema-project.org

between each day of training so that the teachers participating in the course have the opportunity to integrate the contents of the training into their lessons.

BASIC THEORY

Mathematical modelling generally refers to using mathematics to solve realistic and open problems. At the same time, the exact definition varies depending on the aims, which model of the modelling process is being used and the nature of the context assigned to a modelling task (Kaiser-Messmer 1986, Kaiser & Shiraman 2006).

Obstacles to the integration of modelling

In day-to-day school life, modelling still plays a much smaller role than one would wish (Burkhard 2006, Maaß 2004). It appears that at the moment teachers see more obstacles to using modelling than advantages. Blum (1996) has divided these obstacles into four categories: organisational, pupil-related, teacher-related and material-related.

Organisational obstacles: With this Blum (1996) is referring mainly to the short amount of time – 45-minutes – teachers have for class.

Pupil-related obstacles: Modelling makes the lesson too difficult and less predictable for pupils (Blum/Niss 1991, Blum 1996). Pupils can have difficulties carrying out individual steps or even the whole modelling process (Maaß 2004). Standard calculating tasks are more popular with some pupils because they are easier to understand and to solve the problem one simply has to apply a certain formula. This makes it easier for pupils to get good grades in mathematics (Blum/Niss 1991).

Teacher-related obstacles: There appears to be a variety of obstacles for the teachers. The literature on this issue refers repeatedly to the time aspect. Teachers need more time to update tasks, to adapt them to the needs of the respective class, and to prepare them in detail (Blum/Niss 1991). In addition, there are obstacles in relation to the actual lessons: teaching becomes more demanding and more difficult to predict (Blum 1996). Furthermore, a teacher requires other skills and competencies in order to be able to deal with a changed approach to teaching. The latest literature also refers to teachers' beliefs about – or attitudes to – mathematics teaching as being an obstacle to innovation in the classroom (Pehkonen 1999, Törner 2002). Blum (1996) emphasises the fact that teachers do not view modelling as mathematics. Moreover, some teachers do not consider themselves competent enough to carry out modelling tasks when the context is taken from a subject area they did not study (Blum/Niss 1991, Blum 1996). In addition, a significant aspect of the perceived obstacles is the question of how to assess performance, as teachers feel overwhelmed by the increasing complexity of this process (Blum 1996).

Material-related obstacles: Teachers often simply do not know enough modelling examples which they feel would be suitable for their lessons, or they select excessively detailed materials. (Blum/Niss 1991, Blum 1996).

Motivations for integrating modelling

Though there are several arguments against modelling, one can counter these arguments with numerous good reasons why modelling should be integrated into mathematics lessons, despite the existence of the obstacles as described above. A comprehensive representation of these reasons can be found in Blum (1996, p.21 ff.), Galbraith (1995, p.22) and Kaiser (1995 p.69).

The offer-and-use model Figure 1 shows an attempt to integrate influences on the quality of teaching into a more comprehensive model of the effectiveness of a lesson.

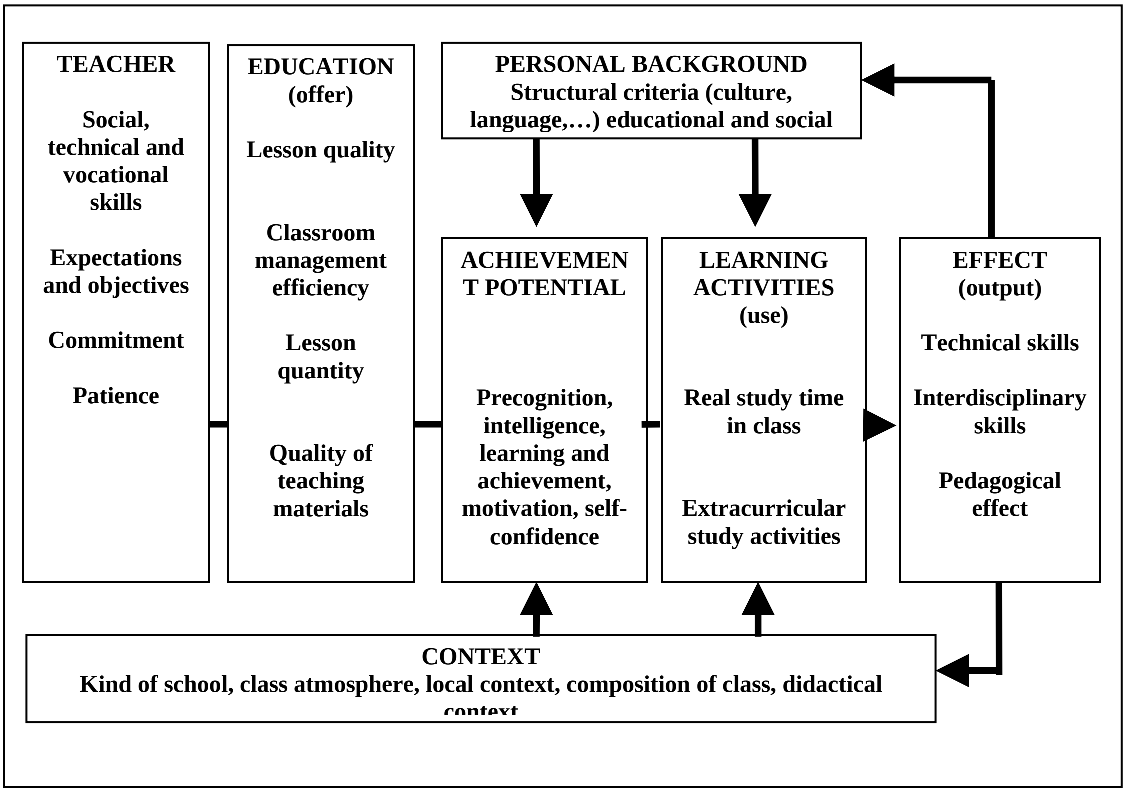

Figure 1: Offer-and-use model; Source: Helmke (2006)

As well as characteristics of the lesson, the model also includes characteristics of the teacher's personality, the classroom context, the individual personal background requirements and the achievement potential and learning activities of the pupils. This model represents a theoretical basis for the obstacles and motives for modelling. At the same time, the model should serve as a basis for systematically organising the reasons for motives and obstacles so as to indicate in which areas of the model the relevant motives and obstacles are to be found. For example, the interviews produced a first indication that the motives belong to the pupil domain and the obstacles with the teachers.

RESEARCH QUESTIONS

The previous section set out some arguments against modelling. However, these are based almost exclusively on experience and have not been subjected to empirical analysis.

This suggests the need of some kind of instrument with which to measure or assess empirically the arguments against modelling. In order to ensure the resulting point of view is not one-sided, this instrument should also analyse the arguments *for* modelling. This has the additional advantage that not only the deficiencies are revealed, but that solutions are also presented and made available. Therefore, the central questions for the survey are:

(1) What are the obstacles and motives? (2) Which obstacles and motives appear meaningful in terms of their being put into practice? (3) Which changes in the obstacles and motives can be identified during training? (4) Can in the process certain types of teachers be identified? (5) Is there a rubric for the offer-and-use model which seems to be especially relevant?

How these questions might be answered is presented in the following.

METHODOLOGY

Survey for the study: To find out which aspects teachers view as obstacles and motives for modelling, quantitative and qualitative methods were applied. Amongst other things, a questionnaire was designed with the aim of ascertaining the obstacles and motives (see next section).

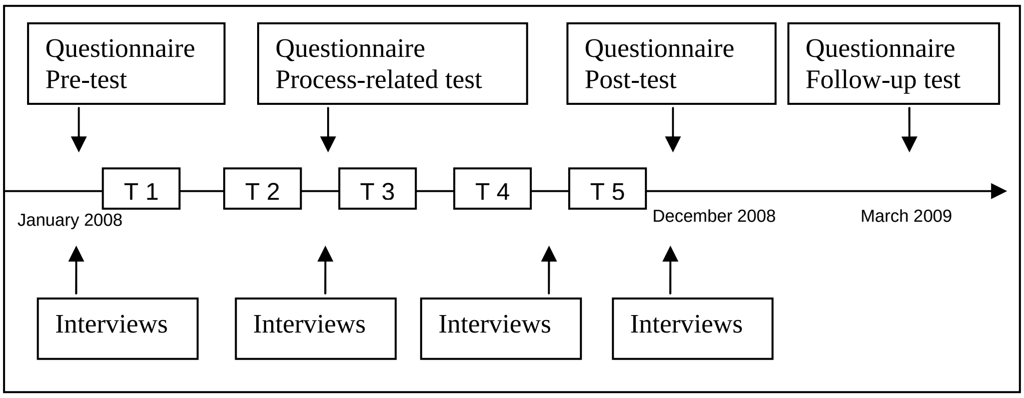

Figure 2: Study schedule

In addition, guided interviews were to be conducted. The advantage of using questionnaires is that a very large number of subjects can be used and that the questionnaire can be highly standardised (Oswald 1997). Only then can the desired generalisation of data be achieved. Significance tests can be applied to test hypotheses and develop a general statement (Bortz & Döring 2006). However, questionnaires are also limited in that data acquisition is not based on a process but mainly only focus on specific points. A further disadvantage lies in the reduction of the information: due to the pre-defined answer format of the questionnaire, the possibilities available to the survey subjects when providing their comments are limited.

Therefore, ideal is the additional use of interviews (Flick 1995). This allows the subjects the opportunity to express their answers in a more open form (v. Eye 1994). Using a set of interview guidelines, the interviewees are granted as much space to provide their own descriptions as possible. Where something is not clear, this type of interview affords the researcher the chance to ask again, to rephrase the question of to explore in more depth spontaneously and associatively things the interviewee might say (Hopf 1995). A central element to research questions is also that in addition to ascertaining obstacles and motives, the interviewer can enquire as to the background behind the arguments.

Study design: The questionnaire was to be implemented at four points in time: pre-test, post-test and follow-up test, as well as a process-related test in the middle of the further training). Four different survey dates were chosen so as to be able later to discover a possible development curve or teacher types. At the same time, additional individual interviews should be conducted with six teachers chosen randomly. So far, the results of the pre-test and process-related test questionnaires are now available for this study. The first and second interviews of the selected subject group are also available. More data will be generated by the end of the year.

Random sample: The random sample includes teachers from two further training courses with a total of 52 participants and a corresponding control group of 47 subjects. The allocation to experimental or control group was random.

The random selection of the teachers for the interviews was based on the results of the pre-tests. This meant that three teachers were selected who saw many obstacles to modelling and three who instead saw many motives for modelling.

Finally, table 1 is intended to show which assessment tools were chosen, their basic structure, their usage during the study period and a brief description of the respective random sample.

QUESTIONNAIRE DEVELOPMENT

To lay the foundations for the study, a questionnaire was developed whose purpose it was to throw light on the obstacles and motives for the teacher regarding modelling in mathematics lessons.

To be able to guarantee this, a three-stage design was developed.

Questionnaire development: The first items were developed from the subjective theories of researchers (deductive item construction). For this, the *obstacles* described above were restated as items. Furthermore, items were also formulated from the identified *motives*. To guarantee the authenticity of the items, the “natural” polarity of the obstacles and motives were retained in the items. The result was a preliminary questionnaire which included a total of 65 items. The answer format corresponded to a 5-level Likert scale (Rost 1996), which ranged from “applies completely” to “does not apply at all”. As the items named on the questionnaire were not expected to prove complete, additional open questions were integrated which allowed the subjects to add any obstacles and motives for modelling which were not mentioned. With the help of these open items, together with the evaluation and optimisation of the closed items, the aim was to create a second and third test version of the questionnaire. This was necessary in order to be able to change the phrasing of items with ceiling effects, thereby minimizing the effect. At the same time, it was important to check the changed items once again in another test version in order to ensure that all ceiling effects were eliminated. If for the third test version no changes can be made to an item, it is removed from the questionnaire. Another reason why the three test versions are necessary is that the open question format generates new items which also have to be checked in a test version for ceiling effects.

The questionnaire was tested on 240 mathematics teachers in three runs. In the end, the questionnaire included 120 items.

Item polarity: The effects of item polarity are a source of controversy in the literature (Bühner 2006, p. 66f). On the one hand, some people are of the opinion that negatively expressed items confuse (e.g. “I am not often sad”). On the other hand, the tendency to say yes should be counteracted. Questionnaires with positive and negative items influence both factors and validity. Other studies have proven, however, that item polarity has only a limited effect on studies (ibid. p.66f). Due to these contradictory points of view regarding item polarity, in this study the natural polarity of the items was retained. This means that a high level of validity for the questionnaire is assumed, as the items in their natural polarity are less ambiguous and clearer. Thus the questionnaire includes both positively and negatively formulated statements about the research topic.

Forming categories: The aim was to organise the 120 items into categories. At the same time, the categories should be formed from the items (inductive categorisation). The first indications for categories were provided by Blum’s classification (1996) as illustrated above. In addition, the items were repeatedly analysed together as a whole, so as to check for more possible category indicators. In so doing, a great deal of flexibility and openness was extremely important. Through this dynamic process new categories of content were constantly being discovered and others rejected. In addition, a categories validation was carried out by an expert rating, whose task it was to check if the categories were consistent in terms of content.

In the end, the items generated 23 categories. In conclusion, the categories were assigned the aspects of the offer-and-use model (fig.1) so as to give them a theoretical base (deductive approach). These are described in the following.

FIRST RESULTS

In developing the questionnaire, the areas in which teachers see obstacles and motives for modelling were indicated. As the data collection is still incomplete, a final evaluation can not yet be given. Instead, it is more important that the categories be seen as a first indicator of to which areas the various obstacles and motives can be assigned. Thus the intention of the following is to outline the categories and to assign them to areas in the offer-and-use model. In addition, the established categories should be supported by quotes from the interviews.

The *teacher personality* area includes all categories which have to do with the personality of the teacher. Categories could be identified which confirm the obstacles found in the literature and described above. For example, there are obstacles in terms of the context of a modelling task. Some teachers appear to be held back by the unfamiliar contexts in modelling ["…how on earth am I supposed to know that? I didn't study biology! I'm certainly not going to add a task to that."]. Another obstacle appears to be the amount of preparation time needed ["I recently had a really good idea for a modelling task. I spent three hours working on it until I was satisfied with it. I simply can't do that for every lesson. After all, I have 6 teaching hours to prepare for every day."]. The belief of some teachers that modelling makes the lesson too difficult for the pupils could also be confirmed ["The pupils had no idea what they were supposed to calculate. This isn't surprising when so much information is missing!"].

However, it is worth noting that these same aspects represent not only obstacles but also motives. For example, some teachers appear to regard an unfamiliar context as a challenge ["What's really exciting is what I learn myself in the process!"], and others see in modelling an opportunity to gain time in terms of the preparation ["I just cut out a newspaper article, think of a suitable question to go with it and I'm finished."], apparently holding the opinion that modelling requires less time to prepare. For this area new aspects could also be discovered which have so far not been mentioned in the literature. According to some teachers, modelling appears to require an increased level of flexibility ["I do try to think about which ideas the pupils could come up with, but it's not possible to predefine all the directions they could go in. Sometimes they ask questions I don't know the answers to myself, and suddenly the lesson takes a quite different direction to the one planned."] The role of the teacher, which changes when using modelling tasks, was regarded by these teachers as a positive role ["The pupils only really call on me when they're lost. Otherwise I can just take a back seat and observe them; the atmosphere is very relaxed."].

In the area *lesson quality* two categories from the literature could be confirmed: some teachers criticize the fact that there is insufficient availability of materials.["At the moment we are looking at functions, and for this I took the task with the bridge. And then another one … and another. But I can't always do bridge tasks; it's too boring for the pupils. But there aren't any other tasks for functions."]. In addition, one's ability to plan the lesson is negatively affected as it is more difficult to predict the way in which the lesson is going to go with modelling. Moreover, three new categories could be assigned to this area: first, teachers appear to regard modelling as being very complex ["The tasks are just too complex for the pupils; they feel really overwhelmed."]; second, as well as the time factor being a problem in terms of the preparation for the lesson, time was also cited as an issue for the actual lesson, as some teachers feel that modelling tasks are very time-consuming [“I haven’t done any modelling recently because quite simply there isn’t the time. When I decide to use modelling tasks, I need more than an hour. Perhaps two, or even better, three. But I don’t have the time.”]; third, concerning methods, both positive and negative aspects could be named, with some teachers holding the view that modelling tasks offer a huge variety of methods [“I can apply absolutely loads of methods; and besides, the pupils are then much more motivated.”], whereas others held exactly the opposite view, claiming that modelling tasks are in terms of methodology extremely difficult to design [“I have no idea which methods I should use for these tasks.”].

In the area *individual personal background,* the category ‘pupil motivation’ could be corroborated. Here, too, as corroborated by the literature, there appear to be two forms of this aspect. Several teachers hold the opinion that pupils are more motivated when doing modelling tasks [“The pupils find the practical work in modelling tasks really interesting. Then they’re fully motivated and have much more fun.”], while others claim that standard, traditional calculating exercises are more popular [“The pupils come to me and ask when we can do normal tasks again.”]. Three further categories could be established: some teachers believe that when doing modelling tasks pupils are more creative in their thinking and calculating [“The pupils have really good ideas that even I wouldn’t have come up with.”]; some teachers are convinced that modelling tasks lead to greater independence in the pupils [“The pupils work much more independently.”], which they view as being a highly positive aspect; and there is the question of the difference in abilities within one class. Here, again, opinions go in two opposite directions. A section of the teachers hold the view that modelling should not be applied in a class where there is too big a difference between the various abilities [“The weaker pupils freeze up even more and the stronger pupils are bored because there isn’t much calculating to do.”], while the others would appear to disagree with this view, arguing that it is exactly then that modelling should be used [“The weaker pupils tend to get lost less and are also more motivated. The stronger pupils can try out new ideas, taking more and more parameters to make the calculations more complex.”].

The area *context* stands for the basic conditions. The influence of colleagues and parents plays a significant role. And here, too, it appears to go in two different directions, which can also be found in the literature. Concerning the cooperation with colleagues and/or parents, the experience of teachers seems to be either good [“I asked the parents at parents’ evening to work out one of the modelling tasks, and after that they thought it was really good!”] or bad [“The parents? They don’t support it at all! They want me to set tasks like the ones they had at school.” Or: “My colleagues are all quite old and they’re not going to change things in their classes now. If I start talking about modelling tasks, they just smile at me patronisingly. So there is no cooperation at all."], both sides obviously having a very different effect on the use of modelling.

The area *effects* describes effects which can be attained from the long-term use of modelling. Here, all of the motives named in the literature and described above could be confirmed. Teachers appear to be aware of the positive effects modelling seems to have. It was also corroborated that teachers consider the measuring of performance as regards modelling somewhat problematic, as it would seem to be more complex ["I found it really difficult to assess the results. One of the pupils perhaps only guessed but got the right result; the other carried out a really complicated calculation but made a mistake. How can I assess that fairly?"]. A new category is the efficiency of the lesson. Some teachers see a more efficient lesson through modelling ["It is quite simply more efficient, because every pupil can contribute to these tasks. The pupils are all constantly occupied when they are modelling. And besides, they can remember the content of the lesson much better when they are actively involved, for example when they have had to measure the playground."], while others claim to see quite the opposite. ["I can't really afford to do modelling in my lessons, as it means giving up so much of the exercises."]

This list shows that as well as the reasons for and against modelling named in the literature, further relevant aspects are to be found. It is interesting that the very same aspects that are viewed positively by some teachers are viewed negatively by others.

PERSPECTIVE

By the end of the year, the data collection from the questionnaires and interviews will be completed. This should provide more information on the obstacles and motives, also highlighting any changes that occur to said obstacles and motives in the course of the further training. The question is whether in the process it will be possible to identify certain types of teachers.

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MODELLING IN MATHEMATICS' TEACHERS' PROFESSIONAL DEVELOPMENT

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One of the chapters of the new Dutch handbook of didactics of mathematics, which is currently being written by a team of didacticians, concerns mathematical modelling. This handbook aims at (further) professional development of mathematics teachers in upper secondary education. In this paper we report about the issues we included: dispositions about modelling, goals, designing aspects, testing, the role of domain knowledge, and computer modelling. We also reflect on the relationship between mathematics, teaching of mathematics and modelling, and on the role of modelling in the Dutch mathematics curriculum.

INTRODUCTION

In this paper we describe how the subject of mathematical modelling is treated in the new Dutch handbook of didactics of mathematics, which is to appear within the next few years. The intended audience of the handbook consists of students in teachers' colleges as well as mathematics' teachers in upper secondary education who want to learn about teaching modelling as part of their professional development. We try to bridge the gap between educational research and teaching practice by bringing together results, scattered about the literature, thus making them accessible to (future) teachers. We highlight those topics which our post graduate courses for teachers have shown to be most urgent for their practical needs.

Many maths teachers are not familiar with modelling or do not want to spend time on modelling in math' class. Therefore we first address the question what modelling is (not) about and why it should be included in the mathematics curriculum. Next, we cover briefly some essential issues concerning the teaching of modelling.

We focus on the non-mathematical aspects of mathematical modelling, since the didactics of the necessary mathematics is dealt with in other chapters of the handbook. Furthermore, experience with professional development courses for teachers shows that these non-mathematical aspects of modelling deserve very careful consideration as they are often ignored.

We restrict ourselves to the question of how to apply known mathematics to non-mathematical problems. In particular, we do not discuss modelling as a tool to learn mathematics.

MODELLING, MATHEMATICS AND TEACHING OF MATHEMATICS

Goal of modelling

As mentioned above, we restrict ourselves to the application of mathematics using mathematical models to non-mathematical problems. Looking through the eyes of a scientist, it is our goal to understand the relations between the variables of our context. Mathematics is an important tool to achieve this goal. Scientists use mathematical models to experiment with variables and possible relations between them and answer specific questions, such as: Which percentage of Rhine water ends up in the ecologically important and sensitive Waddenzee? Of course, everyday life can also be a source of interesting problems, for example: Does it pay out to drive across the border to fill up the car? Students tend to that think models are copies of reality (Sins, 2006). It is important that they learn that models are made to answer specific questions and that the same context can lead to completely different models, depending on the question. A steal ball can be modelled as a point mass or a sphere or a conductor or a lattice or a free electron gas, depending on the question to be answered.

Modelling cycle

We describe the modelling process by a simple version of the modelling cycle. We start with a problem, which is to be solved using tools from mathematics. In the first stage the problem is described in terms of relevant non-mathematical concepts. During this stage one typically has to make some choices about (simplifying) assumptions. The result of this stage is a *conceptual model*. This conceptual model is then translated into a *mathematical model*, which can be analyzed mathematically. The actual translation of the conceptual model and the original question into mathematics may also be subject to certain choices. Next, the mathematical solution is translated back into the context and language of the original problem. We call this *interpretation*. Finally, one *validates* the solution. If necessary, one starts the modelling cycle all over again, adapting one or more of the steps.

Role of mathematics in modelling

The role of mathematics in modelling can vary considerably. It can be elementary or advanced. Sometimes computers are needed to aid mathematical analysis. The mathematics may involve calculus, algebra, geometry, combinatorics or some other field. The modelling problem can be well-defined with clear-cut data, a specific question, a standard mathematical model and ditto solution. In such problems mathematics and context science merge into a very potent mixture. The interplay between mathematics and context is then especially fruitful with techniques like dimensional analysis, where mathematical algebra is applied to physical units. The famous theoretical physicist Wigner was quite right when he spoke about “the unreasonable effectiveness of mathematics”! Conversely, a physical concept like velocity can be helpful to learn a mathematical concept like the derivative.

There may be several possible models and it is not always clear *a priori* which one serves our purpose best. If one doesn't have a complete theory describing the relevant phenomena, one usually fills the gaps by posing simple (e.g. linear) relations. For such models validation is a main point of concern. Most models are not built up from scratch anyway, but emerge as refinements and combinations of existing models.

Applications of mathematics in maths education

Mathematics started as an applied science, dealing with practical problems in trading, measurement, navigation, etcetera. The separation of theoretical mathematics from the empirical sciences is a relatively recent phenomenon, brought about by the development of non-euclidean geometry around 1800. In the middle of the nineteenth century mathematical education followed this trend and its focus shifted from applications to logical reasoning. Since then, the emphasis has swung back and forth between pure and applied mathematics (Niss, Blum & Galbraith, 2007).

Mathematics' education should pay attention to both sides of mathematics. However, many students consider mathematics as a theoretical, abstract subject, which hasn't much to do with reality (Greer, Verschaffel & Mukhopadhyay 2007). They have a blind spot for applied mathematics and the role of mathematics in the sciences or daily life. If students never learn how to apply mathematics, then their mathematical knowledge is indeed useless. Furthermore, it is counterproductive if common sense, intuition and reality are not used to aid mathematical understanding.

Modelling and the Dutch mathematics teaching programs

Non-mathematical contexts have played an important role in parts of Dutch mathematics education since 1985. Since 1998 all mathematics programs for secondary education involve modelling. The experiment which preceded the introduction of the new program indicated that assessment of open modelling tasks was a major problem and was avoided by many teachers. The modelling tasks in the national exams, too, paid little attention to conceptualization, interpretation and validation (De Lange, 1995). To counteract this deficit, the Freudenthal Institute in Utrecht started organizing modelling competitions for schools where these aspects *do* play an essential role.

All these efforts have partially paid off: PISA shows that Dutch students perform well on modelling related tasks. On the other hand, Wijers & Hoogland (1995) and De Haan & Wijers (2000) mention in their evaluation reports of the above mentioned modelling competitions that many students' papers lack in mathematical substance. Students tend to neglect relevant concepts and work by trial and error. Sins (2006) also laments the lack of conceptual thinking and understanding of the purpose of modelling. Future maths education should address these weaknesses more effectively.

PROFESSIONAL DEVELOPMENT

Ongoing professional development is obligatory by Dutch law since 2006. Since many maths teachers in upper secondary education have only scant knowledge of applications of mathematics, post graduate courses for teachers should fill this gap.

We use Schoenfeld's description of complex tasks like modeling (Schoenfeld, 2008, based on his work on problem solving 1985 and 1992). The essence of this framework is as follows. Anyone who takes up a complex task like mathematical modelling starts with certain *knowledge* (not only mathematical knowledge like facts, algorithms, skills, heuristics, but also domain knowledge), *aims* and *attitudes* (opinions, prejudices, preferences). Parts of these are activated, one makes *decisions* (consciously or not, depending on one's familiarity with the problem), one adjusts aims and designs a plan. During the execution of the plan one *monitors* the progress on several levels, going back and forth between the stages of the modelling cycle. Metacognition thus plays an important role in modelling.

We address the issues of aims and attitudes in the sections *Goals, Authenticity, Dispositions* and *Epistemological understanding*. Knowledge aspects are dealt with in the sections *Domain knowledge*, *Authenticity*, and *Computer modeling*. We conclude with a discussion of decisions and monitoring in *Monitoring* and *Assessment*.

Goals of teaching modelling

Modelling isn't easy. It takes a lot of time and is difficult to assess (Galbraith, 2007a) and (Vos, 2007). So why should we take up modelling in mathematics education? First, students have to learn how to apply mathematics, to prepare them for their further education and their jobs, as well as for everyday life. (It might improve their understanding of mathematics as well.) Modelling can help to achieve this (Niss, Blum & Galbraith, 2007). Second, modelling shows that mathematics is useful to scientists as well as practical problems solvers. Third, modelling is useful for students to make their picture of mathematics more complete: it is not a set of ancient, irrelevant algorithms, but an interesting, important, creative, still developing part of science, society and culture (Blum & Niss, 1991). Finally, modelling may help to counteract naïve conceptions like the illusion of linearity (De Bock, Verschaffel & Janssens, 1999; Greer & Verschaffel, 2007).

Authenticity

According to Galbraith (2007b): "Goals and authenticity are in practice inseparable, as the degree to which a task or problem meets the purposes for which it is designed is a measure of its validity from that perspective." Palm (2007) also emphasizes the importance of authenticity. He describes an experiment where two different tasks are distributed randomly among 160 Swedish school children. Mathematically, the tasks are identical: to determine how many busses are needed if 360 students have to be transported and each bus can hold 48 students. One version consisted of just this question, the other was much wordier, paying attention to other aspects of the school trip as well. The second, more authentic version was solved correctly by 95% of the students, whereas the first version was solved correctly by only 75% of the students! Greer & Verschaffel (2007) and Bonotto (2007) also describe how lack of authenticity can hamper students to use common sense in maths class. Authenticity is also beneficial for motivating students. Lingefjärd (2006) found that students are interested in problems concerning health, sports, environment and climate. Van Rens (2005) showed that mimicking scientific research practice in the class room, including writing papers and peer review, enhances motivation and improves the quality of the students' work.

Dispositions about modelling

Abstraction and generalization belong to the core business of mathematicians. Model building, on the other hand, depends critically on the characteristics of the context and the specific research question. This tension (Bonotto, 2007) between mathematics and modelling makes many maths teachers and students feel uncomfortable (Kaiser & Maass, 2007). In their opinion there is no place for modelling in the mathematics curriculum, which should be devoted to "proper" mathematics. We know, however, that even students with solid mathematical knowledge are not necessarily able to use this knowledge outside mathematics (Niss, Blum & Galbraith, 2007). In the minds of many students and teachers there is no connection between the subjects taught during maths class and the topics taught next door by the physics or economics teacher. We are not just talking about superficial problems like different notations, conventions or terminology, but also about deeply rooted opinions about mathematics and reality.

Greer, Verschaffel & Mukhopadhyay (2007) argue that students are trained to expect that problems in maths class are always solvable, that solutions are unique and that reality can be ignored. Students even think that using non-mathematical knowledge is forbidden (Bonotto, 2007). As Schwarzkopf (2007, 209-210) put it:

> The students do not follow the logic of problem solving, but they follow the logic of classroom culture.

This obviously impedes successful modelling in teaching of mathematics.

Understanding what modelling is about is strongly related to dispositions about modelling (Sins, 2006). He distinguishes between three levels. At the lowest level a model is considered a copy of reality. Students at the intermediate level understand that models are simplified representations of reality constructed with a specific goal. Different goals may lead to different models. At the highest level attention shifts towards theory building: Models are constructed to develop and test ideas. Sins experiments show that a higher level of epistemological understanding leads to better models. Students at the highest level use their domain knowledge to analyze the relevant variables and the relations between them. Most students, however, are at the middle level. They try to reproduce measurement data by varying the parameters one by one. They ignore domain knowledge, reason superficially and consequently produce poor models.

Epistemological understanding

Sins (2006) investigated the influence of epistemological understanding of modelling on the quality of models made by students. He advizes to make the goals of a modelling task explicit: what do we want to *understand* or which problem do we want to solve? He proposes that the teacher presents reasonable models to his students who have to analyze and improve them. This way students learn about the tentative nature of models: They are not perfect copies of reality, since they often depend on choices, approximations and incomplete information. Furthermore, this adjusting of existing models and iteration of the modelling cycle gives a fairer picture of the modelling process as performed by experts, who of course have lots of standard models at their disposal and rarely start from scratch.

It is not sufficient to just talk about modelling with students. Indeed, students who model themselves perform significantly better on modelling skills such as using various data, recognizing the limits of applicability of a model and adjusting models (Legé, 2007). However, even if students have a sound epistemological understanding of modelling, in very open modelling tasks they still do not always understand what is given, what is asked and how to attack the problem.

Domain knowledge

Modelling typically concerns extra-mathematical contexts. As a consequence, the maths teacher may find himself in an awkward position, since he cannot be an expert in all possible modelling domains, such as the natural sciences, computer science, economics, arts, sports or other specific (not necessarily scientific) contexts.

The same holds for students. We know, however, that lack of domain knowledge leads to poor models (Sins, 2006). So it is essential to choose a modelling context where students' lack of domain knowledge is not an issue. Furthermore, the teacher has to encourage the students to actually use their domain knowledge. Finally, the teacher has to be familiar with the modelling problem himself. In particular, he has to be aware that a problem can lead to several different models.

Computer modelling

Computers can be useful to in modelling, especially when the mathematics gets complicated. Using a graphic modelling tool it is easy to modify a model, run simulations and display the results graphically. The representation of a model in such a tool reminds one of a concept map in the sense that it indicates the relevant variables and the relations between them.

In Löhner (2005), who summarized claims and results from the literature on computer modelling, we find that computer simulations make validation and adaption of models very natural. It facilitates exploring the limits of validity of a model.

Unfortunately, it also facilitates the superficial ad hoc modifications and data fitting behaviour Sins (2006) warns against. Löhner (2005) finds that students who work with computer models over a longer period of time tend to start working in a *top down* fashion and develop a more mature, qualitative attitude towards modelling, although one shouldn't expect too much in this direction. Simulation results may lead students to new research questions. Computer modelling is challenging and motivating for students, as long as the models are not too complicated and the software is easy to use. It also helps to turn abstract, theoretical models into something more concrete, which makes it easier to discuss these models. Finally, experimenting using computer modelling helps students to understand and remember the phenomena and associated theory.

Monitoring

Monitoring the modelling process of a group of students can be very difficult. Different students make different and often implicit assumptions and simplifications, have different goals and use different data and notations. This makes monitoring the modelling process of a group of students very difficult if not virtually impossible (Doerr, 2007). It is thus very important to force students to make all of the above explicit. The teacher can make life easier by inserting go-or-no-go-moments at certain points of the modelling cycle. However, even if everything is written down neatly, it can still be difficult for teachers and students to compare different modelling results. Are the differences due to different conceptualization or to mathematical errors? This problem can be moderated by discussing and comparing the various conceptual models with the whole group. Monitoring becomes much simpler if consensus is reached about the data, the goal and notations. This also facilitates understanding and comparing the different results, which in turn improves motivation and understanding (Van Rens, 2005; Bonotto, 2007).

If modelling is new to students it is advisable to have them record their modelling process in a pre-structured log. In this log they have to describe all data, assumptions, etcetera. The log can also be very useful for assessment.

Assessment of modelling

One of the main obstacles when teaching modelling is evaluation. The goals of modelling can not be assessed as objectively as is customary in education of mathematics (De Lange, 1987). Maths teachers who take the non-mathematical aspects of modelling seriously have to come to terms with this lack of objectivity. To reduce the subjectivity one can use a team of assessors **(**Antonius, 2007; Vos 2002; Vos 2007**)** and weighted lists of evaluation criteria. One can search for *rubrics* on the internet and adapt them to the assessment at hand. One can use the modelling cycle to generate evaluation criteria: conceptualization (analysis of the original problem, data, relevant concepts, data, variables, relations, simplifications, modelling goal), mathematization, mathematical analysis (completeness, correctness), interpretation, validation, conclusions, adaptions. Other criteria which are mentioned by experienced assessors of modelling are general impression, readability, representation and originality. A common pitfall is to overestimate appearance, so it remains necessary to study and evaluate *thoroughly* the technical contents of students' work (De Haan & Wijers, 2000).

De Lange (1987) argued that traditional written tests are not suited very well to test higher skills like modelling. He mentions several alternatives, which may be more appropriate, like group work, home work, essays or oral examinations. Vos (2007) argues, however, that alternative tests like observation, interviews and portfolio's are often too time consuming and too subjective. She investigated experimentally how teamwork can indeed reduce subjectivity. Furthermore, she shows how alternative, laboratory like tests using manipulative materials can lead to valid assessment of modelling skills. These results are confirmed by Antonius (2007), who adds, however, that this kind of assessment levels out the differences between strong and weak students.

Above we emphasized the importance for teachers of taming excessive divergence for monitoring the modelling process. Similarly, assessment is facilitated by posing authentic "convergent" modelling tasks (Niss, 2001):

> Mathematical modelling involves the posing of genuine, non-rhetorical questions to which clear and specific answers are to be sought.

CONCLUSIONS

To prepare teachers for mathematical modelling teachers' colleges have to take into account (apart from the necessary mathematics and their didactics) the lessons learned from literature about the role and goals of modelling in science and mathematics education, the modelling cycle, dispositions, authenticity, epistemological understanding, domain knowledge, computer modelling, monitoring and assessment. Unfortunately, empirical research on modelling education is mostly restricted to short term teaching experiments. To design effective modelling education it is necessary to gain more experience and to systematically carry out *longitudinal* research into the effects of teaching modelling.

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MODELLING AND FORMATIVE ASSESSMENT PEDAGOGIES MEDIATING CHANGE IN ACTIONS OF TEACHERS AND LEARNERS IN MATHEMATICS CLASSROOMS

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This paper explores how modelling and associated tasks and pedagogies can bring about a refocusing of the nature of assessment which it is argued, when viewed through the lens of Cultural Historical Activity Theory, appears to currently adversely mediate the object of activity in many school mathematics classrooms. An international professional development programme in mathematical modelling has been designed with formative assessment as a key theme. Drawing on data resulting from classroom activity developed from this programme I argue that modelling undertaken with a formative assessment approach can bring about a significant change in classroom activity for learners and teachers that might better prepare students to apply mathematics.

INTRODUCTION

A Cultural Historical Activity Theory (CHAT) analysis of classroom activity suggests that in almost all classrooms, at least in England, the collective activity of teacher and students is mediated to a large extent by "rules" of assessment and "performativity" which ultimately focus on learners' qualifications. Whilst this is not necessarily clearly discernible on any particular day in any particular classroom, recent research (Williams et al, 2008) points to an all pervasive culture of "performativity". Systemic measurements of performance and accountability are seen to drive the curriculum in our classrooms to the extent that this can be detected in classroom discourse with teachers making regular reference to the demands of assessment and examiners. In terms of CHAT, assessment and performance measures are part of, and lead, the "rules" that mediate the activity of the classroom activity system with its object of learning mathematics. These rules have culturally and historically evolved affecting, for example, the texts and pedagogic instruments used by teachers and learners. They also help mould and define the expectations of what lessons in mathematics should be, in the sense of Brousseau's didactical contract (1997).

The current culture is, therefore, such that a teacher's enactment of the curriculum does not necessarily match his or her espoused beliefs about the nature of the subject they teach, and consequently how it should be taught and learnt (see for example Tobin and McRobbie (1997)). Boaler (1997) documents case studies that illustrate how, in England, this has led to a narrowing of professional practice and risk taking, leading to a normative cultural script (Wierzbicka, 1999) where many lessons comprise of an initial period of transmission by the teacher of key mathematical ideas or rules and procedures followed by a period where students practise these. This is not only detrimental to learning where shallow or surface learning dominates at the expense of deep learning and understanding (see for example, Entwistle (1981)), but can also be responsible for a narrowing of participation. As Brown et al (2008) report this can lead to situations, when students are asked about their likely future participation in mathematics beyond the compulsory curriculum (to age 16 in the UK) to responses such as

> “I hate mathematics and I would rather die.”

This paper explores, how, taking a mathematical modelling approach to classroom practice that incorporates formative assessment introduces a range of new mediating instruments allowing teachers and learners to refocus their classroom actions. The work reported here resulted from classroom experiences that emanated from the work of a professional development programme as part of an EU funded Comenius project, Learning and Education in and through Modelling and Applications (LEMA). Central to the approach advocated by this programme is the focusing of classroom activity on modelling with teachers and learners becoming fully involved with formative assessment practices. An overview of the framework that guided the development of the programme in this respect is outlined in the next section before some resulting classroom experiences are described and analysed in terms of CHAT.

ASSESSMENT FOR LEARNING

Following a thorough review of research relating to assessment, Black and William (1998a) claimed that focussing on formative assessment, i.e. assessment with the purpose of informing teacher and learner about learner progression, raises student attainment. Thus the assessment *for* learning movement, as it became, conceptualised assessment as crucially providing feedback at all stages of day-to-day classroom activity and promoted this in favour of summative assessment, or assessment *of* learning, where the focus is on measuring outcomes, often being used to give grades. In follow-up studies that involved teachers and their pupils working with researchers Black and Wiliam (1998b) clarified the key areas that need to be considered if classroom assessment practices are to be effective in improving learning. These are identified in the diagram of Figure 1 and outlined below.

This emphasises an overarching pedagogic philosophy in which teachers and students strive together to ensure that, as a community, they will use their monitoring, at every stage, of the mathematical modelling taking place in their classroom to inform them of how to improve students’ learning. Fundamental to this is the clarifying of learning objectives so that all know what it is they are trying to achieve.

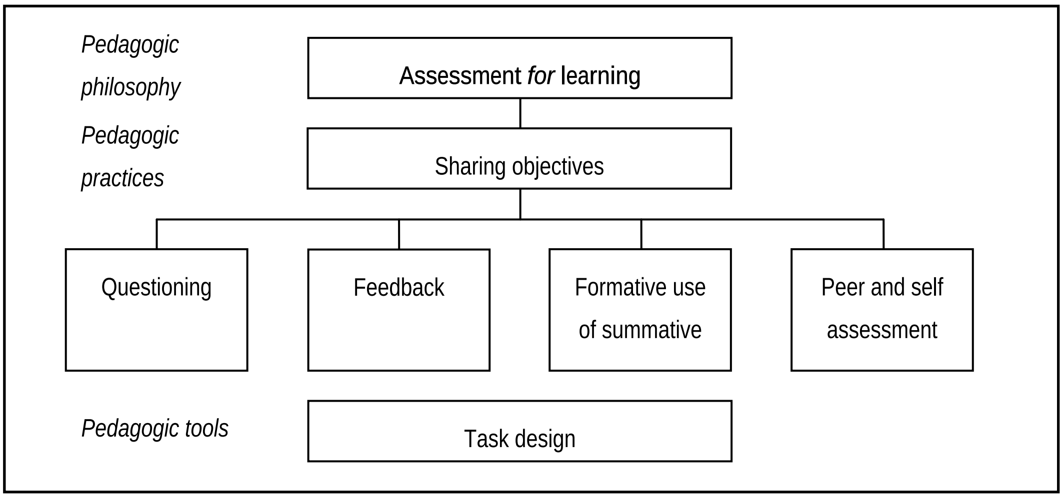


Figure 1. Schema illustrating key aspects of formative assessment

In terms of mathematical modelling this requires that students understand the overall nature and aim of modelling and the key sub-competencies they need to acquire. In supporting assessment *for* learning four key underpinning aspects of classroom activity were identified by Black and Wiliam:

(i) *Questioning*. Classroom discussion between teacher and students and between students is crucial in the learning of mathematics (see for example, Ryan and Williams, 2007) and fundamental to this are the questions that teachers pose. In summarising research in this area Tobin (1987) points to findings that suggest that the time between a teacher asking a question and intervening, perhaps to re-phrase the question, (often referred to as "wait-time"), is in many classrooms very short, and if lengthened leads to more effective learning. However, he points out that it is the quality of the question that is crucial in opening up opportunities for thinking and consequently learning.

(ii) *Feedback*. How teachers best give feedback to students to scaffold their learning (in the sense of Vygotsky) is always an issue of concern but this is possibly even more problematic when developing new pedagogic practices such as those associated with mathematical modelling. The research in this area that informed the development of good practice in formative assessment is clear in suggesting that the best feedback focuses on the task, is given immediately and is given orally rather than in writing. An important study by Butler (1988) reached the conclusion that as soon as teachers give a grade for a piece of work their comments about how to improve are ignored and that feedback that comprises of comments about how to improve instead of grades is more effective in raising student attainment.

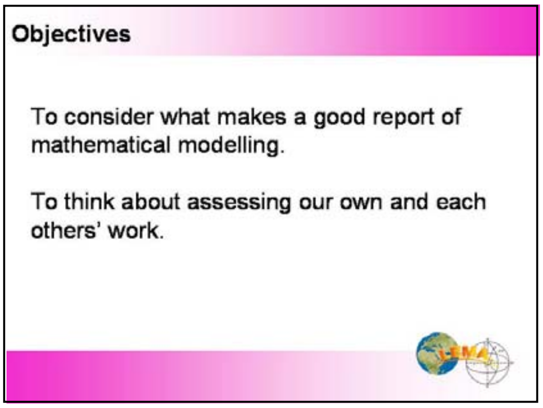
(iii) *Formative use of summative assessment*. Much work has been done in developing ways in which such summative assessment of mathematical modelling can be carried out: see for example many of the bi-annual proceedings of the ICTMA. Whilst this has had little impact on summative assessment that leads to qualification at a national level, the frameworks and structures that have been developed may well provide suitable structures to inform formative assessment in classrooms.

(iv) *Peer & self assessment*. Of course, learning is most effective when the learners themselves have a clear understanding of what it is they are trying to achieve, can measure their progress against clear objectives and know how to proceed to achieve their aims. Hence, the important focus on clarity of learning objectives. Peer assessment, where students assess each others' work, provides a valuable direct source of feedback for students, often using a language and given in a manner they readily understand, and also allows them to start to reflect on their own work and learning.

In addition to these important pedagogic practices one further key area that needs to be considered is the design of the tasks that are used. Here, where assessment is being refocused and considered as being an integral part of daily classroom activity, the tasks students are asked to engage with are therefore absolutely critical. If, for example, the teacher wants students to focus on their ability to interpret from mathematical model to reality, the tasks used need to be designed to allow a range of possible and appropriate interpretations to be made by the students being taught. On other occasions other particular modelling sub-competencies or meta-cognitive awareness of the modelling process as a whole may need to be the focus of attention of classroom activity, requiring tasks to be designed accordingly.

A MODELLING CLASSROOM

Here I describe some detail of a lesson that was designed to involve students in mathematical modelling incorporating formative assessment approaches. Due to restrictions of space I focus on just two aspects of the lesson particularly related to formative assessment practices: namely teacher questioning and peer assessment. The lesson was one of a sequence taught by both the teacher of the class and researcher following the teacher's partial attendance at the LEMA professional development programme in England, which the researcher had led following his work as part of the development team. The lessons were developed using materials and approaches advocated by the programme, and in the particular lesson outlined here the intention was to involve students in peer assessment as a prelude to future self assessment. The students were aged 13-14 and in an upper mathematics set in a comprehensive school catering for students of all abilities (aged 11-18), in a town in the north west of England. The teacher started the lesson by introducing its objectives (Figure 2a). The emphasis of the first objective was on the development of good communication skills about mathematical modelling rather than on the mathematics itself; additionally the remaining objective of the lesson was for students to focus on their assessment of their own modelling activity and that of their peers. Following this the teacher reminded the class of the sub-competencies of mathematical modelling to which they had previously been introduced, and which had been clarified using the schema of Figure 2b. This is based on that used as the theoretical basis of the PISA study (OECD, 2003); here it has been adapted to highlight processes that are used in developing a solution to a modelling task as the "modeller" moves from one key stage to the next. The teacher highlighted these suggesting that the students might wish to think about them when making a poster of their "solution".



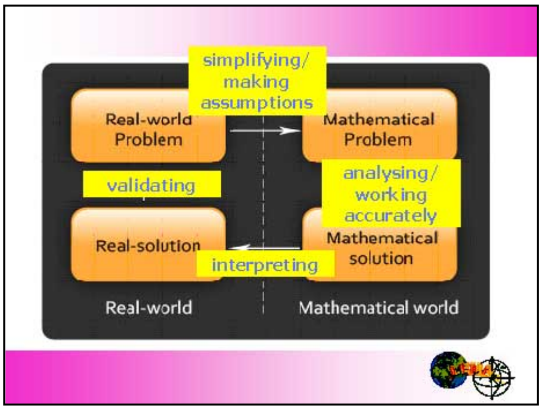


Figure 2a. Lesson objectives. **Figure 2b. Schema outlining modelling cycle**

Finally in this introduction to the lesson the teacher set the task:

> In a school playground there are two trees: one is small and one is large. There is also a straight wall.
>
> A group of pupils organise a race: each pupil starts at the small tree; then has to touch the large tree; followed by the wall; before finally running back to the small tree.
>
> Where is the best place for a pupil to touch the wall?

The pupils started to tackle the problem, working in groups of four or five: as the lesson was shorter than usual, the pupils had only about half an hour to complete their work and poster. The teacher circulated the room as the groups worked. Here I illustrate the teacher's interactions with one group. He approached their cluster of tables and discussed where they had got to.

Teacher: OK, what's your group doing?

Pupil 1: Going for the middle point of the wall (gesturing to a diagram of the situation)

Teacher: And you think that's the solution?

Pupil 1: Yeah

Teacher: How could you convince somebody that's the solution?

Pupil 1: I don't know.

Pupil 2: Does it have to be in a triangle [referring to the path taken by someone in the race]

Teacher: [reflecting the question to other members of the group] Does it have to be in a triangle?

Pupil 3: Yeah, because there are three points….

Pupil 2: Yes, that's the only way you can do it.

Teacher: Well, I suppose somebody could run

Pupil 2: If the wall was there, then they could just go like that [pointing to a sketch diagram]

Teacher: [indicating to the rest of the group Pupil 3's sketch with a section of wall lying along a straight line joining the two trees] oh right, so if the wall was there…. so the first thing you are doing is making some assumptions. So you have to say what your assumptions are: you've assumed everything is in a straight line [indicating this on Pupils 3's diagram] and you've assumed that it's like that [indicating the triangle path on Pupil 1's diagram]. What is it you actually want to….

Pupil 1: Find out where the wall is.

Teacher: Right, so at first you have to decide what the situation looks like…..

The teacher continued circulating the room encouraging groups as they worked on the problem and towards the end of the period completing their posters which explained what they had done to arrive at their solution. Following this the teacher focused the whole group on the second objective of the lesson: "To think about assessing our own and each others' work". This was "operationalised" by adopting the pedagogic practice of asking each group to consider the poster of another using pink sticky notes to identify up to 3 positive features of the poster being considered and 3 or fewer features where there could be improvements using yellow sticky notes (see Figure 3 below). As these early attempts demonstrate much of the feedback focused on issues relating to communication ("Not enough diagrams") and aesthetics ("Cool trees! [referring to drawings] and "Colourful"). In many ways this was a disappointing outcome, but this was the first time the class had been asked to take part in such formative assessment processes, and in a lesson a week later students gave slightly more attention to issues of mathematical content but there still remained room for there to be more of a focus on the processes involved.

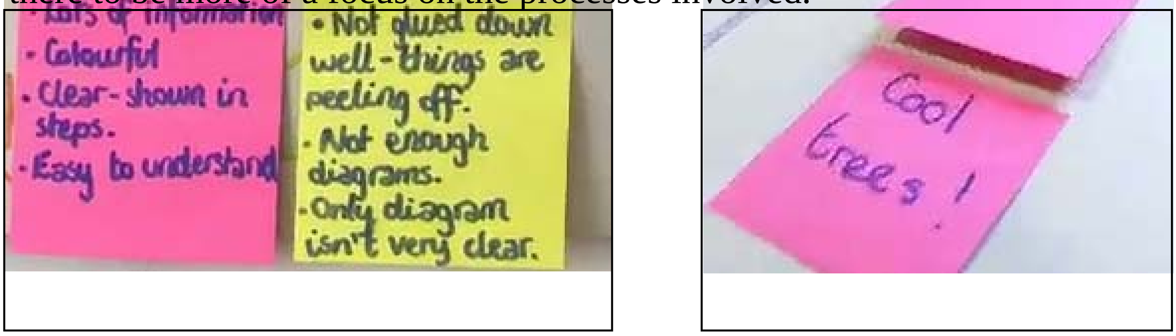


Figure 3. Peer feedback on modelling task.

DISCUSSION

In the brief extracts with which I illustrate a modelling lesson here we observe activity that is very different from the normative script of lessons that I describe earlier and which a recent nationwide inspection report corroborates as being the norm (Ofsted, 2008). Consider, for example, the interaction of the teacher with the group of students, where the teacher prompts discussion and problem solving rather than “transmitting” rules and procedures. I now consider how Cultural Historical Activity Theory might enlighten our thinking about the nature of such lessons and highlight potential areas of conflict for teachers who attempt to follow such approaches.

CHAT builds on the fundamental thinking of Vygotsky, who suggested that the action of a subject is mediated by ‘instruments’ which may include artefacts and tools, or in the case of communicative action, as is often the case in classrooms, by cultural tools, concepts and language genres (see for example, Engestrom, 1995). This is indicated by the top triangle in the schema of Figure 4.

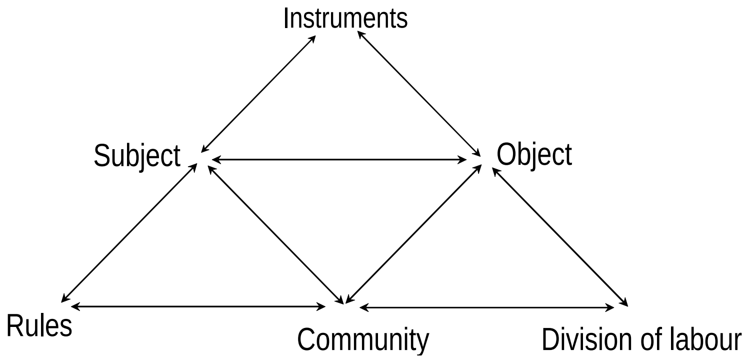


Figure 4. Schema of activity system

Leont’ev extends thinking to take account of the communal nature of activity: the schema of Figure 4 thus indicates the additional nodes of mediation in a culturally-mediated and historically-evolved Activity System. These indicate the importance of the ways in which the division of labour and associated norms/expectations/rules mediate the subject’s activity in relation to the community.

I suggest that in the modelling classroom which attempts to involve formative assessment practices there is a shift in the attention of both teacher and students to view assessment in terms of informing learning and this in turn considerably alters the dynamics of the learning community. Highly visible in bringing about this refocusing are the pedagogic tools that the teacher employs. Crucial in this regard is the use of a rich modelling task, but equally important are (i) the sharing of learning objectives that in this case (at an early stage of the students’ development as mathematical modellers) focus on the object of the activity (the learning of mathematics), (ii) the teacher’s decision to involve groups of students in working on this, (iii) their need to develop a poster communicating their solution together with their way of working and (iv) the peer assessment activity which clearly refers back to the shared learning objectives.

Greater insight might be gained into the nature of the classroom activity by exploring further Leontev’s (1978) theoretical development of Vygotsky’s thinking in which he explores the nature of a subject’s action in relation to the communal activity and the manner of the operation that achieves this. He suggests three parallel hierarchies shown schematically in Figure 5.

ACTIVITY	--	ACTION	--	OPERATION
COMMUNITY	--	SUBJECT	--	INSTRUMENTS
MOTIVATION	--	GOAL	--	METHODS

Figure 5. Schema illustrating the nature of the action of a subject in relation to communal activity.

Thus in terms of classroom mathematical activity we need to understand how things are “normally” for the subject and how the modelling classroom differs from this. In both classrooms the activity has as its object the learning of mathematics: normally this is motivated for the community, as I suggest earlier, by the pressure to perform well in summative assessment and with institutional measures of performance having a major influence in defining goals related to achieving high grades in national assessments. This has led over time to a use of a restricted range of instruments: in particular, reflecting the highly structured nature of the summative assessment (Wake, 2008) the texts used involve students in practice exercises that in the main involve students in the recall and use of instrumental understanding (Skemp, 1978). Equally pedagogic practices are in general restricted with “the teacher doing most of the talking, emphasising rules and procedures rather than concepts or links with other parts of mathematics” (Ofsted, 2008 p. 20), and with teacher talk constituting “a substantial proportion of pupils’ time for learning mathematics” (ibid. p. 20). Thus, the actions of teacher and student might to a large extent be considered as active and passive respectively.

In the modelling classroom, however, the introduction of new instruments (tasks and pedagogic practices) brings about a change of motivation and goals. On these occasions the motivation for teacher and learners, as encapsulated in the learning objectives of the lesson illustrated here, has been, perhaps only temporarily as I shall discuss below, re-focused on the students’ learning. This alters the nature of the actions of both teacher and learners: *both* are now active with learners struggling to solve a task and make reflective judgements about their ability to do so using new rules of assessment that focus on process as opposed to outcomes. At this early stage of this class of students working on modelling the *operation*, the method by which the action is instrumentally accomplished, requires careful attention by both teacher and students. The introduction of new instruments for use by both teachers and learners destabilises their usual ways of operating, introducing new challenges for all. Thus the development provides a ‘break-down’ in the usual routine of the classroom activity which now becomes the focus of attention and hence conscious action. Previously we (Williams and Wake, 2007) and others (eg Hoyles et al, 2001) have recognised this in workplace activity. Here in classrooms, I propose, this as a useful way of deliberately provoking a means of mediating changes in the *actions* of teachers and learners.

Finally, a word of warning! The developments in classroom activity arising from the LEMA programme, such as described here, are in many ways encouraging, demonstrating the potential to enrich the learning experience of students of mathematics. The claim by Black and Wiliam that a focus on formative assessment practices will ultimately lead to increased attainment in summative assessment is helpful to teachers working in a system where measurement of performance is so pervasive. However, bringing about the necessary changes in teacher and student actions involves teachers, either individually or as a collective, in considerable risk taking: when all around are following the “safe” option there is a great deal of pressure to conform to the “norm”. Additionally, as Hodgen (2007) points out the simple messages often associated with “assessment for learning” are not necessarily sufficient in allowing teachers to make the shift. Perhaps programmes of professional development such as that developed by LEMA will help in this regard. However, it seems unlikely that teachers will be able to sustain developments in such a way unless summative assessment is realigned to support this. Elsewhere, (Wake et al, 2004) our research has shown that attention needs to be paid to each mediating node of an activity system if curriculum development of this sort is to be effective: in paying such attention there needs to be alignment of purpose and an awareness of how each part of the system interacts with each other.

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TOWARDS UNDERSTANDING TEACHERS' BELIEFS AND AFFECTS ABOUT MATHEMATICAL MODELLING

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Work in progress on a framework aiming at capturing teachers' beliefs about mathematical models and modelling is presented. It is suggested that the belief structure of mathematical models and modelling as perceived by teachers fruitfully might be explored as partly constituted of the teachers' beliefs about the real world, the nature of mathematics, school mathematics, and applying and applications of mathematics. Some aspects of the suggested framework are explored using two case study interviews. It is found that the two teachers do not have any well formed beliefs about mathematical models and modelling, and that the interpreted beliefs structure of the teachers contain inconsistencies which are made explicit within the framework. The empiric findings also suggest some modifications of the framework.

INTRODUCTION

Since the mid 1960s gradually more emphasis has been put on mathematical modelling in the written curricula documents governing the content in Swedish upper secondary mathematics courses (Ärlebäck, in preparation). In the latest formulation from 2000, *using and working with mathematical models and modelling* is put forward as one of the four important aspects of the subject that, together with *problem solving*, *communication* and *the history of mathematical ideas*, should permeate all teaching (Skolverket, 2000). Indeed, it is stressed that "[a]n important part of solving problems is designing and using mathematical models" and that one of the goals to aim for is to "develop their [the students'] ability to design, fine-tune and use mathematical models, as well as critically assess the conditions, opportunities and limitations of different models" (Skolverket, 2000). However, as noted by Lingefjärd (2006), "it seems that the more mathematical modeling is pointed out as an important competence to obtain for each student in the Swedish school system, the vaguer the label becomes" (p. 96). The question naturally arises what mathematical models and modelling are and mean for the different actors in the Swedish educational system. Ärlebäck (in preparation) concluded that the governing curricula documents, the *intended curriculum* (Robitaille et al., 1993), do not give a very precise description of the what a mathematical model or mathematical modelling is, but rather describe the concepts in an implicit manner as exemplified above. Therefore, focus is turned to teachers who interpret and realize the intended curriculum, and thereby have a big impact on which mathematical content and what view of mathematics students in classrooms are exposed to. One way to try to understand part of the process of what ends up in the classroom, the *(potentially) implemented curriculum* (ibid.), is provided by studying teachers' beliefs.

The question of how teachers' knowledge, beliefs and affects towards the learning and teaching of mathematics influence and relate to their practice is a highly active field of research (Philipp, 2007). Thompson, acknowledging the dialectic nature between beliefs and practice, argues that "[t]here is support in the literature for the claim that beliefs influence classroom practice; teachers' beliefs appear to act as filters through which teachers interpret and ascribe meanings to their experience as they interact with children and the subject matter" (Thompson, 1992, p. 138-139). Indeed, the six authors of the chapters on teachers' beliefs in the book edited by Leder, Pehkonen and Törner (Leder, Pehkonen, & Törner, 2002) all infer a strong link between teachers' belief and their practice, working from a premise that could be expressed by "to understand teaching from teachers' perspectives we have to understand the beliefs with which they define their work" (Nespor, cited in Thompson, 1992, p.129). In particular in connection with mathematical modelling, while discussing four different categories of mathematical beliefs, Kaiser (2006) concluded that depending on the mathematical beliefs held by a teacher, it is more or less likely that they build up obstacles for introducing applications and modelling in their mathematics teaching. Furthermore, Kaiser and Maaß (2007) looking at "what are the mathematical beliefs of teachers towards applications and modelling tasks?" (p. 104), found that for the group they studied, applications and modelling did not play a significant role in their beliefs about mathematics and mathematics teaching. The investigated teachers rather created/modified and adapted application-oriented beliefs in line with their existing mathematical beliefs.

In a research project aiming to design, implement and evaluate sequences of lessons exposing students to mathematical modelling in line with the present governing curricula carried out in collaboration with two upper secondary teachers, initial individual interviews was held with the participating teachers. The purpose being first to provide information about the teachers' background and their views and beliefs on the nature of mathematics, about their teaching, views on problem solving and mathematical modelling, as well as their opinion for the reasons and aims for mathematical education. Secondly, the interviews also intended to end up in a common understanding and agreement of key concepts among the researcher and the two teachers, laying the foundation for the collaboration project. The aim of this paper is partly theoretical in that we seek to develop a framework trying to capture and conceptualize beliefs about mathematical models and modelling and relate these to other types of beliefs studied in the literature. Nevertheless, it also aims to provide background about the two teachers participating in the research mentioned above and hence to feed in to the bigger analysis of that project.

BELIEFS, BELIEF STRUCTURES AND BELIEF SYSTEMS

Reviews on research on different aspects of beliefs in connection to mathematics knowing, teaching and learning often conclude that there is a great degree of variation of the involved concepts and their meaning used by different scholars (Leder et al., 2002; Pajares, 1992; Philipp, 2007; Thompson, 1992). The motive with the following small theoretical exposé is to establish the vocabulary used in the paper and to relate some of different concepts used in the literature.

As a point for theoretical departure we start from the work, and use the vocabulary, of Goldin (2002), who defines beliefs as one out of four "subdomains of affective representation[s]" (p. 61), distinguishing between *emotions*, *attitudes*, *beliefs*, and *values*, *ethics and morals*. More specifically, beliefs are "multiply-encoded cognitive/affective configurations, usually including (but not limited to) prepositional encoding, to which the holder attributes some kind of *truth value*" (p. 64, emphasis in original). For an individual, a collection of mutually reinforcing or supporting non-contradictory beliefs taken together with the individual's justifications for this constitutes a *belief structure*. Törner (2002) argues that beliefs generally are about something and introduces the notion of this something as a *belief object*, to which a set of beliefs, the *content set* is associated, which can be seen as the analogue of Goldins' beliefs structures. Other scholars often refer to similar constructs as *belief systems* or *cluster of beliefs*, but in Goldins' framework, a *belief system* is an "elaborated or extensive belief structure that is socially or culturally shared" (Goldin, 2002, p. 64). This terminology makes it easy to talk about and distinguish between beliefs held by an individual contra shared beliefs within a community, as well as the dialectic and tension between these types of beliefs.

Many authors deepen their discussion on beliefs drawing on Rokeach (1968) or Green (1971), or a combination of the two, introducing different *dimensions* of beliefs. Rokeach talks about a *dimension of centrality* for the individual, where a *central belief* is a belief which is non-contradicting within a persons' belief structure, whereas beliefs with some disagreeing features are less central for the individual. Green on the other hand introduces the construct of *psychological centrality* and uses *peripheral* and *central* to describe beliefs that the individual holds more or less strongly. Both Rokeach and Green argue that the more central a belief is, the harder it is to change it. Green also talks about *quasi-logicalness,* which captures the fact that some beliefs only are in consensus within a belief structure provided that a non-standard and personal logical explanation is provided. In connection to quasi-logicalness Green also proposed to differentiate *primary* beliefs from *derivative* believes. Returning to Goldins' framework of beliefs, part of the dimensions above are captured by the notion of *weakly*- or *strongly-held* beliefs. The two factors determining to what strength a belief is held are *importance for the individual of the belief being true* and the *degree of certainty the truth-value of the belief is attributed*.

MATHEMATICAL MODELLING

The literature on the aims, use and results of different approaches to incorporate and use mathematical modelling in the teaching of mathematics has steadily been growing since the beginning of the 1980s. The theoretical perspectives invoked display a great variety (Kaiser & Sriraman, 2006) as does the research methods used to explore this vast field of research; see for examples the recent 14th ICMI study (Blum, Galbraith, Henn, & Niss, 2007) and the published proceedings from ICTMA 12 (Haines, Galbraith, Blum, & Khan, 2007).

Mathematical modelling is often perceived as a multistep or cyclic problem solving process using mathematics to deal with real world phenomena. The student or modeller is supposed to use his mathematical modelling skills or modelling competencies (Maaß, 2006) to work through the steps, stages, phases or activities of the process. In this paper mathematical modelling refers to the complex and cyclic-in-nature problem solving process described for instance by Blum, Galbraith & Niss (2007), here illustrated in figure 1.

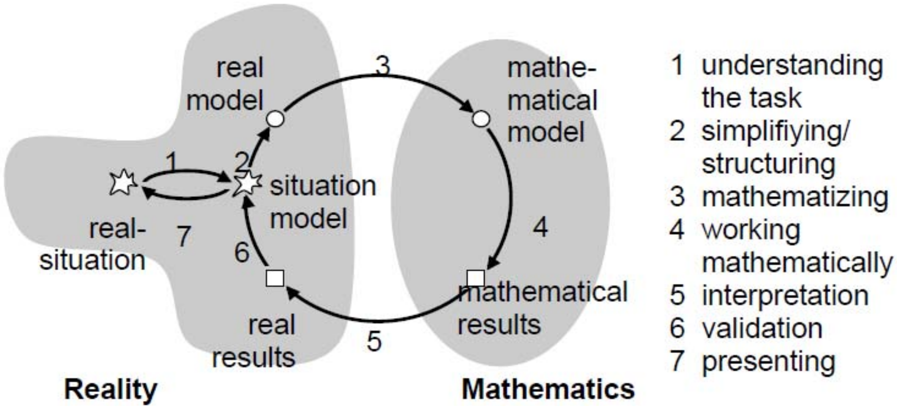

Figure 1. The modelling cycle from Borromeo Ferri (2006, p. 87)

It should be noted that this is only a schematic, idealised and simplified picture of the modelling process. For instance, in an authentic modelling situation the modeller normally jumps between the different stages/activities in a more non-cyclic, but rather unsystematic, manner (Ärlebäck & Bergsten, 2007).

A SUGGESTED BELIEF STRUCTURE OF SOME ASPECTS OF MATEMATICAL MODELLING

In setting out to investigate teachers' beliefs about mathematical models and modelling it is important to be explicit and specific about what object the beliefs should be about. Using the terminology of Törner (2002), the *belief object* under study in this paper is defined to be *mathematical models and modelling as perceived by upper secondary mathematics teachers*. For clarification we stress that the focus at this stage in the research process is not on the teachers' beliefs of the teaching and learning of mathematical models and modelling.

The literature review suggests the importance and influence on teachers' practice of their beliefs about mathematics and its teaching and learning. Hence, the validity of the framework suggested here steams both from analyzing the view taken on mathematical modelling in this paper and from research on mathematical beliefs of various sorts. A teachers' belief structure of mathematical models and modelling is suggested to be constituted of the beliefs of the following *(sub-)belief objects*:

Beliefs about **the nature of mathematics**. This is without question the most general of the constituting sub-belief objects, assumed to serve as a primary and central belief in the belief structure of modelling. The perspective taken on the nature of mathematics might radically change the interpretation and meaningfulness of fig. 1.

Beliefs about the **real world (reality)**. In our view, it is important that the problems used in connection with modelling to the greatest extent possible be from real problem situations in the real world. Different views, both philosophical and pragmatic, potentially influence the way one might think about mathematical modelling and models. In addition, how reality is perceived, especially in contrast to the nature of mathematics, can make a difference when it comes to the interpretation and validation of ones' modelling work. In fig. 1, beliefs about the real world might especially influence the phases 1, 2, 5, 6 and 7.

Beliefs about **problem solving**. In principle, depending on perspective, modelling is about problem solving or problem solving is about modelling (see Lesh & Zawojewski, 2007 for an overview). Regardless of which view adopted, the meaning of and role played by problem solving as a mathematical activity, seen as part of one's practise of one's mathematical knowledge and skill/competence might have important implications for how mathematical modelling and models are perceived. In connection to fig. 1, (mathematical) problem solving beliefs are important for the phases 3, 4 and 5.

Beliefs about **school mathematics**. Thompson (1992) concluded that the consistency between teachers' beliefs about the nature of mathematics and beliefs about the subject mathematics taught at schools are of varying magnitudes. Therefore, school mathematics beliefs are incorporated in the bigger belief structure to capture the potential influences they might have on other beliefs of the teachers.

Beliefs about **applying, and applications of, mathematics**. The application of mathematics is sometime synonymous with different views taken on modelling, and hence it is important to include beliefs about applying and applications of mathematics in the belief structure of mathematical models and modelling. Depending on point of view, beliefs about applications of mathematics are significant for phases 3 and/or 5 in fig. 1.

The five categories of beliefs above are suggested to constitute a way of describing the belief structure of mathematical models and modelling. This framework is initially based on the indicated links to the modelling cycle and will need empirical investigations to be further developed and validated.

This framework does not set up isolated beliefs but, by the discussion above, these beliefs are rather overlapping belief structures in themselves. Hence, an indication of the validity of the framework would be that the substructures display inner coherence, that is, display an inner quasi-logical structure. However, it is possible that taken all together as constituting the belief structure of mathematical models and modelling, incoherencies appear and then the question is which beliefs are more central, primary, and in line with official guidelines.

SOME EMPIRICAL FINDINGS

Although the empirical data used here was not collected primarily with the testing of the above framework in mind, due to its focus on teachers' views on mathematical modelling, we see it as relevant for discussing the viability and usefulness of the framework. As a result, it may also point out directions for how to develop it further.

Method

The interviews with the two teachers (here called Lisa and Sven) in the projects briefly described in the introduction were partly structured around five mathematical problems to serve as a basis for the discussion and reflection. Three of these were standard text problems from a widely used textbook in Sweden, one the so called *Fermi Problem* studied in (Ärlebäck & Bergsten, 2007), and one was *The Volleyball Problem*, a so called *modelling-eliciting activity*, described in (Lesh & Doerr, 2003). The interviews were recorded, transcribed and analysed using what may be called a contextual sensitive categorization scheme based on the five sub-beliefs object in mind. Due to the nature of the data, beliefs about the real world and applications and applying mathematics surfaced only sporadically and can therefore not be fully accounted for here. To economize with respect to writing space, the accounts of the teachers' beliefs are here given mostly in narrative form.

Lisa

Lisa, 36 years old, has been an upper secondary teacher in mathematics and physics for 13 years and is now working in her second school going on her 5th year. She teaches on a 70% basis and the other 30% she spend on administration, marketing and teacher education networking. She became a mathematics teacher because it seemed to make a lot of fun and as far back she can remember she always enjoyed doing and thinking about mathematics.

Beliefs about **the nature of mathematics**: Lisa talks about mathematics as a *tool* and something that *develops and strengthens ones' thinking* (logic). She connects mathematics to *structuring* and *organizing*, and a number of times talks about *geometrical pattern*, *forms and shapes in nature* and *mathematics as an art form*.

Beliefs about the **real world**: Lisa's comments in the interview seem to imply that the most prominent consequences of working on real problems are that the numbers occurring in the calculation are messy and that the calculations should be preformed and answered using better accuracy (more decimals).

Beliefs about **problem solving**: For Lisa problem solving is about *solving puzzles* and she associates feelings of satisfaction and happiness with the success of solving a hard problem. Problem solving is for Lisa something that preferably takes place in a technological environment with free access to every source of information possible. She also stresses the importance for the problem context to be familiar to the students.

Beliefs about **school mathematics**: Lisa repeatedly states the importance for school mathematics to be experienced as an entity, *a well defined course*, but also comments on the written governing curricula documents as theoretically formulated and hard to understand both for students and teachers. Lisa regretfully confess that some areas of mathematics (such as ordinary differential equations) only are taught as a set of procedures and recipes although the areas really have a great potential for making the subject more interesting and intriguing.

Lisa's direct talk about **mathematical modelling**: When asked about mathematical models and modelling, Lisa first seems to have a clear conception of what this means; without any time for consideration she says: *"Well, it might be a whole lot of things… a mathematical model… it might be that you describe a course of events or situation, or really just to make an assumption is a mathematical model, although a very simple one"*. Then she retreats and only considers a made connection/relation to constitute the model, not an equation or an algebraic representation of the relationship, but changes her opinion on this and clarifies that a mathematical model does not have to be expressed in mathematical terms. Rather, it should be the need of the situation that decides which degree of mathematization to use. The goal however, she continues, should always be a formulation of the model using mathematical symbols and ways of writing. Lisa also draws parallels between modelling and generalizing, and gives numerous of examples of what she considers to be different types of models when discussing the problems. She considers all five problems except The Volleyball Problem to be about, and include different aspects of, modelling.

Sven

Sven, 58 years old, has been teaching mathematics and physics (and computer science and chemistry) at the upper secondary level for 33 years and has been working at four different schools and last changed workplace in 1981. He teaches on a 60% basis and plans/manages the school schedules the rest of his working hours. It was mere coincidence that Sven became a teacher, following his personal fascination of mathematics, which led to physics and later also to teacher education.

Beliefs about **the nature of mathematics**: Sven describes mathematics as a *pure, exact and axiomatic science,* enabling to *part right from wrong*. It is about *logic*, the *relations between different quantities*, and it has a central *aesthetic component*. He emphasises that *"knowledge of the tools open up for the realization of the beauty"*.

Beliefs about **problem solving**: Sven talks about mathematical problem solving as an *exercise for the intellect*, as something decoupled form other subjects and contexts. When discussing the problems he carefully places them in a syllabus context; where, when, and how the topics touches in the problems are treated within the course.

Beliefs about **school mathematics**: When talking about school mathematics Sven expresses the importance to *learn to think logically* and to *prepare for learning in other subjects as well for higher education*. He thinks the aesthetic side of mathematics is something only a few students can appreciate and hence it plays only a minor role in the classroom.

Beliefs about **applying, and applications of, mathematics**: For Sven, application of mathematics is *"a tool used in other sciences; physics, chemistry and economics"*.

Sven's direct talk about **mathematical modelling**: When asked to describe mathematical models and modelling Sven answers, *"Yes, well… no, I don't know…"*

turning to the five problems and try to use them helping him to form and formulate his perception of mathematical models and modelling. To begin with Sven talks about a model as something to use solving problem, a tool, but elaborates his thinking further: *"I think it [a mathematical model] is something you create… in a more or less obvious manner…and there can be more than one model to use to solve a given problem."* Sven then describes different ways of working with a model; creating, using, and exploring it. He also strongly connects making assumptions and modelling, and considers all five problems used in the interviews as related to modelling. Sven also mentions that it is important for the students to learn to use and apply mathematics.

Discussion and conclusion

Although Lisa initially seemed to have a clear conception of mathematical models and modelling, it became clear throughout the interview that this was not the case. She rather, like Sven, had to make up and formulate her views as the interview went on. One explanation why neither of them had a clear conception of modelling might be the vague formulations found in the curriculum documents that provide no support and only circumstantial guidance. However, since they volunteered to participate in a research project about mathematical modelling, one could suspect that they had been doing some thinking about the project, and thus had some firm ideas about the central concepts. If they had, this was nothing that surfaced during the interviews. However, when talking about mathematical modelling, directly or indirectly during the interviews, the different categories of beliefs in the framework are touched on, as described above.

No flaws in the quasi-logic holding together the different sub-beliefs structure where detected in neither teacher's sub-beliefs structures. Sven for instance expressed the school mathematical belief that it is important for the students to learn to use and apply mathematics, and professed a similar belief about the application of mathematics. Lisa, when discussing The Volleyball Problem, on the other hand, strongly rejected it as a modelling problem since *"it is more about comparing advantages and disadvantages, structuring and organizing [than modelling]"*. This is in conflict with her beliefs about the nature of mathematics and a direct contradiction to what she said previously in the interview. One possible way to interpret this is that Lisa strongly held conflicting primary beliefs about the nature of mathematics on one hand, and mathematical modelling on the other.

Although the data was not initially collected for the testing of the suggested framework, the analysis indicates that it may be useful for exploiting beliefs about mathematical models and modelling, other professed beliefs, and relations between them. However, a thing to consider is to follow up the point made by Thompson (1992, p. 130-131), who lists a number of studies in mathematics education indicating the important impact teachers' beliefs about mathematics on the one hand, and about teaching of mathematics on the other, have on their practice. Including the teachers' beliefs on the learning and teaching of mathematics in general, and mathematical models and modelling in particular, seems to be the next logical step. A perhaps as urgent dimension to add to the framework is to include more actively affective considerations, which Goldins' (2002) framework make possible.

If indeed beliefs can be seen as filters influencing the teachers' practice, it is important to try to get a better understanding of beliefs about mathematical models and modelling if we want teachers to integrate it more in their mathematics teaching. Kaiser (2006) concluded that "beliefs concerning mathematics must be regarded as essential reasons for the low realisation of application and modelling in mathematics teaching" (p. 399), and we believe, like (Törner, 2002, p. 80), that higher consciousness about one's beliefs lead to a higher degree of integration of the beliefs in ones' practice. A question that we feel needs priority is how beliefs are formed.

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THE USE OF MOTION SENSOR CAN LEAD THE STUDENTS TO UNDERSTANDING THE CARTESIAN GRAPH

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Abstract. *This paper shows the experimental results of a didactical lesson conducted in three classes of Upper Secondary School using motion sensor. It is an example of modelling practice, in which the students are involved in mathematics representations of real phenomena. Our research corroborates works about the use of MBL-tools, according to which the use of motion sensor allows the students to reading, understanding and interpreting kinematics graphs. Besides our analysis shows that the students acquire these competence respect to graphs of other type too. These results emerge from the implicative statistical analysis of the pre-test and the post-test and from the qualitative analysis of the lessons.*

Key words: *teaching, learning, Cartesian graph, motion sensor, modelling*

INTRODUCTION

This research work consists of the analysis of a didactical situation conducted in three classes of Upper Secondary School. The didactical activities were developed using a motion sensor to visualize, to understand and to interpret space-time and velocity-time graphs, representing moving bodies. Motion sensor is one of MBL-tools (Microcomputer Based Laboratory). In the late 1980's, these tools were produced by the project ''Tools for Scientific Thinking'' in the Center for Science and Mathematics Teaching at Tufts University. The central objective was to help students in order to recognize the connections between the physical world and the abstract principles presented in the classroom (Krusberg 2007). Motion sensor is used in Physics laboratory to study rectilinear motion of bodies moving in front of it.

Our research proves that not only the students improved reading, understanding and interpreting motion graphs but they also improved these graphing practices (Roth 2004 p.2) in other types of Cartesian graphs. We believe that this is an interesting result because learning mathematics means that a person acquires aspects of an intellectual practice, rather that just acquiring any information and skills (Roth 2004 p.7). These interdisciplinary activities give the opportunity to optimize available time in classroom and to increases the student's motivation.

We chose this argument of research because graphing practices are part of the mathematics curricula of all school levels. Moreover, they can become prerequisites for other mathematical subjects. For instance, Cartesian graph is one register of semiotic representation of a function. Besides, graphing practices are central to scientific communication and to the scientific enterprise more broadly (Roth 2004 p.2). Moreover graphing practices have many applications in everyday life as the comprehension of an economy graph printed on a newspaper, the understanding of a temperature graph hanged on a hospital bed, etc.

RESEARCH QUESTIONS

Research hypothesis: *Motion sensor is a learning tool to reading, understanding and interpreting kinematics graphs.*

Research questions:

1) *Using motion sensor to reading, understanding and interpreting kinematics graphs, do students learn to reading, understanding and interpreting other types of Cartesian graphs and, in particular, function graphs representing a statistical phenomenon?*
2) *How can modelling activities aid for the understanding of Cartesian graphs?*

THEORETICAL FRAMEWORK

MBL tools collect physical data and allow visualizing them in tables and Cartesian graphs in real time (Thornton & Sokoloff, 1990). So MBL tools can facilitate the comprehension of abstract representations of physics phenomena and can give long lived conceptual understanding (Bernhard, 2001). Besides collected data can be manipulated, analyzed and fitted, studying the characteristics of the phenomena and testing the relationships between the variables. The efficiency of motion sensor compared to traditional methods for helping students to learn basic kinematics concepts has been proved by several researches, as Thornton & Sokoloff (1990), Redish et all. (1997), Liljedahl (2002), Arzarello & Robutti (2004). Our research wants moreover to show that when the students are involved in activities with sensor motion they become able in graphing practices, not only in kinematics field.

The idea of using motion sensor to improve graphing practices finds strong theoretical support in the cognitive theories *of the Embodiment of the mind*, for which «the detailed nature of our bodies, of our brains, and of our daily functioning in the world structures human concepts and reasoning» (Lakoff & Núñez, 2005, p.27). So it's fundamental in this kind of activity as the students can visualize and analyze in real time the graphs of bodies. Beside according to *Metaphorical Thought* «for the most part, human being conceptualize abstract concepts in concrete terms, utilising ideas and models of reasoning founded on a sensor-motor system» (Lakoff & Núñez, 2005, p.27). Particularly «the functions on the Cartesian plane are often conceptualized in terms of motion on a route» (Lakoff & Núñez, 2005, p.70) and motion sensor induces this type of conceptualization as the students see the graph constructed under their own eyes as *motion of a point that leaves a wake*. It can be explained through a historical-epistemological analysis of the concept of function, which finds its origins in the ambit of kinematics and geometry.

This analysis shows that the representations of the function are: verbal, Cartesian, analytical and tabular (for numerical values). So the laboratory activity with sensor motion could be utilized as kinematics approach to the concept of function (Arzarello & Robutti 2004) because it allows studying all the representations of a kinematics function and to pass from one kind of representation to another. A representation cannot describe fully a mathematical construct and each representation has different advantages, using multiple representations for the same mathematical situation is at the core of mathematical understanding (Duval 2002). The representations of the function developed in different historical periods. Before tables of functions appeared (2000 B.C.), then geometrical representation (middle of the 14th C.) and later analytical form (17th C.) (Youschkevitch, 1976). Using motion sensor the chronological introduction of the representations of the function is respected (Piaget & Garcia, 1985). Besides it involves the students in a historical process that conducted to the function concept: modelling process. In fact it allows analyzing the motion of a body as a point in moving along a straight line respect to the reference point, studying all its mathematical representations (Gilbert, 1998).

In this activity the modelling is a transversal objective, reached by the study of other matters of the mathematics curriculum (Lingefjärd, 2006). Modelling practise can be a way to increase *thinkers*, who can use their mathematics for their own and for society's purposes (Burkhardt, 2006). To conclude we want to point out that motion sensor is an artefact. As referring to mathematical meanings it may be seen as «tool of semiotic mediation» (Bartolini Bussi & Mariotti 2008). The role of the teacher becomes fundamental in the use of this tool to reach the graphing practices.

Some didactical considerations

To clarify the connection between the graphing practices in motion graphs and in any Cartesian graphs, we made the following comparison between competences:

C	MATHEMATICS	PHYSICS
C1	Reading the coordinates of a point of the graph	Reading the values of a kinematics variable in relation to the values of the temporal variable
C2	Reading the extremes and the size of intervals	Reading space and time of departure and arrival, the covered space and the spent time
C3	Distinguishing among increase, decrease and constancy of a function	Distinguishing between motion of approach, motion of separation and still bodies
C4	Individuating absolute maximums and minimums of a function	Individuating absolute maximum and minimum distance with respect to the position reference system
C5	Individuating relative maximums and minimums of a function	Individuating relative maximum and minimum distance with respect to the position reference system
C6	Confronting the different degrees of rapidity of increase or decrease of tracts of a curve	Confronting the velocity of differing tracts of motion
C7	Forming hypothesis and conjecture	Forming inferences on experimental data

EXPERIMENTAL WORK AND RESEARCH METHOD

The experimental work consisted of two laboratorial lessons[1] of two hours each one. It was leaded in three Italian classes[2] of Upper Secondary School (43 students). It is a homogeneous sample because before the experimental work they possessed the same competences in graphing practices and necessary prerequisites for this activity:

- Knowing the real number field and representing them on a straight line
- Representing points on the Cartesian plane
- Knowing motion concept and kinematics variables

The research methodology adopted is *Theory of Didactic Situations* by Brousseau (Brousseau, 1997). The laboratorial lesson was preceded and followed by the administration of a test, with the aim of evaluating the a priori and a posteriori students' behaviours. We made the qualitative analysis of the didactical activities analyzing the teaching/learning process through the analysis of the involved semiotic register. It refers to *APC space and Semiotic Bundles* by Arzarello (Arzarello & Robutti 2008). We made also a quantitative analysis of tests through *Statistical Implicative Analysis* by Gras (Gras et all, 2008). Cause of limited space, in this paper we show only the main results of our analysis.

Statistical Implicative Analysis

It is a non-parametric statistic, so it uses small samples and it is appropriate for this kind of research. We use the method of *implication* that establish the implication intensity between variables and the method of *similarities*, that classifies variables and groups them according to hierarchical levels (similarities) (Gras et all, 2008). Data were analyzed by using C.H.I.C.[3] software that visualizes *implication graphs and similarity tree,* working on Excel tables. We studied the implication of the *students' behaviours variables* by tables like this:

	Behaviour 1	…	Behaviour *n*
Student 1			
…			
Student *m*			

The values of this table are 0 or 1, depending if a student doesn't follow or follows the behaviour that corresponds in the table respectively. We analyzed the similarity of the *students' variables* using the *supplementary variables* method (Spagnolo 2005), (Fazio & Spagnolo, 2008). Here we use the supplementary variables as models of

[1] Lessons was conducted by the teacher-researcher M. L. Lo Cicero in her curricular classes.

[2] 1. December 2007, 4th class of Classical Liceo (17 years), (*Liceo Classico "Scaduto", Bagheria (PA), Italy*)
2. April 2008, 2nd class of Commercial Technical Institute (15 years), (*"Jacopo del Duca", Cefalù (PA), Italy*)
3. May 2008, 4th class of Classical Liceo (17 years), (*Liceo Classico "Scaduto", Bagheria (PA), Italy*

[3] Classification Hiérarchique Implicative et Cohésitive. Information regarding the software can be found at the following site of the A.R.D.M. (Association de Recherche en Didactique des Mathématiques): http://www.ardm.asso.fr/CHIC.html

student's behaviour, so the outcomes of our research depend from the similarities of the *students* respect to the *correct models of students' behaviour*. The correct models of students' behaviour are selected by combination of the correct behaviours. To obtain the similarity trees we used tables like this, with binary values:

	Student 1	…	Student *m*	model of student's behaviour 1	…	model of student's behaviour *p*
Behaviour 1						
NOT behaviour 1						
…						
Behaviour *n*						
NOT behaviour *n*						

Phases of the didactical activity

The phases of the didactical activity were the following ones:

1. Prediction, reading and comprehension of the graphs of rectilinear student's motion of three types:
    a. Leaving motion from the sensor
    b. Approach motion to the sensor
    c. Still body with respect to the sensor
2. Prediction, reading and comprehension of various rectilinear student's motion, with leaving and approach with respect to the sensor.
3. Study of rectilinear uniform motions of a train on tracks.

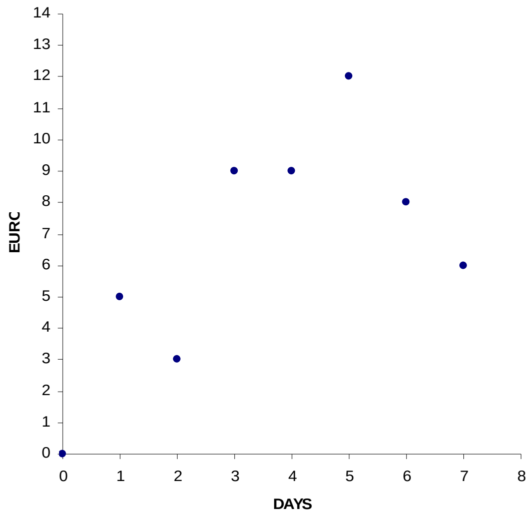
During phase **1** the students made a reflection on the variables studied by the sensor. They observed and calculated space and time of departure and arrival, the length of space and the time spent. Not all the students immediately realized the relation between abscises and ordinates. After the study of the leaving motion the students correctly predicted the other types of graphs. In phase **2** the topics of the previous phase were consolidated for every piece of curve of leaving, approach or stilling. Also the maximum and minimum distance reached with respect to sensor was read. The students noted that the slope of every piece of curve depended on the corresponding velocity of the student. Then the students were asked to make a relationship between spatial intervals and temporal intervals about pieces of a curve and to make comparisons. Besides the students calculated the mean velocities and compared them and the observations about the slopes of the pieces of the curve with the graphs velocity-time. In the phase **3** they studied the analytical representation of a uniform rectilinear motion by the fit of the data. The students noted that this is a particular type of straight line equation.

After the laboratory activity, the students were involved in a metacognitive reflection about the development of the lesson. The students reconstructed the phases of the modelling process and reached the devolution of these processes (Brousseau, 1997). They realized that physics phenomena, belonging to everyday life, could be representable by mathematical representation. In particular, uniform rectilinear motion can be represented by algebraic equation, commonly studied in scholastic mathematics. So this modeling process was an occasion to realize that mathematics is a tool to read the existence of mathematics in our everyday life (Lingefjärd, 2006), (Kaiser & Schwarz, 2006). During the didactical activity it was noted that motion sensor induces curiosity and desire of learning in students. They were encouraged to experiment several typologies of motion to compare the graphics produced with their own predictions. It was noted that the process of prediction is important to acquire the skill of *forming hypothesis on the base of experimental data.*

Test

A test was administered before and after the laboratorial lesson. The students worked individually, they were not allowed consulting books or notes. They had sixty minutes to accomplish the task. The test contained items concerning reading and understanding of space-time graphs representing motion of bodies, contextualized in real life. So the students had to interpret models of kinematics phenomena. The students' improvements in kinematics graphical practises were remarkable, so they corroborated our research hypothesis. Besides the test contained the following exercise (*Sara's test)* concerning reading of not kinematics graph:

Sara's dad decided to reward his daughter every time she got a good grade at school by giving her five euro, which she could decide to spend or save as she pleased; but this would be her only source of income. The adjacent graph shows the money Sara possessed on each day of a week. Observe it and answer the following questions:



a. How many euros did Sara posses on the 4th day? (C1→COORD)

b. On which day/days did Sara surely get a good grade? (C5→R-MAX)

c. Knowing that on the second day Sara didn't get a good grade, how much money did she spend that day? (C2→INT)

d. Could she have gotten a good grade on the 6th day? (Justify your answer.) (C7→HP)

e. On which day/days did Sara possess the most money? (C4→A-MAX)

A priori analysis of students' behaviours of *Sara's test*

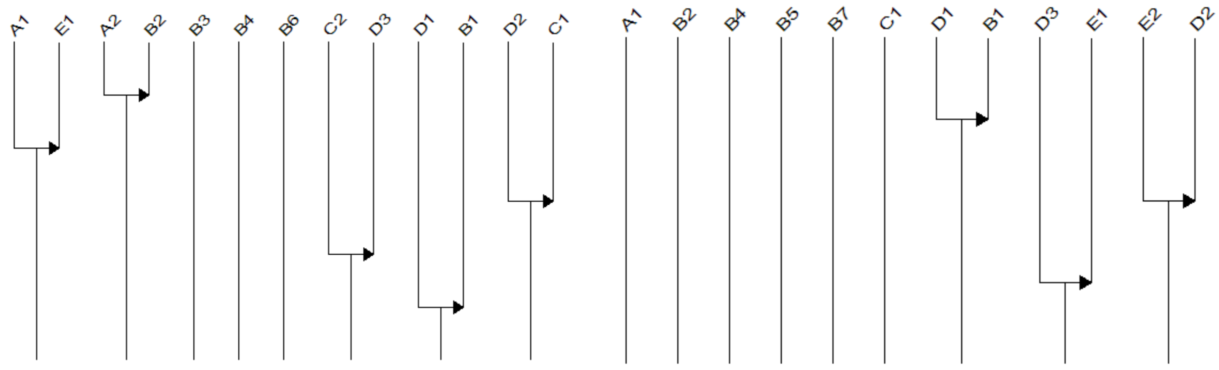
As it is indicated by *Theory of Didactic Situations*, we made an a priori analysis of students' behaviour in working out the test:

Q[4].	A.	BEHAVIOURS
a.	9	A1: Correct reading of the value of the ordinate in correspondence with abscissa
	26	A2: Sum of the euro that Sara possessed in the first four days. Interpretation of the graph like earned money.
b.	1,3,5	B1: Correct identification of the days corresponding to the relative maxima
	5	B2: Confusion between the concept of relative maximums and of absolute maxima
	1,3,4,5	B3: Writing, besides of the days which correspond to the relative maxima, also, of the 4th day, in which the euro remained constant
	3,4,5	B4: Writing of the highest values
	3	B5: Writing of the day corresponding to the major growth
	1,3,4,5,6,7	B6: Writing of all the days except the absolute minimum
	3,5	B7: Writing of the highest relative maxima
c.	2	C1: Correct identification of the width of the interval
	3	C2: Confusion between the concept of interval and of value of the coordinate. Wrongly interpretation of the graph like spent money
d.	Yes, …	D1: Affirmative answer to the question *d*, justifying with the affirmation “she could have spent the earned money”: forming correct hypotheses on the base of experimental data
	No, …	D2: Negative answer to the question *d*, justifying with the affirmations “she spent 4 euro” or “her budget would have become 13 euro”: not forming correct hypotheses on the base of experimental data
	Yes, …	D3: Affirmative answer to the question *d*, justifying with the affirmation “because she earned 8 euro”: not forming correct hypotheses on the base of experimental data and wrongly interpretation of the graph like earned money
e.	5	E1: Correct identification of the absolute maximum
	3,4,5	E2: Writing of the highest values

EXPERIMENTAL RESULTS AND CONCLUSIONS

We classified the behaviours of the students in tables. Using *Chic* software we obtained the following implicative graphs of the student’s behaviours:

PRE-TEST POST-TEST



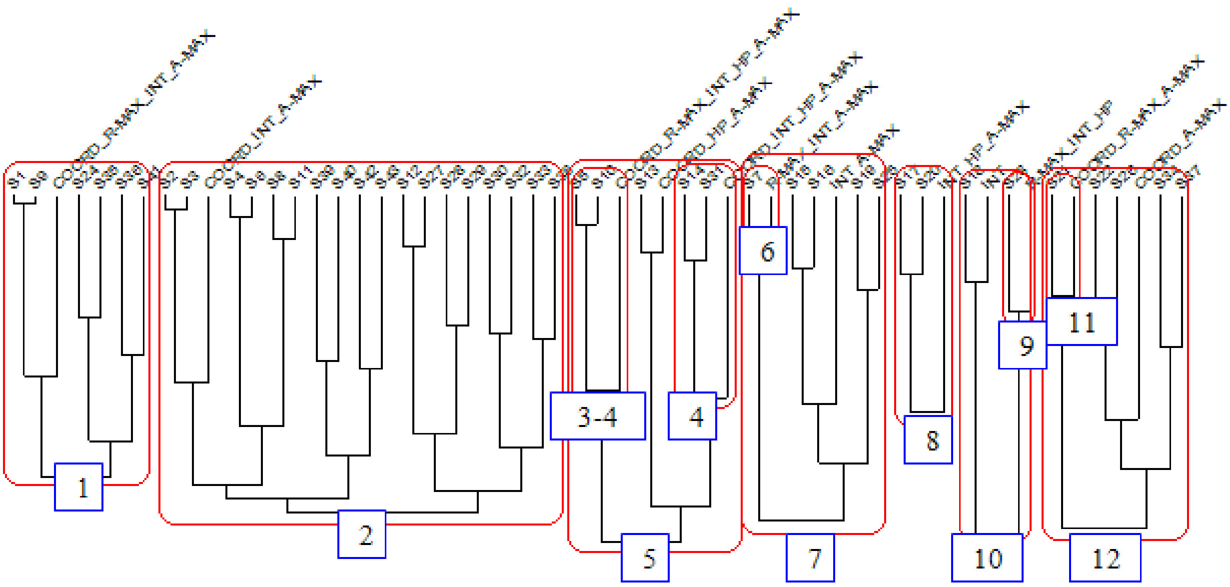
[4] Q=Questions. A= Students’ Answers

In the implicative graph of the pre-test there is a strong implication of A2 towards B2: all the students that follow the behaviour A2 follow the behaviour B2 too. They represent two mistakes in reading of graph (reading of coordinates and relative maxima respectively). The implication A1→E1 inverts the expected implication between the reading of the coordinates and of the absolute maximum. It is due to the wrong interpretation of the graph like earned money in the answer *a*. D2 implicates C1 because the behaviour D2 includes the competence of reading of the width of intervals. The implication C2→D3 points out a wrong interpretation of the graph like spent money and earned money respectively. So the same students gave two wrong opposite interpretations of the graph. Since D1→B1, the students that form correct hypotheses on the base of experimental data are able to read relative maxima.

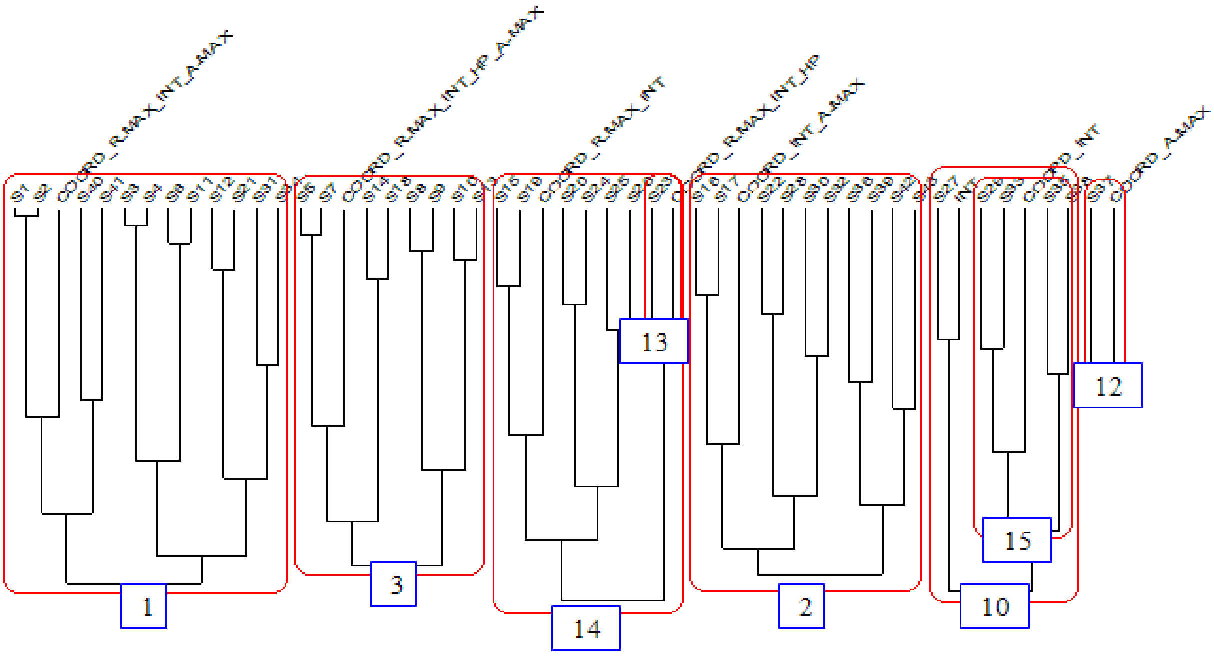
In the graph of the post test the implication D1→B1 is stronger than in the pre-test. Given that E2→D2, if the students don't read correctly the absolute maximum then they don't form correct hypotheses on the base of experimental data. Finally, since D3→E1, the students that don't form correct hypotheses on the base of experimental data and interpret the graph like earned money are however able to read absolute maximum.

We analysed the similarity of the variables *student* respect to the variables *models of students' behaviour*. Below we report the graphs obtained by C.H.I.C.:

PRE-TEST



POST-TEST



In these graph we can observe the improvements of the competences of each students. The general improvements for each competence are:

	COORD	R-MAX	INT	HP	A-MAX
N° correct answers, pre-test	34	11	37	8	41
N° correct answers, post-test	43	27	42	9	31

Below we report a table extrapolated by the similarity trees. It shows the numbers of the students that possessed 5 or 4 or 3 or 2 or 1 competences in the pre and post-test.

	5 comp.	4 comp.	3 comp.	2 comp.	1 comp
N° stud, pre-test	2 (group 3)	8 (groups 1,4)	24 (groups 2,5,6,8,9)	8 (groups 7,12)	1 (group 10)
N° stud, post-test	8 (group 3)	13 (groups 1,13)	16 (groups 14, 2)	5 (groups 12,15)	1 (group 10)

In particular, in the similarity trees we note that the group n. 3, representing the students that possessed all the competences, is increased by 6 students in the post-test. The group n. 1, representing the students that possessed all the competences except the forming hypotheses, is increased by 6 students in the post-test.

Conclusions

The experimental results show that a laboratory activity with the use of motion sensor develops the competences of the students in reading, understanding and predicting of kinematics graph. This tool allows studying the steps of the modelling process of the phenomena *rectilinear motion* and to make metacognition reflection of their own learning. Modelling activities aid for the understanding of Cartesian graphs because they are the bridge between the real phenomena and the mathematical representations. According to the theory of *Embodiment* the students construct their knowledge observing the real phenomena and connecting it with its graphical and tabular representations. Our mind conceptualizes a function as a point that is moving on the plane and the use of motion sensor induces this kind of conceptualization. So, using motion sensor, the students acquire competences in reading, understanding and predicting Cartesians graphs not representing only a kinematics phenomena. In particular, our research shows improvements of the students in reading of the correspondence between abscises and ordinates, of maxima and width of intervals of a statistical function.

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INTERACING POPULATIONS IN A RESTRICTED HABITAT– MODELLING, SIMULATION AND MATHEMATICAL ANALYSIS IN CLASS

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This presentation will introduce an authentic modelling process for two interacting species which is well accessible to high school students. Based on an analysis of ecological systems, a simple conceptual model leads to simulation software tools and the derivation of a mathematical model. A wide range of systems, e.g. predator-prey, competition or parasitism can be investigated. The approach also allows independent modelling activities and in silico experimentation by students. As the presented modelling process builds on authentic research by Johannson and Sumpter (2003) it allows to give students an insight into current research of Theoretical Biology.

MODELLING

The importance of modelling in the teaching of mathematics is universally accepted. But often work with models in education consists only in the usage of formulas or the fitting of parameters. There is not much suitable teaching material about reality-based mathematical models. Some groundbreaking efforts were made by Sonar and Grahs (2001, 2002). Gotzen (2003) created well comprehensible, reality-based one-species-models for school use in his doctoral thesis. The two-species models presented in this paper are based on his work.

We will use the following modelling process:

1. Definition of the purpose of the model.
2. Analysis of the real situation.
3. Establish a conceptual model, from simplified description of the real situation.
4. Simulation software and mathematical model equations on the basis of the conceptual model.
5. Predictions and validation using the simulations and/or the mathematical models.

Except for some slight modifications these are the five modelling steps presented by Gotzen, Liebscher and Walcher (2008). See also earlier work of Schupp (1988). Results gained during the modelling process have to be compared to reality and the intention of modelling and thus must eventually be corrected. Hence the modelling process is rather a modelling-cycle as Blum and Leiß (2007) presented. Nevertheless the main idea of modelling will be comprehensible for students following the five steps.

We will introduce population models which are very suitable for educational use because of their relevance, authenticity and traceability for students.

- The models are relevant as they are built on current research of Theoretical Biology (Johannson & Sumpter, 2003).
- As the modelling process is based on capturing the most relevant features of a population development, observed on an ecological level, it ensures a strong biological foundation. This kind of modelling is called “bottom up” modelling. A detailed description of the advantages of “bottom-up” models and a separation from classical “top-down” models is given by Sumpter & Broomhead (2001).
- They are suitable for educational use because the whole modelling process is comprehensible with means of school education. Furthermore the models provide explanations of the observed phenomena and allow predictions.

The models are applicable in mathematical and biological classes in secondary school as well as in education at university (e.g. classes of Biomathematics).

The software and a workbook, which gives all necessary instructions and allows self-contained work of students, are allocated for free use in the internet (Roeckerath, 2008).

PURPOSE OF THE MODEL

We want to derive a bottom-up model of two interacting species which is capable to give information about their development over time. The model shall capture the main important ecological patterns and phenomena affecting the development of the species. Thus we are looking for a model, which gives the size of each population at every generation.

THE ECOLOGICAL SYSTEM

The basis of the modelling process must be an analysis of the ecological system in order to capture the main important structures concerning the development of both populations.

We look at two interacting species which share a restricted habitat. The populations have non-overlapping generations. This ecological phenomenon is common for insects and annual plants and means that at every time there is only one generation alive. Thus parents and children never live together. Parents distribute their offspring randomly over the entire habitat. The offspring is during the first development state (nearly) not able to move (eggs, larvae, seeds).

Individuals interact with individuals of their own as well as with individuals of the other species. These phenomena are called intra- respectively interspecific interactions and affect the individuals’ ability of reproduction.

We want to include several kinds of intra- and interspecific interactions appearing in ecology. In the following we want to capture them in formulating interaction laws.

Intraspecific Interactions

Intraspecific interactions appear mostly as competition for resources like food, territory or sunlight. The availability of such resources is mainly responsible for the ability of an individual to reproduce itself. We want to distinguish two kinds of intraspecific competition: exploitation and interference competition, which Nicholson describes as “scramble” and “contest” (1954).

Exploitation competition can appear when individuals share a restricted quantity of resources. In this case a high density causes a lack of resources which prevents individuals from reproducing. Ecological examples of this phenomenon are weakness because of hunger or lacks of breeding or germination areas. We capture the main idea in the interspecific law

> **INTRA 1**. If there is a sufficiently high population density no individual will be able to reproduce.

In the case of interference competition individuals deal directly with each other. There is one dominant individual, which is able to gain enough resources and to reproduce, even if there is a high population density. Ecological examples are cainism, where cubs kill each other until only the strongest cub is still alive, or allelopathy, where plants spread poison into the ground in order to prevent other plants from growing. A simplified description of these phenomena gives the reproduction law

> **INTRA 2.** There is a dominant individual which is able to reproduce even if there is a high density.

More detailed biological background concerning intraspecific interactions and concrete biological examples can be found in the article of Gotzen, Walcher, Liebscher (2006).

Interspecific Interactions

Interactions between individuals of different species can have a positive, negative or no influence on their development. There are many ecological examples showing these kinds of influences. For example an individual of a predator-population needs prey. Thus an interaction with individuals of the prey species will cause positive effect on the predator’s reproduction. A suitable reproduction law for positive influence is

> **INTER +.** If an individual interacts with at least one individual of the other species, then it will be able to reproduce.

On the other hand interspecific interactions can also cause negative influence. A prey animal is only able to survive and reproduce if it will not be killed by a predator.

Also in the case of competition for resources between different species interspecific interactions have a negative influence on the reproduction. A simplified summary is the reproduction law

> **INTER -.** If an individual does not interact with any individual of the other species, then it will be able to reproduce.

In eco-systems there can be populations which share a habitat and interact but one species is not affected by the other. For example huge plants which take daylight from small plants. There is an interaction, but the huge plants are not affected by the small plants. This is captured in the reproduction law

> **INTER 0**. Individuals reproduce independently from the other species.

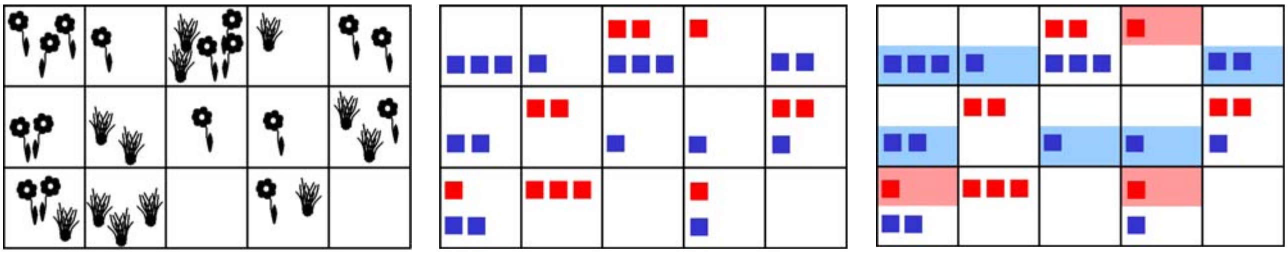
Using the specified interaction laws we will be able to describe a wide range of two interacting species. A concrete example is the following ecological system.

Example: Amensalism
There are two populations of plants which use the same resources. The first species shows the following dominant behaviour. It affects the second species negatively without any influence for it self. Thus it not affected by the second species. This ecological phenomenon is called amensalism. Within the species 1 obtains exploitation competition and within the species 2 interference competition. Using the interaction laws we can determine that species one follows **INTRA 1** and **INTER 0** and the species 2 follows the reproduction laws **INTRA 2** and **INTER-**.

FROM THE ECOLOGICAL SYSTEM TO THE CONCEPTUAL MODEL

After the ecological observations, we will now capture the main important structures affecting the populations' development in a conceptual model. A conceptual model is a (partly very strong) simplified description of the reality. The challenge is to distinguish the relevant and the irrelevant factors. The conceptual model is often only a caricature of the real system but it is clearly arranged and practicable.

The habitat is displayed on a field with a fixed number of sites. Each site represents an area of the habitat. As shown in figure 1(a) and 1(b) for each area the containing individuals are displayed by a dot in the corresponding site. To distinguish the different species the dots are differently coloured.



(a) (b) (c)

Figure 1: Conceptual model

Individuals displayed at the same site are close to each other und thus interact. Due to the non-overlapping generations the development of the real system can be described with discrete time-steps and it is only affected by the number of reproductions. A site provides enough resources for at most one reproduction per species. As parents deposit their offspring randomly somewhere in the habitat, for every new generation the concerning number of dots will be randomly distributed over the field.

Interaction laws

The sites provide a basis to comprise the concept of "high density" for the intraspecific, and the concept of "presence" for the interspecific interaction laws in the conceptual model.

We assume that we have a high density at a site, if it contains more then one individual. Using this understanding of density we can integrate the introduced interaction laws in our model.

> **INTRA 1.** At a site there will be a reproduction for a species, if it contains exactly one individual of the same species.
>
> **INTRA 2.** At a site there will be a reproduction for a species, if it contains at least one individual of the same species.

The concept of "presence" can easily be realized in the conceptual model. The other species is present, if there is at least one of its individuals. Thus we get the following interaction laws for the conceptual model.

> **INTER +.** At a site there will be a reproduction for a species, if it contains at least one individual of the other species.
>
> **INTER -.** At a site there will be a reproduction for a species, if it contains no individual of the same species.
>
> **INTER 0.** At a site there will be a reproduction for a species, if it contains any number of individuals of the other species.

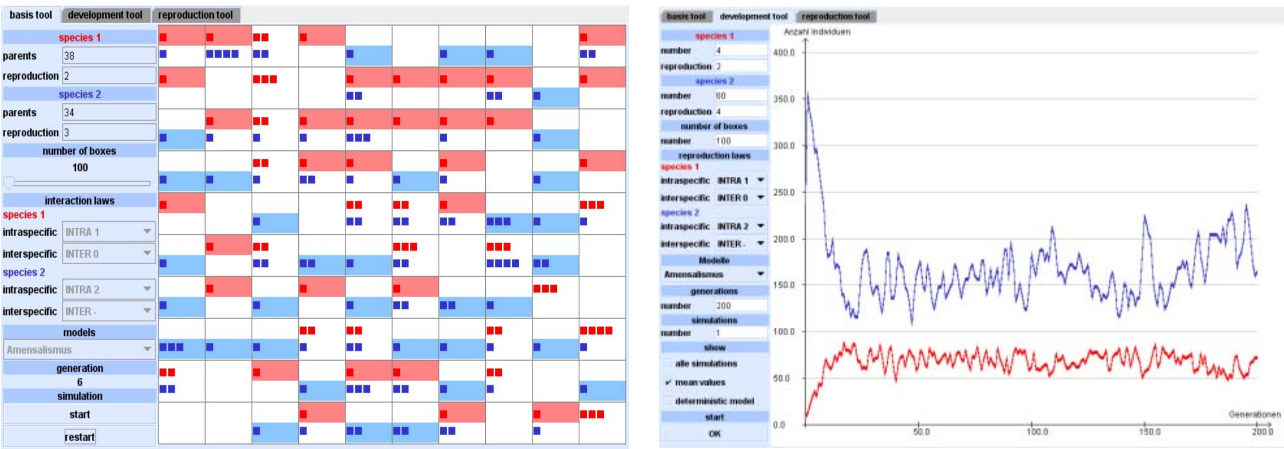
Now the means to determine if there is a reproduction for a species at a site are available: If a species follows the interaction laws **INTRA** and **INTER** then there is a reproduction for this species at a site if and only if **INTRA** and **INTER** are both fulfilled at the site.

In order to get a species' population size of the next generation the reproduction laws must be applied at each site. Multiplying the resulting number of reproductions with the mean number of offspring per reproduction we get the population size of the next generation. The generation cycle repeats by spreading this number randomly over the field. On the basis of this conceptual model, software was created which simulates the development of the species.

Example: Amensalism
Species 1 follows **INTRA 1** and **INTER 0** Species 1 will reproduce at a site, if and only if it contains exactly one individual of species 1 and an arbitrary number of individuals of species 2. Species 2 follows **INTRA 2** and **INTER-** Species 2 will reproduce at a site if and only if the site contains at least one individual of species 2 and no individual of species 1. Figure 1(c) shows the evaluation of the field concerning the reproduction laws of species 1 and 2. A light blue respectively a pink mark of a box represents a reproduction of the blue respectively the red species.

FROM THE CONCEPTUAL MODEL TO THE STOCHASTIC MODEL

The simulation tools provide excellent observation and exploration possibilities to students. Furthermore it should be mentioned that in silico investigations using simulations are very common in modern biological research.



(a) (b)

Figure 3: (a) The basic tool; (b) The development tool

Figure 3(a) depicts the graphical surface of the basic tool, which implements the simulation of the described generation-cycle. For each species students can enter the start sizes of the population, the mean number of offspring per reproduction and the intra- and interspecific interaction laws. Starting the first simulation the entered number of individuals will be spread randomly over the field. The program evaluates for each site and each species if there is a reproduction according to the selected interaction laws. Thus, the program computes the population sizes for the simulation of the next generation.

The development tool, pictured in figure 3(b), was created to get a better insight of the species' development. The tool simulates the development over a longer period of time and displays the resulting population sizes of each generation in a coordinate system. This offers a clear depiction of the long term development for both species.

Example: Amensalism
In figure 3(b) a simulation of the amensalism system (species 1: **INTRA 1** + **INTER 0**, species 2: **INTRA 2** + **INTER-**) is shown. In this case the two species are able to live in coexistence. Changing the parameters, students can determine values for the initial populations and the mean numbers of offspring per reproduction which cause an extinction of one species or which allows coexistence. Thus students are able to explore the biological role and of the parameters.

FROM THE CONCEPTUAL MODEL TO THE DETERMINISTIC MODEL

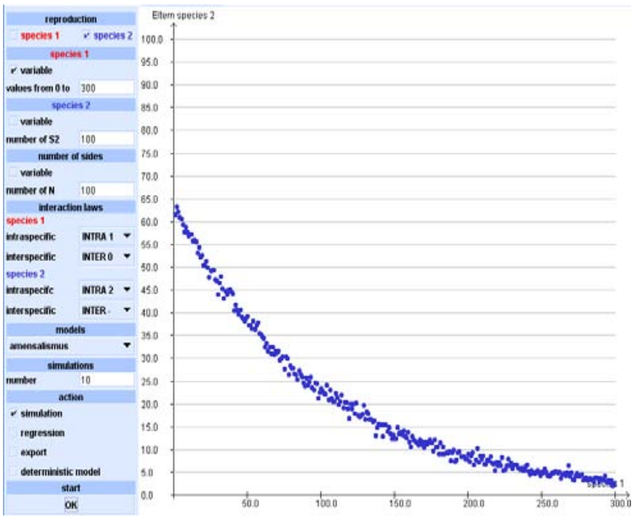
Using the conceptual model students are able to derivate a mathematical description of the systems. We define the number of individuals at a time t as $S_1(t)$ and $S_2(t)$. Due to non overlapping generations, the change of population size from generation t to generation $t+1$ exclusively depends on reproduction. The function of reproduction $R_1(S_1,S_2)$ respectively $R_2(S_1,S_2)$ indicates for species 1 respectively for species 2 how many individuals are able to reproduce, when S_1 individuals of species 1 and S_2 individuals of species 2 are randomly spread over the field. We define the number of mean offspring for each reproduction, as r_1 for species 1 and r_2 for species 2. Thus we get the following mathematical description of the population sizes.

$$S_1(t+1) = r_1 \cdot R_1(S_1(t), S_2(t))$$

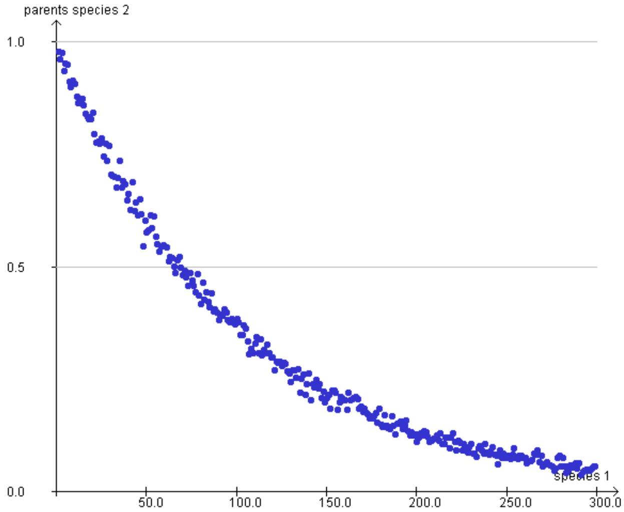
$$S_2(t+1) = r_2 \cdot R_2(S_1(t), S_2(t))$$

To derive the whole mathematical description we need the reproduction functions. With the reproduction tool students can derivate reproduction functions via regression.

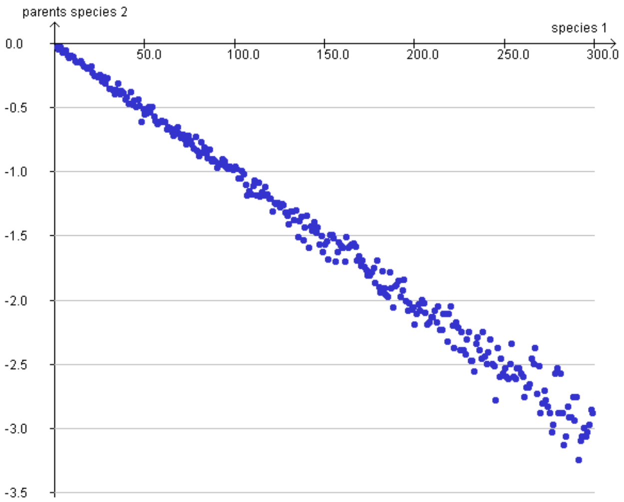
The reproduction tool, shown in figure 5(a), allows simulations of the functions $\mathrm{R}_{1,\mathrm{S}_2}(S_1)$, $\mathrm{R}_{2,\mathrm{S}_2}(S_1)$, $\mathrm{R}_{1,\mathrm{S}_1}(S_2)$ and $\mathrm{R}_{2,\mathrm{S}_1}(S_2)$, which determine the number of reproductions of one species depending on a fixed number of individuals of one species and a variable number of individuals of the other species.



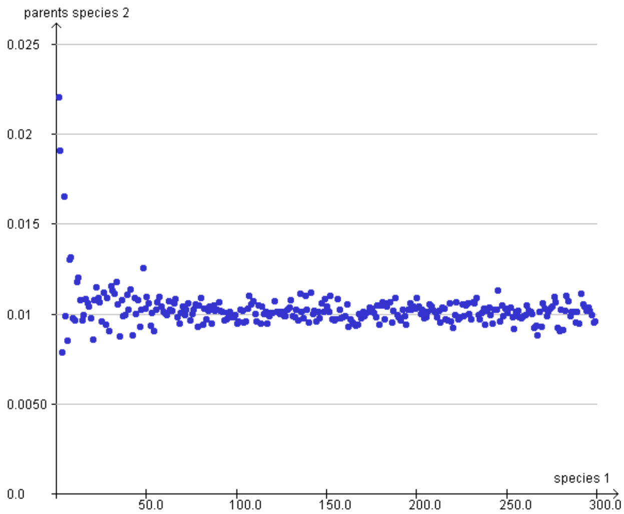
(a)



(b)



(c)



(d)

Figure 5: The reproduction tool: (a) Simulation of $\mathrm{R}_{2,S_2}(S_1)$ ($\tilde{R}(S_1)$ simulation values); (b) $\tilde{\mathrm{R}}^{(1)}(S_1) := \tilde{R}^{(2)}(S_1)/64$; (c) $\tilde{R}^{(2)}(S_1) := \log(\tilde{R}^{(1)}(S_1))$ can be approximated by a linear function; (d) $\tilde{R}^{(3)}(S_1) := -\tilde{R}^{(2)}(S_1)/S_1 \approx 0.01$

The tool provides the possibility to modify the simulated values in order to determine the reproduction functions. In the following $\mathrm{R}_2(S_1, S_2)$ of the amensalism system with $\mathrm{N} = 100$ will be derived.

Example: Amensalism
Simulating $\mathrm{R}_{2,S_2=100}(S_1)$ with the reproduction tool we get the graph $\tilde{R}(S_1)$ shown in figure 5(a). $\mathrm{R}_{2,S_2=100}(S_1) = M\, e^{-K S_1}$ seems to be a proper approach to approximate $\tilde{R}(S_1)$. With the software the simulation values can be linearized in order to check if a

certain function is suitable to approximate them. Figure 5(b) shows the resulting graph $\tilde{R}^{(1)}(S_1)$ after dividing the simulation values by M, which is approximately 64. In the next step the logarithm will be applied to $\tilde{R}^{(1)}(S_1)$. As it is shown in Figure 5(c) the resulting graph $\tilde{R}^{(2)}(S_1)$ can be approximated by a linear function. This verifies that the approach is suitable. The constant $K = 0.01$ can be obtained by dividing $\tilde{R}^{(2)}(S_1)$ by S_1, as it is shown in Figure 5(d).

In order to figure out how R_2 depends on S_2, it has to be checked how the remaining constants in $R_{2,S_2}(S_1) = M\,e^{-K S_1}$ depend on S_2. Determining $R_{2,S_2}(S_1)$ for different values of S_2, shows that M depends on S_2, while $K = 0.01$ remains constant. Thus, we obtain $R(S_1, S_2) = M(S_2)e^{-K S_1}$. If $S_1 = 0$, then $R_2(0, S_2) = M(S_2) = R_{2,S_1=0}(S_2)$. Using the tool $R_{2,S_1=0}(S_2) = L(1-e^{-K S_2})$ with $L = 100$ can be determined. Thus $R_2(S_1, S_2) = L(1-e^{-K S_2})e^{-K S_1}$ is the reproduction function of species 2. With the derivation of the reproduction function of species 1 we can determine the following model equations for the amensalism system:

$$S_1(t+1) = r_1 \cdot L \cdot S_1(t) \cdot e^{-K S_1(t)}$$
$$S_2(t+1) = r_2 \cdot L \cdot (1-e^{-K S_2(t)})\,e^{-K S_1(t)}$$

Detailed descriptions and instructions for many different systems can be found in the workbook (Roeckerath, 2008).

PREDICTIONS

A good model offers predictions for the real system. Many systems develop over time from different states into a relatively stable final state, the climax state, like coexistence of both species or extinction of one or both of them. They can also develop cyclic or even chaotic behaviour. As mentioned above for the amensalism system, students can use the development tool to explore which values of $r_1, r_2, S_1(0)$ and $S_2(0)$ yield to different kinds of systems' behaviours. Doing these kinds of predictions students are able to explore the ecological meaning of parameters.

As the derived models are dynamical systems in form of difference equations, next to the development tool students from a higher educational level can gain predictions with analytical or numerical investigations. A stable fixed point for example gives information about population developments which reach a climax state.

CLASSROOM USE

The models and tools offer various options for classroom use. They were tested successfully in a mathematics workshop for 12th grades students and a 13th grade biology class. During the workshop students worked independently with a workbook (Roeckerath, 2008). Most of them were able to derive the model equations for the modelled systems using the workbook. In the biology class the models were used to introduce population dynamics and to do some in silico experimentation. A derivation of the model equations was not part of the lessons.

The introduced models give a realistic insight into scientific research and real mathematical applications. Authentic modelling processes are always complex. The introduced models cannot be discovered by students autonomously. But they convey the basic processes of “real” modelling. A reasonable use of the introduced models in education requires that the teacher tries to find a proper balance between leading students in certain situations and encourage them to explore and experiment independently.

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ASPECTS OF VISUALIZATION DURING THE EXPLORATION OF „QUADRATIC WORLD“ VIA THE ICT – PROBLEM „FIREWORKS“

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Abstract

This paper deals with several proposals for the modelling of physical phenomenon of projectile motion (angled-launched projectile) in the Earth gravitational field. The problem, which we solve in this article named "Fireworks", is situated in the discipline intersection of mathematics, physics and informatics at the secondary schools. We compare the utilisation of the graphic calculator, the mathematical software WinPlot and spreadsheets in the solving process of this problem. It offers a large space for the unconventional approaches of teaching, for the use of information technologies and for the creation of interdisciplinary relations. The paper lays emphasis on the innovative process in mathematics teaching in Slovakia that incites stimulating discussions in this field new modern methods, ICT and e-learning.

Key words: modelling, quadratic function, graphic calculator, spreadsheets, teaching.

1 INTRODUCTION

Some Mathematics becomes more important because technology requires it.
Some Mathematics becomes less important because technology replaces it.
Some Mathematics becomes possible because technology allows it.
Bert K. Waits [1]

There are many arguments for and against the use of Information and Communication Technology (ICT) in mathematics teaching. This paper sets out some aspects of visualization, which is favourable to the exploration in mathematics learning.

The most considerable didactic aspects of the utilization of ICT in mathematics teaching are [3]:

- Aspect of *visualization* that relieves the conception of thinking process and keeps the learning process shorter,

- Aspect of *process simulation* that enables to create an adequate model on the basis of diverse input values (parameters) as well as to understand their hierarchy,
- Aspect of *interaction between an IC technology and a user* that represents one of the most important attributes of multimedia.

In the following text, we would like to focus primarily on the aspect of *visualization (demonstration)* in mathematics teaching. Problematic of the demonstration in mathematical research and also in mathematics teaching is considered to be one of the most important in the development of mathematical thinking. In relation to that, the literature remarks the notion of *visual thinking*. It is well known that the development of human cognition in the certain field relies on the groups of specific separate models of a future notion or knowledge [10].

Mental operations with the images can be complemented by real experimental manipulations and they lead to the concrete practice manipulation. In the frame of visual thinking, we can assert not only the *algorithms*, but also *heuristics*. *Visualization* represents one of the fundamental strategies in the field of creativity, discovering, inventions and abilities for problems solving. The importance of visualization is affirmed by the fact that the biggest part of brain cortex is aimed at *vision* and *visual analysis.*

Today there is no one to argue about the importance and significance of the development of visual thinking for the school mathematics. In spite of this, the visual methods of problems solving are moved at periphery and they are rare in school mathematics teaching. This reflexion is also underlined by the statement of contemporary mathematician and known popularizator of mathematics, Ian Stewart: *„Images transfer much more information than the words can transfer. Many years, we tried to unteach our students to use the images, because „they are not exact“. It is the sad misunderstanding. Yes, the images are not exact, but they help to think and we could not despise this aid.* “[3]

The main objective of the innovative process in the mathematics teaching in our country is to show the pupils that the mathematics education is not purposeless. The mathematics is the science, which has various important applications in real life that are inevitable for the development of other scientific and technical disciplines. The process planted into the long term horizon must respect the pupils´ mental abilities oriented at the discovering and the cognition of mathematical notion, the development of pupil’s creativity, critical thinking and team-work, but also the need of scientific discussion in the class. The international comparative studies TIMSS and PISA show the actual deficiency of these pupils´ abilities in Slovak school system [5]. That is the reason why in this paper we would like to offer one physical problem together with the possibilities of its solutions including the utilisation of mathematical knowledge and convenient ICT that is accessible to schools. First section outlines the central problem of this article named „Fireworks“. The next sections detail the ideas of the central problem solution with the help of the mathematical software WinPlot, the graphic calculator TI 83+, and MS Excel (spreadsheets).

2 PROBLEM „FIREWORKS“

This part of article was inspired by the mathematics teaching at secondary schools in USA, especially by the implementation of IMP (Interactive Mathematics Program). The aim of this program is to teach the mathematics differently and to prepare a pupil in the constructive way to encounter the world where he lives. The objective is not to let the pupil receive the knowledge in the inactive way, but above all to let him experiment, search, ask, look for the answers, create and test his own hypothesis, consider, work in teams, share and communicate his ideas and inventions.
The principal topic of the following sections is a quadratic function, whose concept is presented from the several points of view (functional, algebraic and geometric). We consider the choice of the „Fireworks“ problem as very suitable, because it includes not only the mathematical problem, but also the physical problem, which the pupils are able to solve effectively by the aid of ICT [4].

Problem definition

High school football team has just won the championship. To celebrate this triumph, the young football players want to put on a fireworks display. They will use rockets launched from the top of a tower near the school. The height of the tower is 50 metres off the ground. The automatic mechanism will launch the rockets with the initial velocity 28 metres per second.

The team members want the fireworks from each rocket to explode when the rocket is at the top of its trajectory. They need to know how long it will take for the rocket to reach the top, so they could set the timing mechanism. Also, they need to know the best place for spectators to stay (they need to know how high the rocket will go).

The rockets will be oriented to an empty field and shot at an angle of 65 degrees above the horizontal. The team members also want to know how far from the base of tower will the rockets land, so that they can fence off the area.

Theoretical background (several formulations)

The problem includes the physical phenomenon of projectile motion named angled-launched projectile [6]. This motion consists of a uniform rectilinear movement in the direction of axes x with velocity v_1 and a vertical displacement with initial velocity v_2. Vector of initial velocity v_0 and direction of projectile motion contain the angle α which is named *elevation angle*. The horizontal distance of the projectile range depends on this angle (the distance is biggest when $\alpha = 45°$). The range distance depends also on the initial velocity v_0.

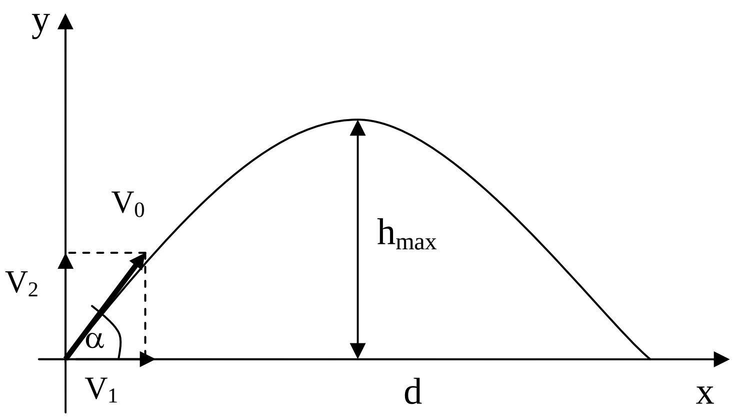


Figure 1 Scheme of angled-launched projectile

$v_1 = \cos(\alpha).v_0$
$v_2 = \sin(\alpha).v_0 - g.t$
$x = v_1.t$
$y = v_2.t - ½.g.t^2$

David is member of the football team. He is also high school student and he is good in mathematics and physics. He would like to help his team to solve the „Fireworks“ problem. He says that there is a function ***h(t)*** that gives the relation between the rocket's height off the ground and the time ***t*** elapsed since launch. This relation can be represented by the equation (in metres and seconds): $h(t) = 50 + 28.\sin 65^o\, t - 5t^2$, $h(t) \cong 50 + 25\,t - 5t^2$.

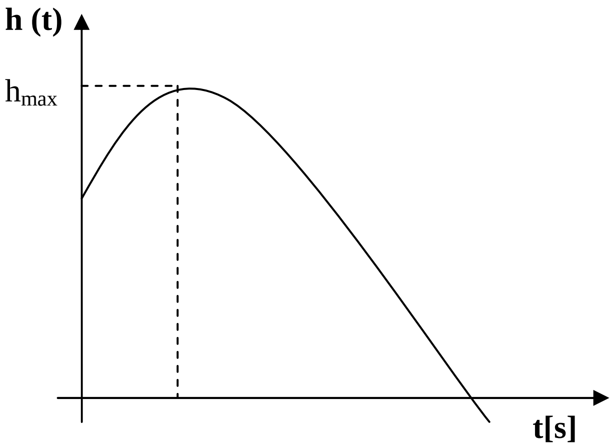


Figure 2 Sketch of the problem situation

We can probably see where the numbers 50, 28, 65 come from. The coefficient -5 in the quadratic component $-5t^2$ coheres with the force of gravity done by the relation: G = $\frac{1}{2}gt^2$.

David also says that it is possible to find a relation describing horizontal distance. The rocket travels with this function: ***d(t)= 28 t .cos 65º.***
Again, ***t*** is the time (in seconds) since the rocket was launched and ***d(t)*** is the distance (in metres).

Tasks:

1. To draw a sketch of the situation.
2. To find answers to the partial questions of the football team players:

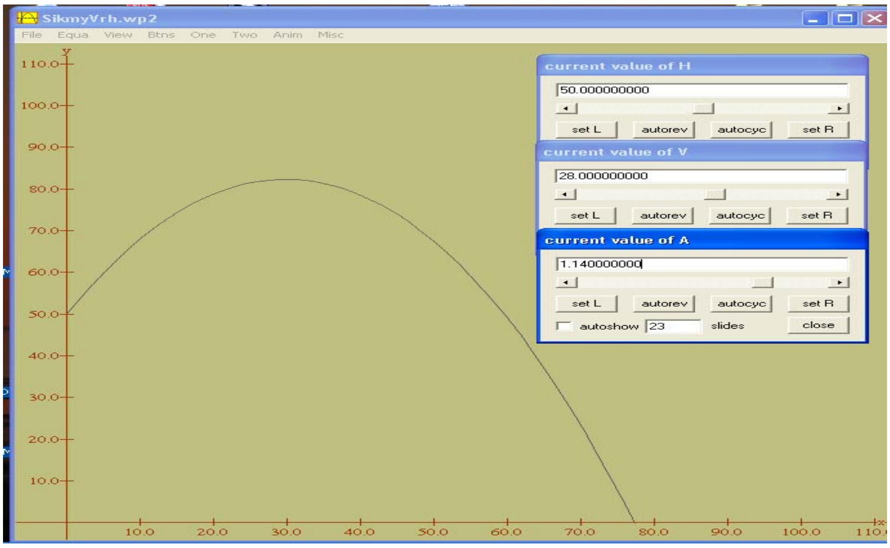
A) What time does the rocket need to reach the top of its trajectory (to find the point where does the function ***h(t)*** reach its maximum)?

B) Where (horizontal distance of the rocket from the tower) does the rocket reach its maximum height?

C) How far (horizontal distance) from the base of the tower does the rocket land?

3 PROBLEM SOLUTION BY THE AID OF THE SPECIAL MATHEMATICAL SOFTWARE WINPLOT AND THE GRAPHIC CALCULATOR

Software WinPlot enables to create an interactive programme, which describes our „Fireworks“ problem. The pupil can change the parameters of the tower height, the initial velocity and the elevation angle. He can observe how these parameters influence the trajectory of the rocket motion. The image underneath represents the trajectory of the rocket since its launch from top of the tower until its landing. The ordered pair **[d(t), h(t)]** express the coordinates of the projectile (fireworks rocket) moving in the frame of its trajectory in terms of the time ***t***.



Variable parameters:

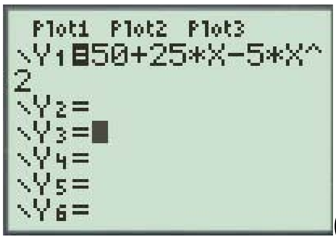
H – Height of tower
V – Initial velocity
A – Elevation angle

Figure 3 Trajectory of the rocket’s motion made in WinPlot

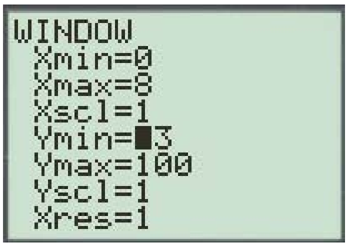
Solving of the „Fireworks“problem by the aid of graphic calculator (TI 83+)

To solve the problem, we can chart the graph of the quadratic function $\boldsymbol{h(t) = 50 + 25t - 5t^2}$, which represents functional dependence of the launched rocket height ***h*** on the time ***t***, with the help of graphic calculator. The task is solved graphically [2].

In order to graph the quadratic function, firstly we have to insert its formula to the function editor **Y=** (a). We must also adapt the window editor to see the whole graph of the quadratic function (b). Than we can let the calculator draw the graph of the function (c).



(a)



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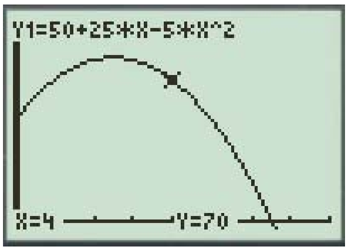
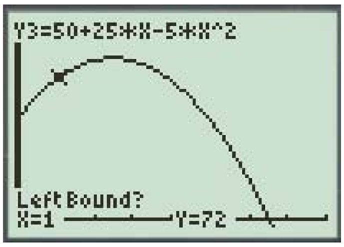
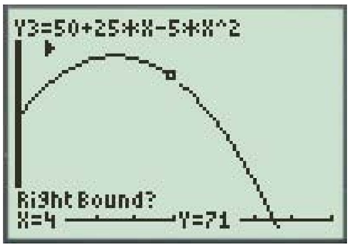
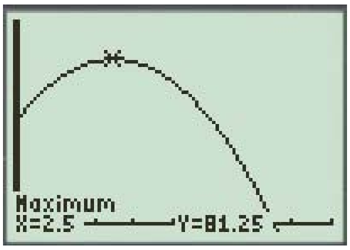


(c)

We can answer the question 2A) “What time does the rocket need to reach the top of its trajectory?” by finding the top, the highest point of the graph of quadratic function (it means to find a point where the function reaches its maximum). The calculator function 2nd [CALC] 4: maximum, enables us to count the maximum of the quadratic function h(t) with the corresponding value of t, so we get e.g. the maximum height $\mathbf{h_{max}}$ **= 81,25 m** and the time when the rocket reaches this height **t = 2,5 s**.

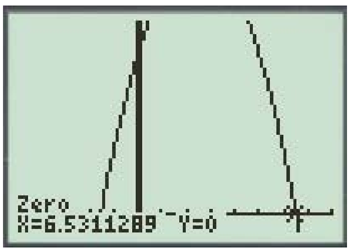






In the task 2B) we have to find the place (horizontal distance of the rocket from the tower), where the rocket reaches its maximum height. We can calculate this position simply by putting the obtained value **t = 2,5 s** into the equation ***d(t)= 28* t*cos 65º***, so we receive: d(t) = 28*2,5*cos 65º. Therefore, the place where the rocket reaches the maximum height off the ground level is approximately **29,6** metres far from the tower.

The following calculation will answer the last question 2C): where should we look for the area (place, point) of the rocket’s landing.



At first, we enumerate the time of rocket landing on the ground (it is one convenient positive root of the equation h(t)=0, or the intersection of the quadratic function graph with the x coordinate axe).

By this calculation, we obtain the time **t = 6,53 s** of rocket landing. Finally, we are able to take the time t = 6,53 s and insert it into the function pattern ***d(t)= 28*t*cos65º***. We acquire the distance of the rocket landing, **d = 77,28 m** far from the base of tower.

The graphs we have demonstrated by the aid of graphic calculator can offer the image and a lot of information about the rocket movement. However, they do not simulate the trajectory of this movement. For this purpose, it is better to use the interactive program made in WinPlot.

4 PROBLEM SOLVING WITH THE HELP OF SPREADSHEETS

During teaching, it is suggested to utilize a spreadsheet processor as a tool to model various possibilities that could occur and to analyze data. The spreadsheet processor and graphical processing of data can be used during education as tools to model and simulate the dynamic processes. These tools are known to the students as quite standard. By the application of spreadsheet programs' features, we can gain quantitative modeling tools, which are suitable for the use during elementary and high school education.

The spreadsheet programs allow us to use one of their important features – an ability to put calculations' results into the graphs [8]. An adequate example could be a modeling of mathematic functions x^2, sin(x), cos(x) or modeling of the angled-launched projectile.

Using the formulas for individual parameters, we put values of α (an angle), v_0 (initial velocity) and constant g (gravitational acceleration) into the cells with absolute addresses. Then we generate a table of calculations for the sequence of values of time parameter t. With the help of functions, we find the values of the maximal height and distance of fall.

Together with the students, we can experiment with the model by changing the angle of throw or initial speed and then observe how it affects a trajectory. The results are visually displayed on the graph.

For the "Fireworks" problem, we use David's equations d(t) = 28*t*cos 65° and h(t) = 50+25*t-5*t^2. We create a table of time values together with functions $h(t)$ and $d(t)$. Based on values we graphically represent relation of the time t to the height function h and distance function d. The students can use the graph to approximate the maximal values of height and fall together with the corresponding time moment. These values can be determined also by utilization of the spreadsheet calculator's function for the maximum.

t [s]	h(t)	d(t)
0	50,00	0,00
0,3	57,05	3,55
0,6	63,20	7,10
0,9	68,45	10,65
1,2	72,80	14,20
1,5	76,25	17,75
1,8	78,80	21,30
2,1	80,45	24,85
2,4	81,20	28,40
2,5	**81,25**	**29,58**
2,7	81,05	31,95
3	80,00	35,50
3,3	78,05	39,05
3,6	75,20	42,60
3,9	71,45	46,15
4,2	66,80	49,70
4,5	61,25	53,25
4,8	54,80	56,80
5,1	47,45	60,35
5,4	39,20	63,90
5,7	30,05	67,45
6	20,00	71,00
6,3	9,05	74,55
6,5	**1,25**	**76,92**
6,6	-2,80	78,10

Table 1 Values of variables

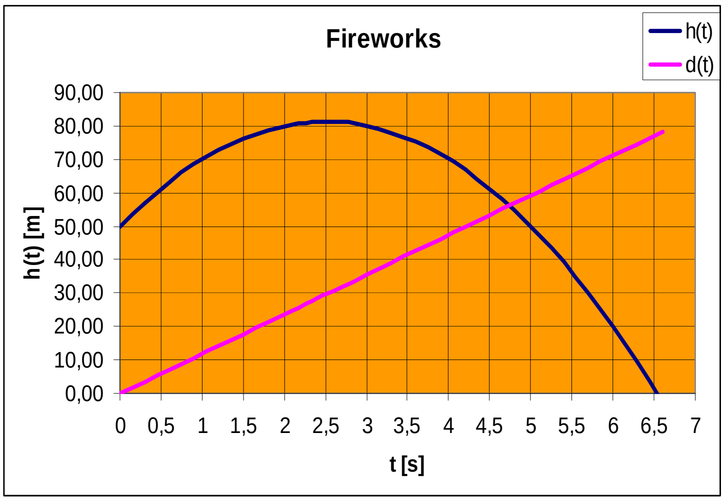


Figure 4 Graphs of functions in the MS Excel program

A representation of the calculations' results by the column of numbers together with the graph allows students to get a deeper insight into the observed phenomenon. The advantages, which result from this kind of spreadsheet programs' utilization in education, are the following ones:

- complicated, repeating calculations are cut down to minimum
- more models of "what happens, if" type can be checked out
- models can be tested by the greater amount of data
- it's possible to graphically represent the examined relations

5 CONCLUSION

Creation and application of the models for the purpose of real world's phenomena demonstration is the subject of teaching process. These models take a significant part in application of didactic principles of science, demonstration and activity [9]. Scientific knowledge is related not only to the content of teaching process, but also it represents the method of its acquirement. Modeling and simulation of the systems, as a scientific method, helps students to gain new information by examining the various systems, based on their models [7].

The graphic calculator, software WinPlot and the spreadsheet processor could be the appropriate tools for the creation of visual and graphically high implemented animation models. A very important function of the models is an enhancement of visual demonstration. A purpose of this demonstration is to create the conclusive ideas for the student. At the age of 12 years, when students acquire an ability to accomplish the formal operations and to think abstractedly, it is desirable to arouse their visual feeling of abstract representations and descriptions of real processes and devices. Various symbolic models, such as diagrams or graphs, can also be used during the mathematics teaching.

A model used for the didactic purposes helps us to demonstrate and discover all the significant features of examined phenomenon. It is appropriate for students not only to get prepared models of the reality, but also to create some themselves. Thus they have to reproduce a structure of the model and to reveal all of its features. Consequently they can improve it or work it over. As a result, the students can learn in a more creative way. This approach creates an area for the use of educational software and tools, which gives us an opportunity to teach the students a given topic with the help of ICT.

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MATHEMATICAL MODELLING IN CLASS REGARDING TO TECHNOLOGY

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Based on the well known modelling cycle we develop a concept of modelling in mathematics education using technology. We discuss the specifics of modelling with computers and handhelds and show some technical possibilities of software tools for mathematics classes. Exemplary we show different modelling cycles using technology based on the three major types of software tools for mathematics.

INTRODUCTION – MODELLING WITHOUT HELP OF TECHNOLOGY

The concept of modelling can be found as a basic concept in some areas of natural sciences, especially mathematics. Therefore it is not remarkable that this basic concept can be found in several curricula all over the world. In mathematics the concept of modelling and the application of real-life-problems in education has been discussed intensively over the last years – see for example Kaiser & Sriraman (2006, p. 304), Siller (2006). It is possible that students of all ages are able to recognize the importance of mathematics through such problems because real-life problems...

- ... help students to understand and to cope with situations in their everyday life and in the environment,
- ... help students to achieve the necessary qualifications, like translating from reality to mathematics,
- ... help students to get a clear and straight picture of mathematics so that they are able to recognize that this subject is necessary for living,
- ... motivate students to think about mathematics and computer-science in a profound way so that they can recall important concepts even if they were taught a long time ago.
- ... allow the teaching of mathematics with a historical background.

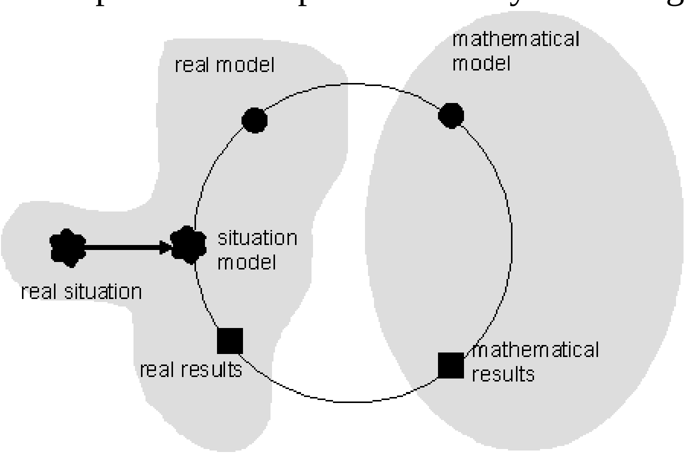

Figure 1: Modelling cycle of Blum & Leiß (2007)

If we look at the concept of modelling (figure 1) designed by Blum & Leiß (2007) in mathematics education we will be able to find three important points:

- Design & Development: Comparable to “Finding the real model” and to the step of “Translation” – Real situation to real model by including the situation model.

- Description: Comparable to “Finding the mathematical model”.
- Evaluation: Comparable to “Finding (Calculating) mathematical results” and to the step of validating.

In curricula the usage of technology and the aspect of modelling very often is demanded. For example you can read in the Austrian curriculum (2004):

“An application-oriented context points out the usability of mathematics in different areas of life and motivates new knowledge and skills. [...]

The minimal realization is the acquiring of the issue of application-oriented contexts in selected mathematical topics; the maximal realization is the constant addressing of application-oriented problems, the discussion and reflection of the modelling cycle regarding its advantages or constraints. […] Technologies close to mathematics like Computer algebra systems, Dynamic Geometry-Software or Spreadsheets are indispensable in a modern mathematical education. Appropriate and reasonable usage of programs ensures a thorough planned progress. The minimal realization can be done through knowing such technologies and occasional applications. In a maximal realization the meaningful application of such technology is a regular and integral part of education.”

So each of us has to ask where the usage of technology can be best implemented. The integration of technology in the modelling cycle can be helpful by leading to an intensive application of technology in education. We have thought about a way that the use of technology could be implemented in the modelling cycle. Our result can be seen in figure 2. The “technology world” is describing the “world” where problems are solved through the help of technology. This could be a concept of modelling in mathematics as well as in an interdisciplinary context with computer-science-education.

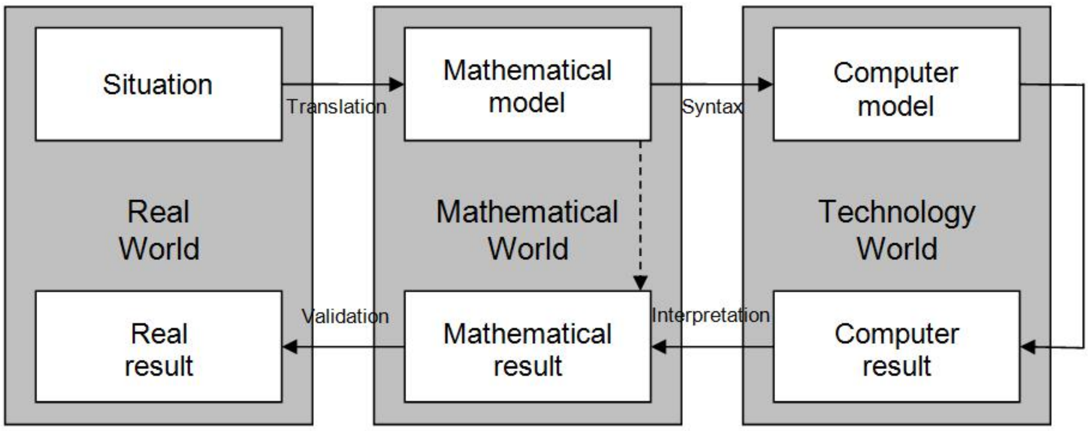

Figure 2: Extended modelling cycle – regarding technology when modelling

The three different worlds shown in figure 2 are idealized; they influence each other. For example the development of a mathematical model depends on the mathematical knowledge on the one hand, on the other it is affected by the possibilities given in the technology world. Using technology broadens the possibilities to solve certain mathematical models, which would not be used and solved if technology would not be available. At this point we want to mention, that successful modelling demands mathematical knowledge and skills in certain software tools.

Based on this graphical illustration we have to discuss the use of technology in terms of modelling in a more detailed way.

MODELLING WITH THE HELP OF TECHNOLOGY

Through the usage of computers in education it is easier to discuss problems which can be taken out of the life-world of students. Through such discussions the motivation for mathematical education can be effected because students recognize that mathematics is very important in everyday life. If it is possible to motivate students in this way it will be easy to discuss and to teach the necessary basic or advanced mathematical contents such as finding a function or calculating the local extreme values of a function.

Unfortunately a lot of teachers and educators prefer not to work with real-life problems. The reasons for this are manifold, e.g. teachers do not want to use CAS or other technology in class or the preparation for such topics is very costly in terms of time. There are however, lots of reasons to combine modelling and technology. Fuchs & Blum (2008) quote the aims of Möhringer (2006) which can be reached through (complex) modelling with technology:

- Pedagogical aims:
  With the help of modelling cycles it is possible to connect skills in problem-solving and argumentation. Students are able to learn application competencies in elementary or complex situations.
- Psychological aims:
  With the help of modelling the comprehension and the memory of mathematical contents is supported.
- Cultural aims:
  Modelling supports a balanced picture of mathematics as science and its impact in culture and society (Maaß, 2005a, 2005b).
- Pragmatically aims:
  Modelling problem helps to understand, cope and evaluate known situations.

As we can see the use of technology can help to simplify difficult procedures in modelling. In some points the use of technology is even indispensable:

- Computationally-intensive or deterministic activities,
- Working, structuring or evaluating of large data-sets,

- Visualizing processes and results,
- Experimental working.

With technology in education it is possible not only to teach traditional contents using different methods but it is also very easy to find new contents for education. The focus of education should be on discussion with open, process-oriented examples which are characterized by the following points.

Open process-oriented problems are examples which …

- … are real applications, e.g. betting in sports (Siller & Maaß, 2008), not vested word problems for mathematical calculations.
- … are examples which develop out of situations, that are strongly analyzed and discussed.
- … can have irrelevant information, that must be eliminated, or information which must be found, so that students are able to discuss it.
- … are not able to be solved at first sight. The solution method differs from problem to problem.
- … need not only competency in mathematics. Other competencies are also necessary for a successful treatment.
- … motivates students to participate.
- … provokes and opens new questions for further as well as alternative solutions.

The teacher is achieving a new role in his profession. He is becoming a kind of tutor, who advises and channels students. The students are able to detect the essential things on their own. Therefore we want to quote Dörfler & Blum (1989, p. 184): “With the help of computers (note: also CAS-calculators) which are used as mathematical additives it is possible to reach a release of routine calculation and mechanical drawings, which can be in particular a big advantage for the increasing orientation of appliance. Because of the fact, that it is possible to calculate all common methods taught in school with a computer, mathematics education meets a new challenge and (scientific) mathematics educators have to answer new questions.”

ENABLING TECHNOLOGY

The use of technology in mathematical education always depends on the enabling technology. For mathematical education there are many different hard- and software tools. The three major types which have emerged in this area are - see for example Barzel et al. (2005, p. 36):

- Computer algebra system (CAS): With the help of such a tool it is possible to work symbolically, algebraically and algorithmically.

- Dynamic Geometry Software (DGS): With the help of such software it is possible to create geometrical constructions interactively and work with digital work sheets.
- Spreadsheet Program (SP): With its help it is possible to organize and/or structure data for easier handling, calculating in tables and common analysis.

New developments in the area of technology try to combine these three aspects, although it is difficult to combine all three points and form unique software for each characteristic.

For example the CASIO Classpad, respectively the associated software package Classpad Manager, offers a real interactivity of geometry and algebra.

The simultaneous application of CAS and DGS of the Classpad is, in our opinion, also a useful application. With the help of CAS it is possible to calculate for example non-linear equations symbolically and at the same time the geometrical aspects can be shown through dynamical geometry.

For these purposes equations have to be transformed from the CAS-part to the geometry part. But this method is – until now – not as effective as it should be. After such a transformation the equations cannot be changed interactively. But this problem is not really important, because such examples can be handled easily with other tools, e.g. Geogebra. In Geogebra it is not possible to use a real CAS-part, but the interactivity can be done easily. And a new feature, which is currently available in a Beta-version, is the implementation of a spreadsheet-tool. With its help it is possible to combine interactivity with numerical solutions, calculated in a spreadsheet. To sum up there are several tools combining two or three of the major types CAS, DGS and SP.

Example

The following example which could be discussed with students can be found in everyone’s life-world:

Dangerous intersection:

Two cars with different velocities are driving on two different streets towards an intersection where those streets meet. One car is going 60 kilometres per hour; the other has a velocity of 50 km per hour. Try to think about the situation at the intersection – is it possible that an accident can happen? It is given that both cars are running with the constant velocities towards the intersection.

The example can be discussed now under the aspect of different didactical principles:

- Haptical discussion: Students model the given situation, for example with some toy cars, and try to find a solution. This could be a starting point for cross-disciplinary teaching with physics (without computers).
- Graphical discussion: Students have to draw a chart or diagram of the given situation, and/or modify a given chart (with paper and pencil or DGS).

- Symbolical discussion: Students have to describe the situation for both cars with the help of a function or functional dependency (with paper and pencil or CAS).
- Numerical discussion: Students compute lots of data to solve the problem (with a scientific calculator or SP).

It is not that important which method students' first use to solve this problem. An important point is that students are working based on experience and the methods used are kept sustainable. But it is important for the students to see the different approaches for this problem. In our course we used the following problem:

A picture which describes the given situation visually can be found in figure 5.

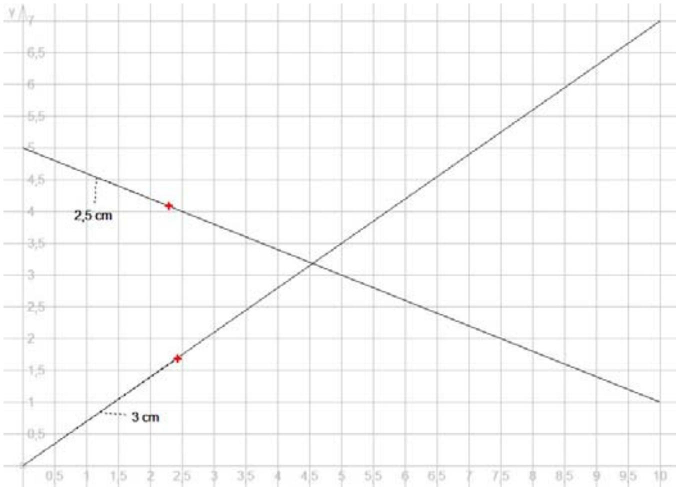

Figure 3: Graphical visualization of the problem

This problem – here in an adequate norm - can be solved in completely different ways. If we use the help of dynamical geometry software, we can move a point for the second car by moving the point for the first car automatically in the right scale and see what happens at the intersection. If we have a closer look at our concept "Modelling with the help of technology" and try to translate the steps which are necessary for solving the problem into our model, it could appear as presented in the following figure (figure 6):

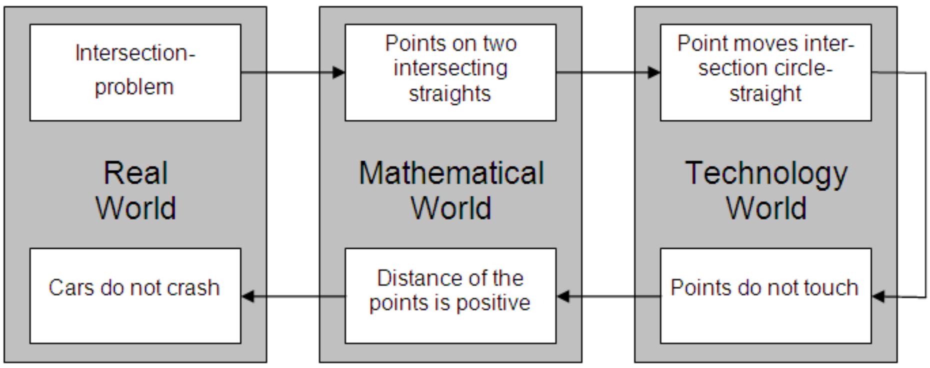

Figure 4: Extended modelling cycle for the problem "Dangerous Intersection"

Alternatively the solution can be calculated with the help of a CAS.

#1: $$f(t) := \frac{t \cdot [10,\ 7]}{\sqrt{(10^2 + 7^2)}}$$

#2: $$g(t) := [0,\ 5] + \frac{\frac{5 \cdot t}{6} \cdot [10,\ -4]}{\sqrt{(10^2 + 4^2)}}$$

#3: $$d(t) := |f(t) - g(t)|$$

#4: $$\text{SOLVE}\left(\frac{d}{dt}\ d(t),\ t,\ \text{Real}\right)$$

#5: $$t = \frac{261540 \cdot \sqrt{149}}{1037531} + \frac{495300 \cdot \sqrt{29}}{1037531}$$

#6: $$t = 5.647806845$$

#7: $$d(5.647806845)$$

#8: $$0.2573238092$$

#9: $$f(5.647806845)$$

#10: $$[4.626863648,\ 3.238804553]$$

#11: $$g(5.647806845)$$

#12: $$[4.369880841,\ 3.252047663]$$

Note: For easier readability we have decided to present the solutions in the CAS in decimal notation.

Figure 7 shows the same mathematical model, but a different computer model in the technology world. A CAS works algebraically so we cannot use a geometrical construction to work on the mathematical model. Therefore we decided to use derivation and distance to solve the problem.

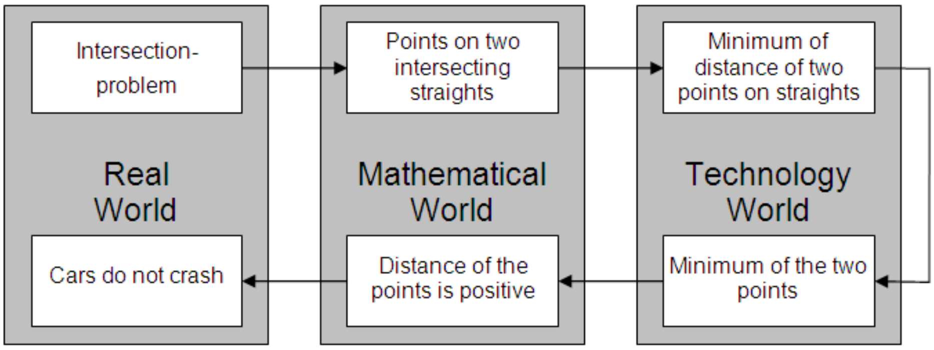

Figure 5: Extended modelling cycle with a different part in the technology world

The solution can also be calculated with the help of a spreadsheet. We will just document this possibility without discussing it. There will of course be a third model in the technology world.

t	x1	y1	x2	y2	dx	dy	d
5,6	4,6	3,2	4,3	3,3	0,3	-0,1	0,2608
5,61	4,6	3,2	4,3	3,3	0,3	0,0	0,2595
5,62	4,6	3,2	4,3	3,3	0,3	0,0	0,2585

5,63	4,6	3,2	4,4	3,3	0,3	0,0	0,2578
5,64	4,6	3,2	4,4	3,3	0,3	0,0	0,2574
5,65	4,6	3,2	4,4	3,3	0,3	0,0	0,2573
5,66	4,6	3,2	4,4	3,2	0,3	0,0	0,2575
5,67	4,6	3,3	4,4	3,2	0,3	0,0	0,2581
5,68	4,7	3,3	4,4	3,2	0,3	0,0	0,2589
5,69	4,7	3,3	4,4	3,2	0,3	0,0	0,2600
5,7	4,7	3,3	4,4	3,2	0,3	0,0	0,2614

Table 1: Worksheet in SP for the example “dangerous intersection”

These three possible solutions (by DGS, CAS, SP) are prototypes for student solutions which represent different mathematical concepts and models. In all three models the assumptions concerning the position of both streets, represented by straight lines, are equal.

The model designed with the help of dynamical geometry software uses only the implicit representation of parameterised straight lines. The main mathematical concept is studying the distance of two (moved) points in the plane. Designing the model as it is shown, presumes the understanding in analytical geometry and connections between the two moving points. The ratio of the velocities of both cars, idealized as points, influences the movement of one point depending to the other. The dynamical visualization allows pupils to experiment with the model (e.g. changing the position of both cars). Thereby possibilities for further developing the model are given (e.g. including the length of the cars).

The models designed by CAS and SP are using parameterised straight lines as algebraic expressions. The distance of both points can be calculated with the help of Pythagoras' theorem. In the CAS model the minimum is calculated with the help of differential calculus, whereas in the SP model the minimum has to be found numerically. One possibility of the CAS model is adding other variables (e.g. different velocities for the cars, changing the starting point of one car) for experimenting or developing the model further. Here more possibilities are imaginable. All of them are very ambitious.

TEACHER EDUCATION

The use of technology in mathematical education does not only depend on technology but also on the knowledge and beliefs of the teacher concerning the different types and usages of technology.

The work with computers in teacher training sets the stage for use in schools. At the beginning and at the end of the course “Computer for Mathematics in School” students who attended were asked about their opinion on the use of computers in class for education. 10 students were present in both interviews comparably. Every question (shown in the diagram of figure 9) had four possible answers: yes, rather yes, rather no, no. In certain cases the beliefs using computers in class changed after attending this course. The topics of the course are the use of CAS, DGS and SP for mathematics in school.

The interviews show a possible change of beliefs while working on topics with computers in class. Some of the positive results concerning computer use can be seen in figure 9, whereas a small bar is closer to the answer "Yes" a bigger bar closer to "No".

The students are asked to say what changes occur using computers in mathematics classes.

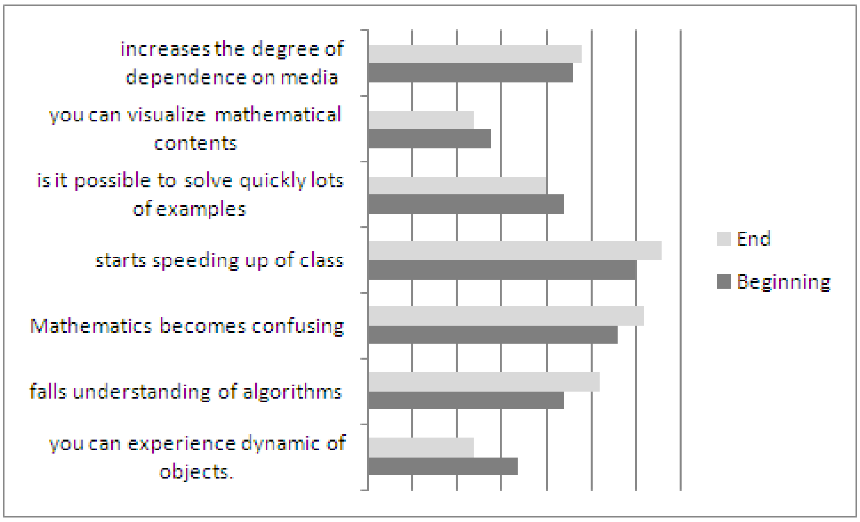

Figure 9: Results of the interview about changes by using computers

This first results show, that it would be interesting to have a closer look at the different strategies of students while modelling with a digital tool. For this research it is necessary to find more examples like "dangerous intersection" with relevance to real life and with different approaches.

Even in teacher education it would be a possible way to discuss examples like "dangerous intersection" by focussing on different computer models. To help the students to reflect upon the role of mathematical software in mathematical modelling processes criteria should be developed and applied (e.g. mathematical content, level of difficulty, possibilities of further developing).

Recapitulatory the use of computers in mathematical education can support and create understanding, in order to improve motivation. The role of technology in the modelling cycle has to be pointed out and examples in education have to be adapted and even created. To implement these points more research in this field needs to be done.

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THE ‘ECOLOGY’ OF MATHEMATICAL MODELLING: CONSTRAINTS TO ITS TEACHING AT UNIVERSITY LEVEL

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Considering the general problem of integrating mathematical modelling into current educational systems, the paper focuses on the study of the institutional constraints that hinder the implementation of modelling activities. The study of these restrictions and the way new teaching proposals can overcome them appear as an unavoidable step for the large-scale dissemination of mathematical modelling activities at all school levels. Within the framework of the Anthropological Theory of the Didactic, it is proposed the use of a hierarchy of levels of didactical determination as a frame to set and analyse from the more specific constraints, related to the usual way of organising mathematical contents, till the more generic ones, linked to the ‘dominant epistemology’ concerning the role of mathematics in experimental sciences.

Key words: ATD, mathematical modelling, constraints, conditions, applicationism.

1. THE PROBLEM OF INTEGRATING MATHEMATICAL MODELLING INTO CURRENT EDUCATIONAL SYSTEMS

Nowadays, there seems to be no doubt about the possibility of introducing students to a mathematical activity orientated towards the study of applied and modelling problems. This agreement is shared by many researchers in the field of mathematical modelling and applications, and supported by the new curricular orientations that have recently been introduced in our educational systems, thus trying to focus mathematical teaching more on the study of ‘real life situations’ than on systems of well-organised mathematical contents. Several investigations from different theoretical perspectives have shown that mathematical modelling activities can exist at school under suitable conditions, at all levels and related to almost all curricular contents.

Beside all the progress of establishing modelling as a normalized activity in some controlled processes of teaching and learning mathematics, the problem of the large-scale dissemination of these processes has recently been addressed as both an urgent and intricate task. Some authors have started pointing out the existence of strong limitations hindering the inclusion and permanent survival of mathematical modelling practices in the classroom. For instance, Blum *et al.* (2002, p. 150) depicts the situation as follows:

> While applications and modelling also play a more important role in most countries’ classrooms than in the past, there still exists a substantial gap between the ideals of

> educational debate and innovative curricula on the one hand, and everyday teaching practice on the other hand.

Kaiser (2006, p. 393) seems to go in the same direction when she states:

> Since the last decades the didactic discussion has reached the consensus that applications and modelling must be given more meaning in mathematics teaching. […] However, international comparative studies on mathematics teaching carried out during the last years, especially in the PISA Study, have demonstrated that worldwide young people have significant problems with applications and modelling tasks.

Related to this state of things, Burkhardt (2008) emphasizes the existence of two realities: on the one hand, the good progress and encouraging results in research about teaching modelling and applications; on the other hand, the difficulties of its large-scale diffusion in the classroom. He states quite brutally (op. cit., p. 2091):

> [W]e know how to teach modelling, have shown how to develop the support necessary to enable typical teachers to handle it, and it is happening in many classrooms around the world. The bad news? 'Many' is compared with one; the proportion of classrooms where modelling happens is close to zero.

To describe the difficulties encountered in the diffusion of modelling, many researchers use expressions such as 'counter-arguments' (Blum, 1991), 'obstacles' (Kaiser, 2006), 'dilemmas' (Blomhoj & Kjeldsen, 2006) or 'barriers' (Burkhardt, 2006), pointing out a new direction of research which moves from the problem of the design, implementation and analysis of modelling practices to the study of the conditions that affect the existence, permanence and evolution of these practices. In a research on teachers' beliefs about mathematical modelling, Kaiser (2006) defines different teachers' profiles to explain how some beliefs can become important 'obstacles' for the implementation of applied and modelling practices in teaching, because the nature of contextual and applied problems does not seem to be compatible with those beliefs. (p. 399). In the same direction, Blomhoj & Kjeldsen (2006, pp. 175-176) point out the existence of different 'dilemmas' that should be faced before widely incorporating the teaching of modelling. These dilemmas refer to: the understanding of mathematical modelling competency from a holistic point of view; considering mathematical modelling as an educational goal in its own right and the dilemma of teaching directed autonomy.

At a more general level, Burkhardt (2006, pp. 190-193) outlines and discusses the existence of 'barriers' that obstruct a large-scale implementation of modelling, such as the systemic inertia, the unwelcome complication of the 'real world' in many mathematics classrooms, the limited professional developments of teachers, the role and nature of research, and the development in education. To overcome these barriers and many others still unknown, he refers to some 'levers' (such as changes in curriculum descriptions supported by well-engineered materials to support assessment, teaching and professional development, etc.) that may show some promise progress in this field. Michelsen (2006) points out an even more general barrier when he questions the common separate vision of scientific disciplines, and states that traditional borders between disciplines suppose a clear constraint for the development of applied activities (op. cit., p. 269):

> The challenge is to replace the current monodisciplinary approach, where knowledge is presented as a series of static facts disassociated from time with an interdisciplinary approach, where mathematics, science, biology, chemistry and physics are woven continuous together.

This situation can be summarized in the formulation of the following didactic problem, which has to be located at the core of all research aiming to integrate mathematical modelling in teaching and learning practices:

What kind of *limitations* and *constraints* exist in our current educational systems that prevent mathematical modelling from being widely incorporate in daily classrooms' activities? What kind of *conditions* could help a large-scale integration of mathematical modelling at school?

Within the framework of the Anthropological Theory of the Didactic (ATD), most of the research related to mathematical modelling and teaching practices[1] (Artaud 2007, Bolea *et al.* 2004, Barquero *et al.* 2008, Barbé *et al*. 2005) takes into account the problem of the 'ecology' of didactic organisations, that is, the study of the conditions needed to implement teaching and learning activities and the constraints that hinder their normal evolution in a given educational institution. The origin of this ecological problematic, which was first applied to mathematical objects and practices before being enlarged to a wider institutional perspective, can be located in the study of the process of *didactic transposition* and its related phenomena (Chevallard 1985, see also Bosch & Gascón 2006).

In our research project on the study of a global modelling process at university level centred on the study of a population dynamics (Barquero *et al*., 2008), we have observed the existence of different kinds of transpositive constraints that hinder the normal evolution of modelling practices in the classroom. We will develop this point further in the next section, preceding it by a short presentation of the 'levels of didactic determination', a key notion introduced by Chevallard (2002) that we will use as a frame to analyse the different kinds of conditions and constraints that affect teaching and learning processes.

2. CONSTRAINTS ON THE TEACHING OF MODELLING ACTIVITIES

2.1. Levels of didactic determination

[1] Several works within the framework of the ATD as Chevallard (1992), Chevallard, Bosch & Gascón (1997) have analyzed and described mathematical modelling activities from this approach. From ATD, it is assumed that doing mathematics consists essentially in the activity of producing, transforming, interpreting and arranging mathematical models.

Mathematics teaching and learning processes can exist because a lot of conditions make them possible: the existence of a social educational project, the choice of a set of contents to be taught, a school organisation with grades, syllabi, teachers and students grouped in classrooms, teaching materials, teachers' training programmes, etc. These conditions are also factors that, while allowing some things to happen, are also impeding others to take place. In the research and design of new teaching proposals, taking into account these conditions and constraints seems necessary if we do not want to have a set of 'ideal' didactic organisations unable to 'survive' under normal conditions, being, as Chevallard (2002, p. 42) put it, only a 'world on paper'. To study the 'ecology' of mathematical practices that exist (or could exist) in a teaching institution and the possible ways of constructing them (the didactic organisations), this author introduced a hierarchy of 'levels of didactic determination' that consists in the following sequence (Ibid.):

Civilization ↔ Society ↔ School ↔ Pedagogy ↔ Discipline ↔ Domain ↔ Sector ↔ Theme ↔ Subject

This hierarchy goes from the most generic level –Civilization– to the most concrete one – the subject or questions that are to be studied by a group of persons. We refer to the lower levels that go from the *discipline* to the *subject* as the *mathematical levels* if the considered discipline is *mathematics*. They refer to the fact that, when a teaching project has been decided on, the contents or the aim of this project should be located in a discipline (or different ones) and, within this discipline, it should be related to the different domains, sectors and themes that structured it in the considered educational institution. For instance, in Spain, a first year course of mathematics for science students at university level is usually structured into three domains: calculus, linear algebra and differential equations. Frequently, the domains are in turn divided into 'sectors', which contain different 'themes', to which every subject or question to study belongs. At secondary school level, the domains are different and can change over time, with each curricular reform: the classical division into 'arithmetic, algebra, geometry' first changed to 'numbers and measure, functions, geometry, statistics', and has now turned into 'change and relations, space and form, statistics, measure, number'. We consider these low levels (as) the 'specific' ones. They are a useful tool to analyse the constraints coming from the didactic transposition process and the concrete way this process organises teaching contents at school: from the division into disciplines and blocks of contents, until (till) the low-level concatenation of subjects.

The upper levels of determination refer to the more general constraints coming from the way Society, through School, organises the study of disciplines (pedagogical level). They concern the status and functions traditionally assigned to educational contents and the general way teaching and learning study activities are organised at school. In effect, there are a lot of common conditions for all disciplines that concretely affect what the teacher and students can do in the classroom. For instance, the amount of hours and sessions assigned to the teaching of a concrete discipline, the possibilities for disciplines to interact more or less easily, the way students are grouped (by age, by level, by gender, etc.), the organisation of the school space, etc. All those conditions and constraints belong to the *school level*, while the *pedagogical level* refers to those only affecting the teaching and learning of ‘disciplines’. The way disciplines are grouped, valued, linked, diffused belongs to this level: the choice of an interdisciplinary way of studying questions or the way of presenting disciplines as independent. Very close to the previous levels, the *society and civilization levels* concern the way our society and civilization understand the rationale, functions, aims, etc. of school instruction.

The next two sections briefly introduce some of the institutional constraints encountered during an empirical investigation concerning to a local implementation of what it is called Study and Research Course (SRC) on population dynamics (see Barquero *et al*. 2008). As it is explained in this work, SRC are proposed as new didactical devices to teach mathematical modelling with a double purpose: to make students aware of the rationale of the mathematical contents they have to learn and to connect these contents through the study of open modelling questions. In more detail, our proposal for the teaching of modelling at university level (Barquero 2006 & Ibid.) consists in the implementation of a ‘mathematical modelling workshop’ that was run in parallel with the ‘usual lectures’ (dedicated to present the main contents of the course and exemplify them through carrying out some exercises on the blackboard). The workshop focused on the study of a population dynamics starting with the question of how to predict the evolution of the population given its size in some previous periods of time. To provide answers to this initial question and to the sequence of questions that followed it, the construction of different mathematical models was required. When studying the links between the questions and the answers provided by the models, new questions appeared that forced to broaden the previous models to more comprehensive, rich and complex ones, which made them continue with the process. At the end, this sequence of modelling activities covered most of the contents of a first-year course of mathematics for natural science students at university level.

Even though this local implementation was able to overcome some of these institutional constraints by setting up a set of suitable local conditions for the workshop[2], the large-scale implementation of such teaching proposals required the study of the real scope of these constraints in order to be able to introduce the appropriate changes at the appropriate level of didactic determination.

[2] For instance, the teacher of the workshop was a researcher in mathematics education and the teacher responsible of the course (the “lecturer”) was a mathematician who does his research in mathematical modelling and was participating in the educational research project. On the other hand, the implementation was developed in an annual course where a group of only 25 students were attending it. Moreover, its program was enough flexible to change of order the introduction of most of the mathematical tools that were required by the development of the workshop.

2.2. ‘Specific’ constraints at the lower levels of determination

If we consider the ‘low levels’ of determination (see figure 1) that are specific of the teaching and learning of mathematics, three main interconnected constraints have been made clear by the workshop experiment(ed). The first constraint is located at the *thematic level* and has been studied in other didactic investigations under the name of ‘thematic confinement’ (Chevallard 2002, Barbé *et al*. 2005). It (comes) stems from the fact that the prevailing culture in educational institutions tends to confine the teacher’s responsibilities at the strict level of the theme, without giving him/her the legitimacy to re-organise mathematical contents in a way different from the one imposed by tradition. In other words, teachers can (and have to) decide how to structure and sequence the themes, what subjects, problems and activities to include in each theme, how much time to spend on each one, etc. but they are rarely asked to decide on the choice of the themes or on the concrete division of mathematics into given domains and sectors. As has been shown by García *et al.* (2006), the problem of the disconnection of mathematical contents and the many efforts to solve it through modelling activities is related to this phenomenon of ‘thematic confinement’.

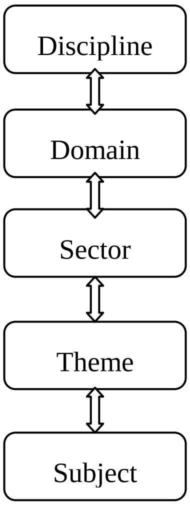

Figure 1

The second constraint is related to the concrete organisation of mathematical contents into *domains and sectors*. As we just said, the modelling activities of the workshop led the students to consider most of the contents of the mathematics course (calculus in one and several variables, basic linear algebra). However, during the workshop these contents appeared in a very different organisation from the one in the syllabus. If the lectures followed the classical ‘logic of mathematical concepts’, the workshop was more guided by the ‘logic of the extra-mathematical questions or types of models’ that progressively appeared. To be more specific, the whole modelling process was divided into three main ‘stages’ that correspond to the main lines of investigation followed during the workshop: the discrete evolution of populations with separate generations (discrete one-dimensional models: recurrent sequences); the discrete evolution of populations with mixed generations (discrete multi-dimensional models: transition matrices) and the continuous evolution of populations (differential equations). This forced the teachers to continuously work in a sort of ‘double curriculum’ project and it seems obvious that, in the long run, much more effort was needed to preserve the new organisation.

Finally, if we move to the *discipline level*, the running of the workshop showed the necessity of strongly modifying the traditional didactic contract that currently exists at universities. To carry out a modelling activity, it is necessary to break with the rigidity of the structure “theory lessons – problem lessons – exams” and to give the students some mathematical responsibilities that are usually assigned exclusively to the teacher: addressing new questions, creating hypotheses, searching and discussing different ways of looking for an answer, comparing experimental data and reality, choosing the relevant mathematical tools, criticizing the scope of the models constructed, writing and defending reports with partial or final answers, etc. Thus, the teacher had to assume a new role of acting like the director of the study process instead of lecturing the students, which highlighted that the teaching culture at university level does not offer a variety of teaching strategies for this purpose.

2.3. 'General' constraints at the upper levels of determination

When we move to the most generic levels (see figure 2), the *pedagogical constraints* appeared when it was necessary to find a suitable timetable for the workshop, with long sessions of two or three hours instead of the usual classes of 50 minutes, as well as some computers available in the class. Organising the students' work in teams, including the assessment of the teams' work and its inclusion in the individual evaluation of the course also appeared as difficult obstacles to overcome.

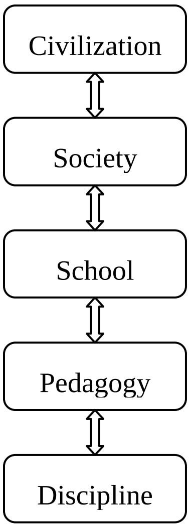

Figure 2

Considering the *society* and *school levels*, by now, we have only studied those related to the 'dominant epistemology', that is, the way our society, the university as an institution and, more concretely, the community of university teachers (and students) have to understand what mathematics is and what its relation is to natural science. Our first hypothesis is that the widespread understanding of mathematics and its relation to natural sciences is what we can call "applicationism". It may be depicted in the following way: a strict separation between mathematics and other disciplines (in particular natural sciences such as biology and geology) is established; when mathematical tools are built, they are 'applied' to solve problematic questions from other disciplines, but this application does not cause any relevant change neither for mathematics nor for the rest of disciplines where the questions to study appeared. For example, in the majority of the Spanish university courses we have examined, the study of population dynamics is a subject located in the sector of differential equations under the label of 'application', as if some dynamics laws could exist without any mathematical tool to describe it and, in the same way, as if differential equations could independently exist without any extra-mathematical problem to solve. One of the main characteristics of this dominant epistemology at university level is that it extraordinarily restricts the notion of mathematical modelling. Under its influence, modelling activity is understood and identified as a mere 'application' of previously constructed mathematical knowledge or, in the extreme, as a simple 'exemplification' of mathematical tools in some extra-mathematical contexts artificially built in advance to fit these tools. To be more concrete, the main characteristics of 'applicationism' can be described using the following indicators:

I_1*: *Mathematics is independent of other disciplines (‘epistemological purification’): mathematical tools are seen as independent of extra-mathematical systems and they are applied in the same way independently of the nature of the considered system.[3]

I_2: *Basic mathematical tools are common to all scientists*: all students can follow the same introductory course in mathematics, without considering any kind of specificity depending on their speciality.

***I_3*:** *The organisation of mathematics contents follows the logic of the models* instead of being built up from considering modelling problems that arise in the different disciplines. All happens as if there were a unique way of organising mathematical contents and different ways of applying them.

***I_4*:** *Applications always come after the basic mathematical training*: the result is then a proliferation of isolated questions that have their origin in the different systems. The first thing is to learn how to manipulate the mathematical concepts and later learn about their use. The models are built from concepts, properties and theorems of each theme independently of any extra-mathematical system.

I_5: *Extra-mathematical systems could be taught without any reference to mathematical models*, that is, there exists the belief that natural science can be taught without any mathematics.[4]

To empirically contrast to what degree ‘applicationism’ prevails in university institutions (see Barquero *et al.* in press), we used these indicators to analyse teaching materials (syllabi, textbooks’ prefaces and curricular documents) and to design an interview and a questionnaire addressed to geology and biology teachers and students of a science faculty in Catalonia. The study was developed during the years 2007 and 2008. The analysis of about 30 syllabi of mathematics for natural science courses of 10 different Spanish universities mainly confirmed I_2, I_3 and I_4. Some of the prefaces of the most recommended books for these courses helped to corroborate I_1 and I_5. A good example is the case of Salas & Hille (1995) (our translation):

> In this edition, you will find some easier applications to physics and, as extra chapters, some more difficult applications […]. Despite the incorporation of more applications, this book is still a mathematics book, not a science book or an engineering book. It is about calculus and its main basic ideas are limits, derivatives and integrals. The rest is secondary; the rest could be left out.

The interview with a sample of 8 geology and biology teachers and researchers and the answers of 30 other teachers to the questionnaire showed the following results: Related to I_1 and to I_3, up to 97% agreed that “Mathematics is introduced independently of geological or biological systems that could be modelled using

[3] This indicator is more general than the other ones as it refers to a characteristic of mathematics as a discipline and not to the way it is taught.

[4] This is an extreme indicator of the independence between mathematics and natural sciences (especially in the case of biology and geology) that is surprisingly widely shared to the point that, in most cases, people state that scientific systems could be studied without any mathematical tool.

mathematics" and that "the teaching of mathematics is more structured according to mathematical notions than to natural science problems". Related to I_4, up to 80% disagree that "mathematics is introduced only after its necessity has been shown and as a tool for the study of science problems". Finally, the most worrying fact (related to I_5) is that almost 40% agree that in natural sciences degrees, mathematics could only be used to analyse the quantitative aspects of science phenomena.

3. CONCLUSIONS

Using this "ecological" metaphor, we can say that for modelling to be able to normally 'live' in a teaching institution, it is necessary to study the conditions that facilitate and the constraints that hinder the type of mathematical activities that can be carried out in this institution. In this sense, the Anthropological Theory of the Didactic appears (as) a prioritary line of investigation to study these institutional constraints that affect the teaching and learning of mathematical modelling in current educational systems. From the ATD, the study of this "ecology" needs to take into account the different levels of didactic determination, not only to reach the variety of constraints acting on the classroom activities, but also to know better at what level – that is, in what intermediate institutions (from the 'mathematical lesson' to the 'Western civilization' in our case) it is necessary to act in order to improve the conditions that make the large-scale development of this activity possible.

In order to carry out this study, it appears necessary to provide a general model of mathematical activity that integrates mathematical modelling into the other dimensions of mathematical practices. Researchers in mathematics education have to emancipate from the dominant epistemologies that are implicitly imposed by educational institutions to which we belong. With this purpose, it is important to set out an alternative epistemological model, that is, an operative definition of what mathematics is and what the main characteristics are of the different mathematical activities that exist in our social institutions. As well as, the integration of a description of mathematical modelling within a general epistemological model of mathematics that takes into account the institutional environment of this activity.

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THE DOUBLE TRANSPOSITION IN MATHEMATISATION AT PRIMARY SCHOOL

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This paper proposes a theoretical framework to analyse the articulation between real world and mathematical world in mathematisation at primary school. This paper is not a report of studies presenting a methodology and results. First we describe this theoretical framework based on Chevallard's anthropological theory of the didactic and on the mathematisation cycle proposed by PISA. Then we illustrate this articulation between real world and mathematic world by using the theoretical framework on some examples, from class or from teachers training, issued from the European project LEMA. In this illustration the teaching of mathematisation is the double transposition of the real world knowledge and of the mathematical one. We conclude by questioning the mathematisation through the double transposition problematic.

THE DOUBLE TRANSPOSITION

Real world and mathematical world

We will differentiate the real word and the mathematical world. “If a task refers only to mathematical objects, symbols or structures, and makes no reference to matters outside the mathematical world, the context of the task is considered as intra-mathematical” (PISA 2006, p.81). A possible construction of this world is axiomatic, on a deductive way. Of course the genesis of parts of the mathematical world is in the real world as shown by history. The plausible reasoning could be a reasoning used as heuristic to find a proof or a mathematical solution, but is not a mathematical reasoning to define or to construct a mathematical object, or to prove on a mathematical way. Jaffe and Quinn (1993, p.10) have proposed to set a new branch of mathematics where plausible reasoning will be used: “Within a paper, standard nomenclature should prevail: in theoretical material, a word like “conjecture” should replace “theorem”; a word like “predict” should replace “show” or “construct”; and expressions such as “motivation” or “supporting argument” should replace “proof”. Ideally the title and abstract should contain a word like “theoretical”, “speculative”, or “conjectural” ”. After a debate in *Bulletin of the American Mathematical Society* this idea was rejected. On the contrary, in the real world the plausible reasoning could be used to define or to construct objects and to validate solutions of a problem. We “focuse on real-world problems, moving beyond the kinds of situations and problems typically encountered in school classrooms. In real- world settings, citizens regularly face situations when shopping, travelling, cooking, dealing with their personal finances, judging political issues, etc., in which the use of quantitative or spatial reasoning or other mathematical competencies would help clarify, formulate or solve a problem” (PISA 2006, p.72).

The double transposition

Using the terminology of Chevallard's anthropological theory of didactics, we consider that the real world is an institution producing the knowledge of real world. In this institution, real world problems have to be solved, using techniques, justifications and validations from the real world. Some of these validations can use argumentations that are not allowed in a mathematical demonstration: pragmatic argument (it is validated because the action is successful), argument of plausibility (as above-mentioned), argument from authority (majority of people, expert ...). The mathematical world is another institution producing a mathematical knowledge (called the scholarly mathematical knowledge). In this institution, mathematical problems have to be solved, using techniques, justification and validations from mathematical world. The mathematisation can be considered as an object to be taught in France (Cabassut 2009), in Germany but not in Spain (Garcia et al. 2007). The process of didactic transposition "acts on the necessary changes a body of knowledge and its uses have to receive in order to be able to be learnt at school" (Bosch et al. 2005, p.4). Here we consider the knowledge of the real world institution and of the scholarly mathematical institution. The mathematisation teaching is the place of a double didactic transposition, one from real world into the classroom and the other one from the mathematical world into the classroom.

MATHEMATISATION CYCLE

Before illustrating this double transposition in mathematisation process, we will present a framework to analyse it. We adopt the mathematisation cycle used in LEMA[1] project. This cycle is inspired by the study Pisa (2006), itself inspired by the works of Blum, Schupp, Niss and Neubrand. As illustrated in the joined figure, we consider five processes in which different competencies are developed:

- setting up the model, what includes "identifying the relevant mathematics with respect to a problem situated in reality, representing the problem in a different way, including organising it according to mathematical concepts and making appropriate assumptions, understanding the relationships between the language of the problem and the symbolic and formal language needed to understand it mathematically, finding regularities, relations and patterns, recognising aspects that are isomorphic with known problems, translating the problem into mathematics i.e. to a mathematical model" (PISA 2006, p.96),
- working accurately within the mathematic world, which includes "using and switching between different representations, using symbolic, formal and technical language and operations, refining and adjusting mathematical models, combining and integrating models, argumentation, generalisation" (PISA 2006, p.96),
- interpreting, validating and reflecting, which includes interpretation of mathematical results in a real solution in the real world, "understanding the extent and limits of mathematical concepts, reflecting on mathematical

arguments and explaining and justifying results […], critiquing the model and its limits" (PISA 2006, p.96),

- reporting the work: this process is more a transversal process which includes "expressing oneself, in a variety of ways, on matters with a mathematical content, in oral as well as in written form, and understanding others' written or oral statements about such matters" (PISA 2006, p.97).

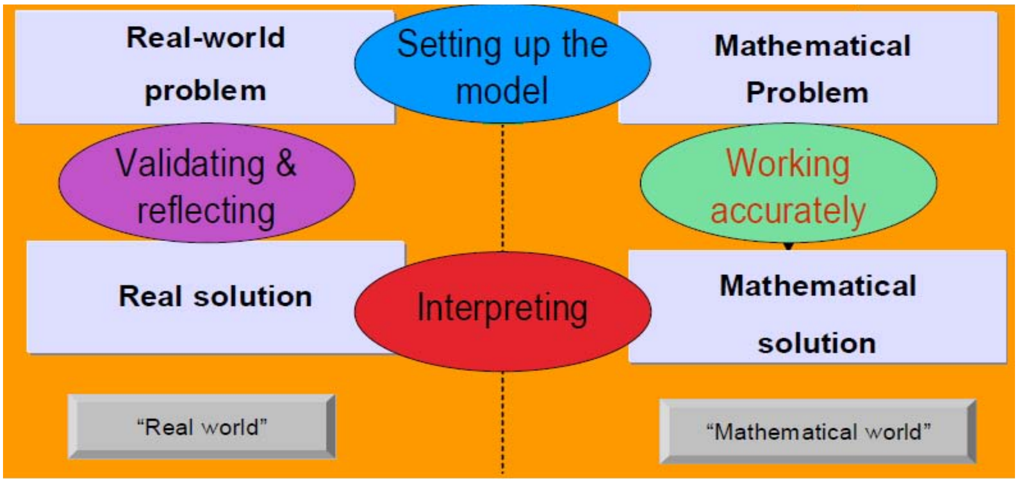

Figure 1: Mathematisation cycle used in LEMA

We illustrate now the double transposition in modelling in the different steps of the modelling cycle. These examples are extracted from the European project LEMA[1]. This project proposes a teacher training course on mathematisation. The information from these examples is from French pupils' observations made when implemented in class. There are also observations done with French primary school teachers or with trainers for primary school teachers.

In these examples we mainly point knowledge and techniques of real world involved in the modelling process. We don't emphasize on knowledge and techniques of mathematical world that are generally well taken in consideration in the related literature.

SETTING UP THE MODEL

Non-mathematical model

The following task was proposed to a French class CP (1st grade: 6-7 years): The class will read a story in a pre-primary school class. How to organize this reading?

In a first-time the pupils must build a mathematical model of the real problem. A possible model is, knowing pupils' number in the class and the number of pages in the book, how to share among pupils the number of pages of the book with the same number of pages per pupil. This model was already practised in class and was suggested by pupils during the discussion. However, in the discussion that takes place in the classroom, some pupils propose a "volunteer" sharing model where pupils read if they are volunteers (for example because they like reading): the distribution of the pages is done until there are no more. This model is not a mathematical model: the problem is solved on a pragmatic way. It is one reason why we have chosen the word “mathematisation” in place of “modelling”. With mathematisation, we clearly indicate that the chosen model has to be a mathematical one. For example (Maass 2006, p.115) suggests considering a real model before considering a mathematical model. **The teaching of modelling has to distinguish mathematical models and non mathematical ones.**

Non-mathematical arguments to choose a model

After discussion, guided by the teacher, it was decided to choose the model of equitable sharing of numbers of pages to read. The main reason of the choice is that this model is more equitable than the other: each pupil gets the same number of pages to read. The choice of this mathematical model is based on a non-mathematical argument (conception of equity: is it more equitable to force to read a pupil who doesn’t like reading than to choose volunteers?). It was not proposed other models, like the equitable sharing of the number of words to read that would have shown the relativity of the concept of equity: is it more equitable to share a number of pages or a number of words? In this phase of choice of some models, arguments of choice could be mathematical or not: taking into account preferences (those who like to read), taking in account equity.

It may happen that the choice of a model is made because of a lack of knowledge of models used in real life, what we illustrate with the following example given in teachers training (Adjiage, Cabassut 2008).

Figure 2 Berliner task[2]

Anne is on holiday in the Black Forest. It is a special offer for a type of pastry called "Berliner" as you can see from the picture. The baker offers the cake € 0.80 each. If you were the baker, would you have proposed the same price on the poster?		

In this situation it is surprising that it is cheaper to buy a single Berliner and three times a bag of 3 Berliners, rather than to buy a bag of 10 Berliners. It is frequent in real life that buying in large quantities is not always cheaper than in small quantities. It is therefore certain that the models of proportionality or decrease in the price with the increase in the quantity purchased are not valid to explain the Berliner prices.

Maybe other models based on the laws of marketing and psychology, justify a price as 1.99€ below the psychological threshold of 2€ or 6.99 € below the psychological threshold of 7€. The trainer didn't know about the models used in marketing or psychology and have chosen the known proportionality model by lack of knowledge of other models. It looks us important to provide to teachers and trainers tasks resources where models used in the real life are described and discussions on the choice of these models are offered in order that the choice of models are done by conscious arguments more than by lack of choice. **The teaching of modelling has to distinguish mathematical arguments and non mathematical ones to choose a model.**

Choice of the data and hypotheses based on non-mathematical arguments

To complete the construction of the model requires data specifying the number of pupils who read and the number of pages to read. All pupils agree on the number of pupils who read by choosing the number of pupils in the class at the present time. It may be noted that this number could change with the day of the reading in the pre-primary school class. But no pupil has considered this problem. Different assumptions about the number of pages to read are made: a group counts all pages (even those where there is nothing to read), others exclude the front page with the title of book, the ones with the single word "end", or having only illustrations. The justification of these different choices is not based on mathematical arguments. **The teaching of modelling has to distinguish mathematical arguments and non mathematical ones to choose data and hypotheses.**

Model to build and model to reproduce

In the process "setting up the model", it has to be differentiated the case where the model is already known by the pupil and the case where the model is new and has to be built by the pupil. In the previous example the pupils have already met equitable sharing problems that they have often solved by using the distribution technique (every pupil receives one after the other an object from the set of objects to distribute so long there is a rest of objects). We have observed that in this example, some pupils have proposed quickly the equitable sharing model. Let us propose an example where the model is new.

Figure 2 Giant task[3]

The task was proposed to a group of French CM1 (grade 5: 10-11 years old).
What is the approximate size of silhouette, which can see only a foot? This photo[2] was taken in an amusement park.

Here pupils have not met the model of proportionality and from this point of view this may be a problem to discover this model.
If the students have a model, they must choose from the stock of available models which accords better with reality. What characteristics of the models must students identify? (And in this case in the study of models which characteristics are putting forward?) What elements of reality must students identify? (And in this case what studies of the reality must be developed by the students?). A part of the heuristic strategies to set up the model comes from the mathematical world (the stock of available models). Of course the real world situation brings also heuristic strategies.
If the students have not an available model, they should build it and make assumptions. What assumptions do they do? How to train pupils to do the "right" assumptions? Here the main part of heuristic strategies seems to come from the real world situation. Of course pupils can use analogies with mathematical available models to set up a model for a real world problem, even if these models are not the right ones for this problem. We see that there are articulations between strategies issued from the real world knowledge and strategies issued from mathematical knowledge of available models. Nevertheless some of the strategies are not specific to mathematisation problems and are more generally developed in problem solving at primary school with or without real world context (Ministère 2005, 7-17).

The teaching of modelling has to organize **the transposition of the knowledge of the mathematical models to reproduce**. Here the traditional process of didactic transposition can be used as suggested in (Artaud 2006 p.374): “the first encounter, the exploratory moment, the technical moment, the technological-theoretical moment, the institutionalisation moment, and the evaluation moment”. For the model to build, if this model is a future model to reproduce, we are in the first encounter or the exploratory moment of the previous case. If not, we have to specify **what knowledge of the real world and of the mathematical world has to be transposed to build a model**.

WORKING ACCURATELY

Working accurately takes place in the mathematical world and produces mathematical solutions of the mathematical problem. So we could think that there is no articulation between real world and mathematical world during this process. Let us come back to the previous example of reading task. Once the equitable sharing model and its assumptions (number of pupils and number of pages) identified, each group of pupils works accurately to solve the problem. Different techniques of distributions are proposed (one by one, two by two ...). Different representations of the situation are worked. Some pupils use cubes representing the distribution to distribute effectively the cubes. Other ones use drawings to represent the set of pupils and the set of the pages and to draw a connection between the two sets. These two techniques show relations with real world: action in the pragmatic technique and visualisation in the drawing technique. **How the mathematical solution is validated?** Is it true because the action has a success (pragmatic validation) or because I see the solution on the representation (visual validation)? More generally we have shown in (Cabassut 2005) how **proofs in the mathematical world articulate mathematical arguments and extra-mathematical ones, especially by using pragmatic, visual, or inductive techniques.**

INTERPRETING

In the reading task, a mathematical solution has to be interpreted as a real world problem solution. The solutions represented by cubes or the drawings have to be re-interpreted in the real situation. This interpretation is fairly simple because the situation looks less abstract than in higher grades. More the mathematical model is abstract more the re-interpretation could present difficulties. (PISA 2006, p.97) points some competencies involved in the interpreting process: “decoding and encoding, translating, interpreting and distinguishing between different forms of representation of mathematical objects and situations; the interrelationships between the various representations; and choosing and switching between different forms of representation, according to situation and purpose […] decoding and interpreting symbolic and formal language, and understanding its relationship to natural language; translating from natural language to symbolic/formal language; handling statements and expressions containing symbols and formulae; and using variables, solving equations and undertaking calculations”. **The use of semiotic representations, and specially the natural language, illustrates the articulation between real world and mathematical world.**

VALIDATING AND REFLECTING

Experimental control

In the case of the reading task, different solutions of the real problem are proposed related to the fact that different assumptions are made to take in account the rest of pages insufficient to distribute one page at every pupils. The common data are 49 pages to share between 17 pupils. In one group, fifteen students each receive three pages and two students each receive two pages. In another group, they add two more pages, the title front page and the last page with the words “the end”, and they distribute three pages to every pupil. Both solutions were validated in the class. In both cases it is possible to control the validity of the solution by playing the distribution in the class and by checking the results of the play.

No possible experimental control

For the giant task, it is not possible to check the giant’s height. There is no complete photo shoving the complete giant and it is not possible to visit the amusement park situated abroad. The validation is made on a consensus criterion. As nobody opposes a critic and no contradiction is discovered, the solution is considered as valid. This way to valid is not specific to mathematisation. Lakatos (1976) has shown the same phenomenon in the mathematical proofs. The validation will be partially based on non-contradiction. But the fact that nobody has discovered a contradiction doesn't mean there is no contradiction, as shown in the history of mathematical proofs. For giant task which is not a familiar situation, the validity is based on the lack of contradictions, which is not a mathematical deductive criterion but a plausibility criterion.

Assumptions and validity of the model

In the case of the giant task, a group of pupils has produced the following data. On the photo the pupils measure 1 cm for a man's foot and 7 cm for his height, what gives a ratio of 7 between both measures. The groups of pupils made the additional assumptions: in the reality an adult's foot is about 30cm and a adult's height is about 180 cm, what gives a ratio of 6 between both measures. With these data it is difficult to use a proportionality model to solve the problem. Here the difficulty is that, as the problem is opened, the pupils have to make additional assumptions to solve it. And it can occur that these additional assumptions are not compatible with a wished model.

In the same task, we can observe solutions proposed by two different groups. In the first solution, pupils measure 9cm for the giant's foot and 1 cm for the man's foot. It means that on the photo the giant's foot is 9 times bigger than the man's foot. They assume that in the reality the ratio is kept. They additionally assume that in the reality the man's foot is about 30cm. Therefore in reality the giant's foot o is 9 times greater what gives 9x30cm= 270 cm. But on the photo, the man's foot measure 1cm and his height 7 cm, which means that the man is 7 times taller than his foot, on the photo and by extension in the reality. They additionally assume that the giant has the same ration on the photo and in the reality. They conclude that the giant's height is 7x270 cm = 1890 cm.

In the second solution, the man's foot measures 1cm and the giant's foot 9 cm; therefore the foot of the giant is 9 times greater than the foot of man. It is assumed that there is the same ratio between the heights. As a man is about 180 cm, the giant's height is about 9 times taller than a man's height. They conclude that the giant's height is 9x180cm = 1620 cm. Both solutions are validated even if they lead to different results because of different assumptions.

It is clear that this validation is similar to that of a conditional statement in the mathematical world: under this condition the conclusion is true, provided that the used reasoning is valid and that the applied theorems are true. In the real world, the role of theorems is played by assumptions like "the ratio on a photo is the same than the corresponding one in the reality" or "the ratio between size of the foot and height is approximately constant". Often such assumptions are valid in approximation or in very accurate conditions. They need a social knowledge of the real world. The teachers have to take in account if pupils have this social knowledge.

We have seen in the above examples that the validating and interpreting step can involve **arguments and techniques of mathematical world** (like hypothetical-deductive reasoning) and of **extra-mathematical world** (like experimental control).

AUTHOR’S POSITION AND IMPLICATION FOR RESEARCH

In the previous examples, we have illustrated in the whole mathematisation cycle that mathematical knowledge and techniques and extra-mathematical ones have to be transposed and interfere. Blum (2002) observes: “In spite of a variety of existing materials, textbooks, etc., and of many arguments for the inclusion of modelling in mathematics education, why is it that the actual role of applications and mathematical modelling in everyday teaching practice is still rather marginal, for all levels of education? How can this trend be reversed to ensure that applications and mathematical modelling is integrated and preserved at all levels of mathematics education?”

We have observed that a lot of resources don’t take in account the double transposition problematic. We propose that teachers training and didactical research give more attention to the double transposition problematic in the mathematisation and try to answer the following questions. In a mathematisation task, what knowledge of real world and of mathematical world has to be transposed? What techniques, justifications and validations from both worlds have to be used? How different knowledge, techniques, justifications and validations are articulated and interfere between the two worlds? What effects on teachers’practice, on pupils’ learning and on class didactical contract have these articulations and interferences?

NOTES

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EXPLORING THE USE OF THEORETICAL FRAMEWORKS FOR MODELLING-ORIENTED INSTRUCTIONAL DESIGN

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Designing modelling processes adapted to school restriction and able to produce a wide, rich and meaningful mathematical activity is far from been unproblematic. That situation seems to be even more problematic if the focus is on Early-Childhood Education. In this paper we explore the possibilities that existing theoretical frameworks can bring us. First, some theoretical consideration about modelling, the lack of sense of school mathematics and the use of theories for instructional design are outlined. Second, the design of a study process under the control of the Anthropological Theory of Didactics is described. Finally, the real implementation of this study process with 4-5 years old pupils is reported, showing how very young pupils can be involved in a wide, rich and meaningful mathematical activity.

INTRODUCTION

Modelling is occupying a central position in the current educational debate, from policymakers and curriculum developers to researchers and teachers. Focusing on research, important efforts in many directions can be observed: students' modelling competence, instructional design, modelling pedagogy, teacher training and support, students' and teachers' beliefs, among others.

On the other hand, research in mathematics education is developing more and more sophisticated theoretical frameworks which aim to understand the complex relations existing in every teaching and learning process. In a simplified way, every theoretical framework can be considered as a model of some *teaching and learning reality*. Structuring and simplifying processes are therefore necessary: every theory focuses on some objects and relations whilst other objects and relations are pushed into the background.

There is an ample consensus about conceptualizing modelling as a cyclic process where a dialectic between an extra-mathematical world and a mathematical one is established, as described by Blum, Niss and Galbraith (2007). Many different versions of the well-known modelling cycle have been developed, depending on researchers' interests and backgrounds. Conceived as different models of the modelling processes, each version tries to capture some features of these processes and/or the modelling-based teaching and learning processes.

However, it seems that there is a gap between research in modelling and applications, on the one hand, and research in mathematics education, on the other hand. That leads us to explore how existing theoretical frameworks not explicitly developed from a modelling perspective could be used to enhance research in the so-called modelling and applications domain. Particularly, we will focus on modelling-oriented instructional design through the Anthropological Theory of Didactics. Moreover, in this paper we will focus on early-childhood education levels, which are normally neglected in the existing research in modelling and applications.

MODELLING-ORIENTED INSTRUCTIONAL DESIGN

The process of designing modelling-oriented teaching sequences optimized to be used at school is far from being unproblematic. In our work with in-service teachers in a European training course in modelling and applications (LEMA project) one of their main concern was how to design interesting and authentic tasks, adapted to their school constraints and students' level and useful to develop the intended mathematical curriculum [1]. Although some teachers can show a great creativity to find real situations and problems and they feel able to adapt them for educational purposes, most of the teachers feel that it is a big, difficult and time-consuming work. Normally, anecdotic and isolated tasks are developed which address to some mathematical topics but these tasks fail in their intention of giving rise to a rich and wide modelling-based mathematical activity.

In the core of this situation a problem of didactic transposition (Chevallard, 1991) can be identified. Normally, real situations do not come themselves with interesting and crucial problems able to develop the desired wide and rich activity. We agree with Lehrer and Schauble (2007, p. 153) in that "models cannot simply be imported into classrooms. Instead, pedagogy must be designed so that students can come to understand natural systems by inventing and revising models of these systems".

Our approach in this paper is that of *applications and modelling for the learning of mathematics*, as Blum, Niss and Galbrait (2007) state and, particularly, the use of a modelling approach to help students to provide meaning and interpretation to mathematical entities and activities (also called *educational modelling* by Kaiser et al., 2007). That agrees with the current Spanish curriculum which reacts again the traditional lack of sense of school mathematics and asks for a meaningful mathematical activity where mathematical topics from different mathematical domains are connected and integrated.

ATD AS A FRAMEWORK FOR INSTRUCTIONAL DESIGN

In the last years, a group of Spanish and French researchers have been exploring and developing the Anthropological Theory of Didactics (ATD from now on) as a reference framework for instructional design. The notion of *Study and Research Course* (Chevallard, 2006) as well as the basic assumptions of mathematics as a human activity linked this effort with modelling and gave rise to a new research agenda.

In brief, mathematics is conceived in the ATD as a human and social construction. Over centuries, mathematics praxeologies have been developed, refined, optimized, rejected, combined, etc. as new problems arose in many different domains: from daily life to natural and social sciences and, obviously, from intra-mathematical problems. In our modern societies, School has the responsibility of the diffusion of a part of this cultural heritage to young people so that they will be able to live and act as responsible and democratic citizens. A common and traditional way of doing that is *showing* to the students already finished mathematic praxeologies, as artefacts they can *visit* and they should preserve. Chevallard (2006) call this the *monumentalistic* approach which directly relates with the lack of sense of mathematics at school.

Opposite to that approach and looking for students' sense-making, Chevallard (2006) advocates for a renewed school epistemology where interesting and crucial problems and questions are in the core giving rise to a meaningful mathematical activity.

The ATD has developed the notion of *Study and Research Course* (SRC) as a model to analyse and design school teaching and learning practices. What are the main characteristics of a SRC? In short: (a) a SRC should start from a *generative* and *crucial* question Q_0 [2]; (b) the *community of study* has to take the study of Q_0 seriously (Q_0, and the situations where Q_0 is inserted, is not the excuse teacher uses to introduce some mathematics); (c) the study of Q_0 will give rise to answers (that is, praxeologies) but also to new questions Q_i, making the study process an open process and, to some extent, undetermined in advance; (d) as far as Q_0 or some Q_i can be extra-mathematical, not only "pure" mathematical answers and questions are expected through the study process but also *mixed mathematical praxeologies* (Artaud, 2007); (e) a SRC gives rise to a collaborative and shared study process, looking for good answers and for good questions (sometimes new answers are developed, sometimes already existing answers are found, depending on the *media* available in the community of study). Finally, it is expected that the community of study develops their own personal answer $A^{\heartsuit}$.

As far as Q_0 and some Q_i emerging from it are of extra-mathematical nature, the subsequent SRC can be considered as a wide modelling process. Indeed, as we reported elsewhere (García, Bosch, Gascón and Ruiz-Higueras, 2006), modelling can be reconceptualised as the progressive construction of a set of praxeologies of increasing complexity. The SRC is therefore a didactic device useful to develop and design wide, rich and meaningful modelling processes with educational purposes.

As far as mathematics will emerge through the process as needed answers for taken as seriously problems instead of an already existing construction, *living* elsewhere and brought to school ignoring the why, the lack of sense of school mathematics will be avoided. Therefore, the SRC is a didactic device useful to make modelling a reality at school fighting against the *monumentalistic disease*.

DESIGNING A STUDY AND RESEARCH COURSE FOR EARLY-CHILHOOD EDUCATION: COLLECTING SILKWORMS

Institutional, pedagogical, curricular and epistemological background

In Spain, the early-childhood education is a non-compulsory educational level for 3 to 6 years old children (3 grades) although almost every child in this age attends to the school. It is not conceived as a kindergarten but as an educational level ruled by a national curriculum. Three are the main domains in this level: *self-knowledge and personal autonomy*, *knowledge of the environment* and *languages: communication* *and representation*. Among the general aims of this level, three of them are of special relevance for our work: *(b) to observe and explore children's familiar, natural and social environment, (f) to develop communicative skills in different languages and forms of expression* and *(g) initiation into logical-mathematical skills* (MEC, 2007)

School activity has to be organized in a holistic and integrated way. Children's reality and near environment should be the starting point for every teaching and learning situation. Therefore, modelling could be present on every teaching and learning situation although it is not explicitly described in the national curriculum.

During this stage, pupils are supposed to develop quantification skills and the cardinal sense of numbers (measure of a discrete set) as well as languages and forms of expression to communicate about quantities. Pupils will develop numbers' cardinal sense through their activity in many situations where the measure of one or several discrete sets is necessary. Numbers (both the meaning and the signs) will emerge as models to deal with this quantification [3]. Validation and interpretation processes as well as communicative needs are crucial to make pupils' knowledge evolve from self-invented representation of quantities to numerals and numbers.

As Ruiz-Higueras (2005) describes, following basic works in Didactics of Mathematics developed by Brousseau and cols. in the University of Bordeaux, the question that should guide early-childhood reconstruction of numbers should be*: why do we need numbers and their representation?* Three would be, at least, the functions of numbers in this level: to measure a discrete set (from the set to the number), to produce a set (from the number to the set) and to order a set (to assign and locate the position of an element in an ordered set). Centred in the first and second function, school situations where numbers emerge as models to express the measure of a set, to verify the conservation of a set, to manage a set, to remember the quantity, to reproduce or produce a set of an already known quantity and to compare two or more sets has to be designed and implemented.

If the design process of teaching and learning situations takes care about the reality and authenticity of the situations considered, then modelling is an optimal pedagogical approach for teachers to develop teaching and learning situations concerning numbers and their representation in early-childhood education.

Design of the Study and Research Course

The Study and Research Course (SRC from now on) reported here has its origins in a real school situation lived by a teacher [4]. She was working with her 4 years old students about butterflies and she thought about introducing silkworms and the transformation process into *butterflies* (metamorphosis). It was spring and pupils are used to collect silkworms and to feed them with white mulberry leaves. So, it was easy to bring a box with silkworms into the classroom and observe its evolution. The teacher, in order to deal with de holistic and integrated principle, decided to make some mathematical work with this situation. She is used to work with a-didactic situations (in Brousseau's sense) and their students are used to face problems, to develop different solutions and representations, compare them, formulate messages in mathematical codes (including self-invented codes), validate the solutions against the *milieu*, discuss about the problem and the different solutions, etc. Students are developing their knowledge of cardinal numbers during this school year and they have been working in many situations where they have to produce a number that measure a discrete set, to build a collection equal to a given number, to compare different collections, to express orally or in a written form how many elements are needed to complete or to reproduce a given collection (both with the collection in front of them and with the collection hidden). However, they do not always use the number as the best way to answer *how many* questions and, depending on the student, they can count up to 20 (or more) but many of this numbers are meaningless.

At the beginning, only an anecdotic an isolated activity (*if we've got N silkworms, how many leaves do we need to feed them?*) seemed to appear. But, as soon as we start working with the teacher and taking the situation seriously, a rich variety of praxeologies emerged.

Compared with other situations used by the teacher, two are the main characteristic of this one. On the one hand, it is a real and authentic situation (silkworms are in the classroom and have to be fed). On the other hand, it is a dynamic system: silkworms will turn into cocoons and, finally, moths (butterfly for pupils) will emerge and die. That means that there are, at least, three different collections to be controlled over the time. In terms of dynamical systems, each state of the system can be described with the vector $(t, n(t), c(t), m(t))$ where t is time, n is the number of silkworms, c is the number of cocoons and m is the number of *moths*. A conservative law rules the system: for every t, $n(t)+c(t)+m(t)=N$, where N is the original number of silkworms.

Working in this kind of systems is quite challenging for 4-5 years old pupils. Techniques to deal with time have to be developed and ways of organizing data are needed in order to record changes. That means that during the study process at school, not only praxeologies around cardinal numbers will emerge but also praxeologies concerning time measurement and data handling. Along the whole study process, silkworms will not only be the excuse teacher uses to introduce some mathematical work, but the centre of the process. Interpretation and validation will be dense during the study process.

IMPLEMENTING THE SILKWORM SRC AT SCHOOL

We will report in this section about the real implementation of the silkworm SRC in two different classrooms (4 and 5 years old students). Data have been taken from a self-report written by the teacher as she was developing the SRC and she was managing the study process at school. The study process took place in spring 2008. There is no space here to explain the study process in detail (both students' work and teachers' decisions). So, we will try to focus on the main issues of this process.

The study process started when the teacher was talking about butterflies in classroom and decided to link that topic with worms and metamorphosis process. She thought that bringing some silkworms into the classroom (figure 1) could be very motivating for their students. No mathematical work was planned in advance but she quickly noticed that a rich mathematical activity could be developed from this situation.

Fig. 1. Silkworms at school.

The first problem arose earlier: silkworms have to be fed, how silkworms' feeding should be organised? Some restrictions into the system had to be introduced: first, a leave for each silkworm each day; second, new leaves are needed each day and third, taking the leaves from the mulberry tree it is not possible (it is dangerous!) but the gardener will do it for us if we ask him. That gave rise to a quantification activity (praxeology around numbers as cardinals): the first and second restrictions were introduced in order to give rise to techniques dealing with cardinals and the comparison among different collections. These are really problematic situations for 4 and 5 years old students and numbers and numerals will emerge as the best models to deal with them (although some intermediate models, for instance, iconic representations, are also used). Although many pupils can recite the number names in sequence and they know numerals up to 9 or even more, many of them are not able to use them in context to measure a collection, to produce a new collection or to compare two or more given collections. For instance, the following dialog was recorded by the teacher:

Student: Teacher, we've got twenty-five silkworms minus two.

Teacher: Why? Can you explain it?

Student: Yes, there are twenty-five pupils in the class and, each day, two of us don't have a silkworm.

Teacher: Then, how many silkworms are there?

Student: Twenty-three.

Teacher: How do you know that?

Student: I don't know.

The silkworm activity offers a rich real situation to develop quantification skills. Moreover, as they have to write a message to the gardener with the leaves needed each day, communicative skills will be also developed.

Time is not a relevant variable for pupils yet, although the teacher asks pupils to write the date on the ordering-sheet. In the 5 years old classroom, students are quite engaged in silkworms care. The class was divided into groups which have to take care of the silkworms each day. A list of things to be done and a diary was made (figure 2): *1st, counting the silkworms; 2nd, cleaning the house; 3rd, bringing as many leaves as silkworms; 4th, filling in the diary; 5th, writing down the numbers; 6th, writing a letter to the teacher* (asking for new leaves). The teacher introduces also a table where pupils record the date, the name of the caring group, the number of silkworms and the number of leaves.

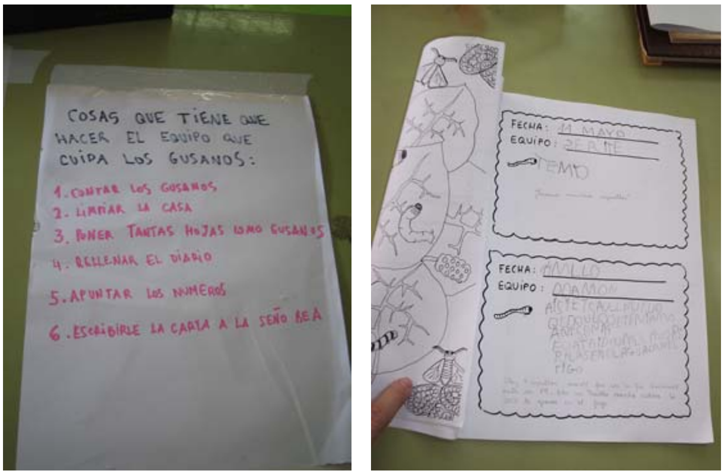

Fig 2. Things to be done and diary

Days were going by until the day cocoons appeared. That caused the first evolution of the mathematical activity (and, obviously, pupils' happiness). On the one hand, pupils decided to put the silkworms apart in other box because cocoons could be damaged when they had to clean the box and fed the silkworms. That caused the division of original collection in more than one collection and additive strategies to control the whole collection were needed. On the other hand, time arose as an important issue: they needed to control time in order to measure how many time will pass until the moth emerges from its cocoon. The static system has changed into a dynamical system. Pupils' quickly asked for time control:

Student 1: When will butterflies emerge?

Student 2: Well, tomorrow.

Student 1: No, they will take more days.

Student 2: Yes? How many?

Student 1: Now, here (the student points at a day in the calendar).

Student 2: Well, we can count the days (in the calendar) and when the butterfly emerges we will know how many days are.

Fig. 3. Table to control system's evolution (1)

Teacher knows that time control can be excessively demanding for 4-5 years old students. She needs to introduce some *tools* in classroom in order to let students control quantity and time together. A table (figure 3) is introduced by the teacher (date, group name, number of silkworms, number of new cocoons, number of leaves and total amount of cocoons). It will emerge as a tabular model of system's variation and records its evolution.

From a mathematical point of view, the original praxeology about quantity is evolving and widening including time measurement and strategies to handle with data (obviously, adapted to 4-5 years old students). From now on, students activity can be characterized as a dialectic between the system divided into different sub-collections (different boxes with silkworms and cocoons) and the tabular model (where system evolution is been recorded).

The day the first moth emerged provoked the necessity of calculating the time passed since the cocoon appeared. Again, this is a problematic task for 4-5 years old pupils. At this level it is usual to introduce some techniques to measure time working over calendars. Pupils are used to work with them and they can get some control on time passed or needed just counting on the calendar.

Student: It's been three days.

Student: No, I said ten days.

Student: Days have gone and they can't be counted.

Teacher: Yes, we can. Let's see, how can we know the days from my birthday?

Student: Well, we look for it in the calendar. We say one, two, three,… (some pupils went to the calendar in the wall and counted, pointing with the finger, since teacher's birthday).

Teacher: Now it is the same. Let's see, Antonio as responsible of the day, tell us how many days. First, you have to look for the day the cocoon appeared.

Student: It has to be one of the first cocoons because it is in the brown box.

Teacher: Ok. Antonio, look for the day in the calendar and count…

The result was twelve days. The next days the same activity gave rise to different results. That was interpreted as there were not a fixed number of days for the moth to emerge but a range. Pupils' interest decayed us they knew the days but they were interested in moths' care.

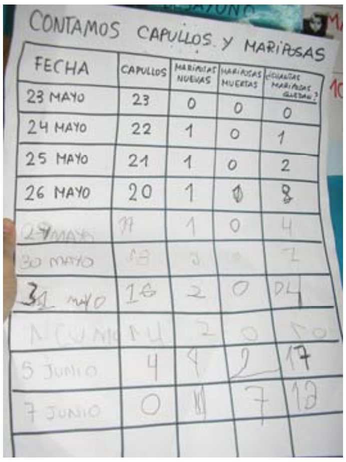

Fig. 4. Table to control system's evolution (2)

As they knew that moths will die very soon, the teacher decided to repeat again the time-quantification activity with the collections: cocoons, new moths, death moths and moths alive (figure 4).

When all the moths died, the system was over and the activity around them finished. However, the class had lots of information about the system and its evolution. The models constructed during the study process recorded this evolution. An interpretation activity was introduced by the teacher in order to make these tabular models useful to recover information about a system that had passed by.

The teacher proposed the pupils to make a poster representing the collection in different stages: silkworms eating leaves, cocoons and silkworms and cocoons and moths (figure 5).

Students had to interpret data on the table and produce the corresponding collections. From a modelling point of view, the fact that the system will never be back again in the classroom makes this interpretation activity completely crucial to summarize what happened and to talk about the system to somebody else. From an educational point of view, this is one of the most important moment of the study process: models can, to some extent, relieve the system and produce information about it even if the system will never be back again. The learning of time-quantity relations is one of the main aims at pre-school. During this final activity, students need to control time and quantity at the same time and the interpretation of the model is the key for that control.

Fig. 5. Reconstructing the system from the model

CONCLUSIONS

Designing modelling processes adapted to school restrictions, able to produce a wide, rich and meaningful mathematical activity is far from been unproblematic. In this paper we argue for the necessity of sophisticated theoretical frameworks for modelling-oriented instructional design. Moreover, for very young students, there is a lack of research concerning modelling-based teaching and learning.

We have described a process of study designed under the Anthropological Theory of Didactics and carried out by 4-5 years old pupils. First of all, the example shows how the theoretical framework allows us to control the design process and its real implementation. Secondly, the study process reported here shows how very young pupils can be involved in a wide, rich and meaningful modelling activity where different praxeologies of increasing complexity emerge as the system is evolving over time. Pupils use, learn and widen their mathematical knowledge as they want to take care of the silkworm collection and to know more about silkworms' transformation: quantification skills, additive and subtractive strategies, time-quantity relations and data handling procedures are brought into play. Finally, once the system has disappeared, models previously developed emerge as tools to reconstruct the system in every stage and to recover time and quantity information. Very young pupils are engaged in a modelling activity, producing and using models, a long time before they are able to really understand what modelling is and the role modelling plays in daily-life, society and science.

NOTES

[1] In Spain, the national and autonomic curriculums are not modelling-oriented. Although many teachers and textbooks are interested in applications and modelling, their main concern is to develop the mathematical topics listed in the curriculum.

[2] When a question can be considered as "generative" and "crucial"? It depends mainly on the institution where the study process will take place, the educational system and, finally, the society. School level and curricular constrains, the way the educational system is organized and the main aims of school within society need to be considered.

[3] Number's ordinal sense will not be considered in this paper.

[4] We will like to thank to the teacher, Mrs. Blanca Aguilar (from "El Olivo" school in Torredonjimeno, Jaén, Spain), her effort and enthusiasm which made the experimentation possible.

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STUDY OF A PRACTICAL ACTIVITY: ENGINEERING PROJECTS AND THEIR TRAINING CONTEXT

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This paper deals with the question about place that should be given to mathematics in engineering training. In particular, we analyze a practical activity: engineering projects. This activity intends to reproduce the working context in industrial engineering. Our research is developed in the frame of the Anthropological Theory of didactics (Chevallard 1999). We use the Expanded Model of Technology (Castela, 2008) to analyze the engineering project. In this paper, we present the analysis of a task of modelling developed in the projects context

Background

What place should be given to mathematics in the training of engineers? Which contents should be approached in this training? How should it be articulated with other domains of the training?

These question have already been asked and treated in different institutions and different times. For example, Belhoste et al. (1994) who studied the formation given by the French *Ecole Polytechnique* between 1794-1994, have shown that different models of training have arisen during XIX century: Monge's model, Laplace's model and Le Verrier's model. These questions which underlie the establishment of training's models and the changes of model, from Monge to Laplace then to Le Verrier, are the fundamental questions of relation between science and application, relation between science and technology.
Nowadays, these questions are modified by the technological development, technology taking an increasing place in the engineers' work:

> Before the advent of computers, the working life of an engineer (especially in the early part of his or her career) would be dominated by actually doing structural calculations using pen-and-paper, and a large part of the civil engineering degree was therefore dedicated to giving students an understanding and fluency in a variety of calculational techniques. For the majority of engineers today, all such calculations will be done in practice using computer software. (Kent, 2005)

In other words, the development of powerful software changes the mathematical needs because this software encapsulates some of the usually taught mathematics. Mathematics may even appear to be useless to some engineers.

During last years, various researches concerning the nature and the role of the mathematical knowledge in the workplace have been realized (Noss et al., 2000; Kent & Noss, 2002; Magajna & Monagan, 2003; Kent et al. 2004). These works point out the existence of gaps between the educational programs and the real world in which the engineers work. For example, the institutional speech asserts that undergraduate engineers need a solid mathematical education, but the researches show that for graduate engineers mathematics is of little use in their professional work.

> Once you've left university you don't use the maths you learnt there, 'squared' or 'cubed' is the most complex thing you do. For the vast majority of the engineers in this firm, an awful lot of the mathematics they were taught, I won't say learnt, doesn't surface again. (Kent and Noss, 2002)

In our research we intend to contribute to the analysis of the observed gaps and to investigate the role that educational practices and technology play in these gaps. We especially study how one innovative practice in a French engineering Institute intends to articulate theoretical and practical knowledge.

Theoretical Framework: The Anthropological Theory of Didactics ATD (Chevallard, 1999)

The general epistemological model provided by the ATD proposes a description of mathematical knowledge in terms of mathematical praxeology $[T/\tau/\theta/\Theta]$

The praxeology has four components: the first type of tasks T or problems T, the technique is a way to solve the problems, the technology is a theoretical discourse to describe, explain and justify the techniques and the theory is also a theoretical discourse to describe, explain and justify the technologies. The praxeology has two blocks:

Practical block or "know-how" (the praxis) $[T,\tau]$ integrating types of problems and techniques used to solve them

Theoretical block or "knowledge" (the logos) $[\theta,\Theta]$ integrating both the technological and the theoretical discourse used to describe, explain and justify the practical block. (Bosch, Chevallard & Gascón, 2002)

As part of ATD, study is seen as construction or reconstruction of the elements of a mathematical praxeology, with the aim to fulfil a problematic task. To represent finely these processes of construction or reconstruction, ATD offers a model of the study of mathematical praxeology. This model so-called: Moments of the study distinguishes six moments or phases. In this paper we only consider the moment of institutionalization: this moment has the object to specify what is "exactly" the worked out mathematical praxeology. It appears de facto that there are not kept in general in the technology "purified" the elements which are not justified or produced by a theory of empirical knowledge they are rather related to the concrete conditions than the usage of techniques.

Castela (2008) proposes that in the technology of praxeology there are two components: theoretical θ^{th} and practical θ^{p}.

> "…the technology of technique is the knowledge orientated to the production of an efficient practice, which has functions to justify and legitimize the technique but also *to equip* and *to make easier* the implementation with it. Beside possible elements of knowledge borrowed from certain appropriate theories (we shall speak following "*the theoretical component"* of technology, noted θ^{th}) this knowledge appears in technology which, according to research domains, is qualified as operative, pragmatic, practical. Collective work was forged in experience; this *practical component* plays technology (noted afterwards θ^{p}) express and capitalize the science of the community of the practitioners confronted in the same material and institutional conditions with the tasks of type *T*, it favours the diffusion within the group." (Castela, 2008, p.143)

There are six functions associated with the practical component of *praxéologie* θ^{p}*:*

1. **To describe** the technique. The verbal description of the series of steps that make up a technique is an important step in the process of institutionalization.

2. **To motivate** the technique and the different gestures which compose it.

To explain why, in which aims. It describes the aims expected by the technique and analyzes the effects, consequences, different gestures and the difficulties that its absence could provoke.

3. **To promote** the technique's utilization. It considers that knowledge allows users to use the technique with effectiveness but also with a certain comfort.

4. **To validate** the technique: it works, it does what is said. It is main goal is to guarantee the technique, when this is used completely it produces a valid solution and the elements were it belongs achieve the expected aims.

5. **To explain** why does technique work? Is about being interested in the causes of effectiveness. Contrary to the second function, the objective is to detail the mechanisms that make that the technique and its components have the desired effect.

6. **To evaluate** the limits, conditions of effectiveness of the technique. The function of validation is positioned on the side of the truth and justified by a theory. In a practical context this function will consider the efficacy.

The institutionalization within different institutions

There are different institutions which maintain a report with a given praxeology. We shall differentiate the institutions with a function of production *P(K)* of knowledge. And the user *UI* institutions of this praxeology, in sense where subjects of *UI* have to accomplish tasks of type *T*. The aim of *P(K)* institutions is to produce and validate the different components of praxeology. But, we asserts that to a praxeology used in a user Institution; this is a part of technology isn't justify for a theory. The technological knowledge validated by an institution *P(K)* do not exhaust technology, which includes in general a component θ^p for which it is also necessary to examine social modes of validation. The question is therefore to reflect upon construction practises as part of UI, tested through the multiplicity of effective achievements and institutionalization (understood as stabilization rather than explicit recognition by a given institution) of know-how and knowledge.

The **Expanded Model of Praxeology** (Castela, 2008) can be simplified in the following way:

$$\left[T,\tau,\begin{matrix}\theta^{th},\Theta\\ \theta^{p}\end{matrix}\right]\begin{matrix}\leftarrow P(K)\\ \leftarrow\ UI\end{matrix}$$

Arrows represent social practices of validation in work in the one or other one of the institutions *P(K)* and *IU* carrying respectively on the block [θ^{th}, Θ] and on θ^p.

Dynamics of mathematical praxeologies

In our work, we focused on mathematical praxeology present in the engineering projects. To account for the way followed by a praxeology from mathematical origin which has to reach the project, we consider different institutions:

<u>Production Institutions</u>

- P(M) Production institution of mathematics
- P(ID) Production of intermediate disciplines

<u>Institutions inside at Vocational Istitute at the University (IUP) (1)</u>

- T(M) Training of mathematics
- T(ID) Training of intermediary disciplines
- Ep Engineering projects

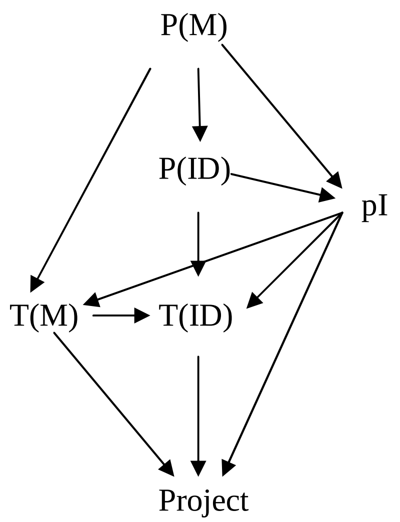
The mathematical praxeologies from production institutions progress to the projects in different ways:

1. **P(M)→T(M)→Ep**
   The first one is from production mathematics to training mathematics until the projects.

2. **P(M)→T(M)→T(ID)→Ep**
   The second one is from production mathematics to training mathematics through training intermediary disciplines and projects.

3. **P(M)→T(ID)→T(ID)→Ep**
   The last one is still production mathematics to intermediary disciplines through training intermediary disciplines until projects.

In our context a vocational training, we shall consider also the profession (professional institution pI). The praxeologies presents in the latter institution are also transposed. These have a specific component θ^p, to promote the use in the professional contexts. We shall take into account the influence from profession to training of mathematics T(M), training of intermediary disciplines T(ID) and Engineering projects Ep. The following schema exhibits the links between the previous components:



Context and methodology of research

In order to realize our study, we have chosen the Vocational Institute at University of Evry (IUP). This Institute uses an educational model of practical education closely related to the industrial world: the university training is combined with training in firm; professional practice takes place during twenty weeks (minimum) over the three years of training. But, the mathematical training is solid, it remains classical at university

The question is: How is the IUP model, which is characterized by a strong nearness with professional middle, inserted in a mathematical training which seems to be designed by this classical model? To answer to this question, our study is focused on an innovative practice, the so-called Projects. These projects intend to connect the official universe of educational disciplines and the professional world of engineers.

The aim of this study is devoted to identify the mathematical praxeology present in the realization of projects and linked with technological tools (TEN).

Therefore, we use this study of praxeology to question the institutional mathematical living in intermediate disciplines or lessons of mathematics.

The projects

The projects are realized by a group of three or four students, very independent, respecting a didactical organization which tries to reflect the real organization in workplaces.

The **engineering projects** are carried out by teams of students in their fourth year of engineering school, over five weeks. The subject of every project is open; there is no previous requirement established by client. The final production and the route towards it have to be built together in the same process. Therefore students have to organize and plan their work, to look for solutions; this generally supposes that they adapt or develop their knowledge.

The projects are realized in two phases. After the first one the students write an intermediary report; in this report they describe the pre-project which is in general justified by a study of the subject. They present the technological solution they have chosen among those they have found during their exploratory work. In the second phase the pre-project must lead to a concrete product.

In this kind of projects, the manager is a college teacher, who plays the role of a client who requests a product from a student's group. All the terms and conditions of the project are described in the schedule of conditions (cahier des charges) which is negotiated between the client (teacher) and the distributor (students). The students are supposed to work on their own to come up to the client's request. The project is assessed from on a double point of view, combining workplace and engineering school requirements. The client must be convinced that the technological solution is the best. But this evaluation is also academic; the students present their work to a jury composed of college teachers. The jury evaluates the use of tools in relation whit knowledge taught in the engineering college. Moreover the students are often asked to justify some of their claims.

Projets Observation methodology

We have realized two observations of the projects. To realize the observation of projects, we used Dumping methodology. In the first phase of project (two weeks) we carried out questionnaires and semi-structured interviews with the students and the clients – tutors. After this phase, we collected institutional data, specifications (document), intermediary reports and documents used for the development of projects. This allowed us to get familiar with projects.

For the second phase we chose only three projects, our aim to be able to realize a deeper and precise observation. To select these projects, we based on the intermediate reports following two criteria: 1) the presence of explicit mathematical knowledge and 2) the project domain such as aeronautics, mechanics, electronics, etc.

In the third week of the project, we met with the students' teams (three teams for three projects) for an interview about the intermediary report; the aim of this interview was to understand the project and to investigate on the role of the identified mathematical contents. We asked the students to do a brief exposition of their project. The aim of this exposition was to identify the role that they were giving to the mathematical content expressed in their intermediary report.From this, we identified the work division inside the team, and we realized that only one student has the responsibility to develop the mathematical activity. After these meetings, interviews were realized with each student individually.

Praxeological analysis of projects

We carried out a praxeological analysis of the projects. In this paper, we present the analysis of one task accomplished in one of the projects: the Development of a conveyor belt for the aerodynamic study of a light ultra vehicle. The aim of this project was to build a conveyor belt to reproduce the velocity floor. For this, it was necessary to model functioning of the motor and simulates it in Matlab (software).

Task: Modelling of motor

The task is to build a model of the motor trough the block diagram. This diagram will allow us to simulate this motor in the Matlab software.

Technique: The modelling of the motor pass by two steps.

1) Mathematical model. The differential equations modelling the electrics and mechanics functioning.

Electrics functioning $u(t) = e(t) + Ri(t) + L\dfrac{di(t)}{dt}$

Mechanics functioning $C_m(t) - C_r(t) = J\dfrac{d\omega(t)}{dt} + f\omega(t)$

The electrics and mechanics functioning are linked by two equations. Every single equation contains a flow and couple constant k: $e(t) = k\omega(t)$ and $C_m(t) = ki(t)$

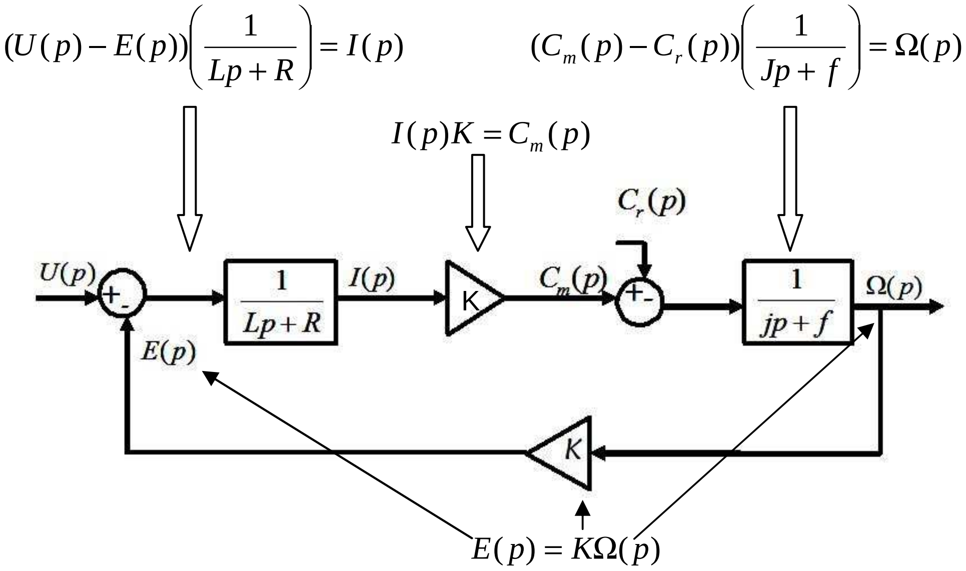
Next, we apply the Laplace transform to every equation:

$$I(p) = \frac{U(p) - E(p)}{R + Lp} \quad (1) \qquad \Omega(p) = \frac{C_m(p) - C_r(p)}{Jp + f} \quad (2)$$

$$E(p) = K\Omega(p) \quad (3) \qquad I(p)K = C_m(p) \quad (4)$$

2) Construction of block diagrams

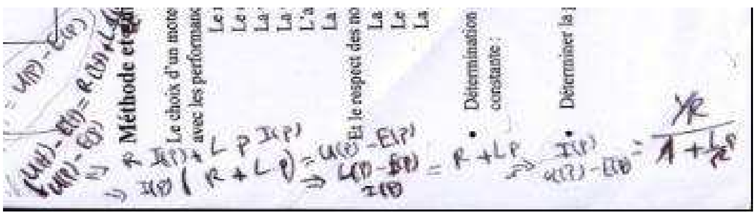
These equations allow us to construct the following block diagrams. Every one element of the equation is represented in the block diagram.



Student techniques:

The student describes the technique utilized to accomplish this task.

"For example if we take this equation (showing $u(t)-e(t)=Ri(t)+L\frac{di}{dt}$) [...] and if we apply Laplace transform we shall have $U(p)-E(p)=RI(p)+LpI(p)$, if we make this (factorize $I(p)$) we shall have this $I(p)(R+Lp)=U(p)-E(p)$, this means that $\frac{U(p)-E(p)}{I(p)}=R+Lp$ and if we make the inverse we shall have $=\frac{I(p)}{U(p)-E(p)}=\frac{1}{R+Lp}$ [...] this equation is modelled by this part" (oral explanation)



Written traces accompanying oral explanation

Technology:

In the description of technique, the student shows the aim of task is to express the "transfer function" of the system. The Laplace transform is for the student a tool which allows to treat an electrical equation as a transfer function. At the same time, Laplace transform allows to pass from temporary domain (algebraic) to a non temporary domain (differential equation).

> "[…] we have $U(p) = E(p) + I(p)R + LpI(p)$ and if we transform $pI(p)$, we apply the inverse Laplace transform, then we obtain the derivative of a temporary function" (Oral explanation)

We see here that motivation appears (function 2 θ^p) by the utilization of the Laplace transform. The student focuses in the derivate term $LpI(p)$, showing interest in using the Laplace transform to pass from differential equation (temporary domain) to transfer function (algebraic domain) or the block diagrams.

From the mathematical point of view, there is a notion justifying the block diagram: the transfer function. This notion considers that the physics systems are described by the differential equation:

$$b_n \frac{d^n y}{dt^n} + b_{n-1} \frac{d^{n-1} y}{dt^{n-1}} \ldots + b_1 \frac{dy}{dt} + b_0 y = a_m \frac{d^m u}{dt^m} + \ldots + a_1 \frac{du}{dt} + a_0 u$$

> "If we apply Laplace transform to the differential equation and assume the initial conditions to be null, then the rational fraction which links the output Y(p) to the input U(p) is the transfer function of the system.
>
> $$L\left(\frac{dy}{dt}\right) = p.Y(p) \Rightarrow L\left(\frac{d^2 y}{dt}\right) = p^2.Y(p) \Rightarrow \ldots \Rightarrow L\left(\frac{d^n y}{dt}\right) = p^n.Y(p)$$
>
> $$\Rightarrow b_n p^n Y(p) + \ldots + b_1 pY(p) + b_0 Y(p) = a_m p^m U(p) + \ldots + a_1 pU(p) + a_0 U(p)$$
>
> $Y(p) = H(p).U(p) = \dfrac{a_m.p^m + \ldots + a_1 p + a_0}{b_n.p^n + \ldots + b_1 p + b_0}.U(p)$" (Automatics course: Introduction à l'Automatique des systèmes linéaires, pp.7 -8)

This notion is part of the Automatics course (intermediary discipline).

Conclusion

The task modelling of the motor is the reproduction the existent model. The students are not created a new model. They adapted a type models a specific situation. This adaptation need mobilize the technological elements. These elements are from different institutions: teaching institution of intermediary disciplines T(ID), teaching mathematics T(M) and practical institution pI. We see the processes of transposition of the praxeologies, which pass from one institution to other institution and are transposed. The functions of the practical component θ^p, allows us to analyze the praxeologies used in the projects. To understand the technologies linked to the students techniques, it is necessary to take in account the intermediary disciplines. These disciplines are intermediary between mathematics teaching and mathematics used in practise.

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FITTING MODELS TO DATA: THE MATHEMATISING STEP IN THE MODELLING PROCESS

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This paper presents a mathematical modelling activity experienced with students of first year university level centred on a problem of forecasting sales using one-variable functions. It then focuses on the back and forth movements between the initial system – a time-series of the term sales of a firm – and the different models proposed to make the forecasting. The analysis of these movements, that are at the core of the 'mathematising step' of the modelling cycle, shows how the initial empirical system is being enlarged and progressively enriched with new variables and mathematical objects. Thus the development of a modelling activity initiated with a real-situation may soon lead to a process where the mathematising affects both the system and the model.

1. THE MATHEMATISING STEP IN THE MODELLING PROCESS

In current didactic contracts, the validity of the mathematical knowledge students have to learn usually has its last guarantee in an external source of the activity: the teacher. It is the teacher who, as a last resort, decides if a result is correct or wrong, if the used tool or technique was the best possible choice, etc. Because of this dominant epistemology underlying current didactic contracts of our teaching institutions, research in mathematics education puts forward an 'experimental epistemology' more in accordance with the Galilean's spirit of modern science. According to this epistemology, scientific knowledge (and mathematics in particular) is building up in permanent contrast with 'empirical facts' that, added to the principles of theoretical coherence, represent the main elements of proof. The reproduction of this 'experimental epistemology' in mathematics underlies the Theory of Didactic Situations (Brousseau, 1997), especially through the notion of adidactic situation and the principle of knowledge construction in contrast with a *milieu*. The recent developments of the Anthropological Theory of the Didactic (Chevallard 2004 and 2006) have introduced the notion of 'media and milieu dialectics' as an analysis tool of the necessary interaction between a *milieu*, i.e. any system devoid of any didactic intention, and the *media* (in the sense of 'mass media') as any source of information or pre-existent knowledge. The aim of this paper is to

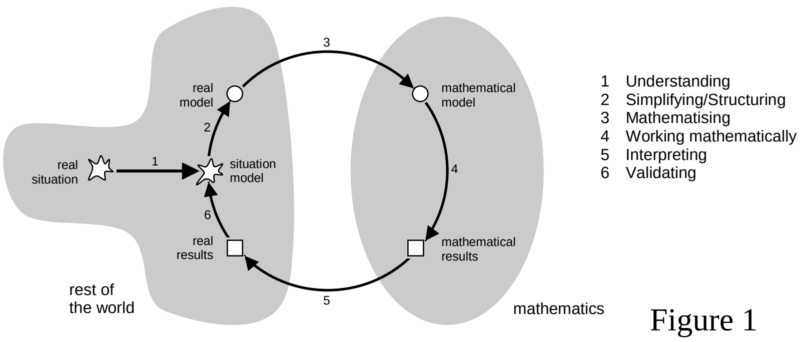


Figure 1

consider how these notions can help analyse a concrete step of the modelling process as it is considered in many research works in the 'modelling and applications' domain using the modelling cycle (Blum & Leiβ 2006), namely the 'mathematising' step (see figure 1).

This paper considers a special modelling activity that has been experimented with first-year students of a mathematics course for economy and business at university level. The real situation that is modelled is a problem of forecasting sales given the historical data or previous term sales. The concrete 'mathematising' of this situation consists in choosing an appropriate mathematical model (a one-variable function) fitting the empirical given data. The possibility of choosing different models and the need for a criterion to select one starts a process of contrast between the models and the empirical system acting as a 'milieu'. The next section presents the conditions of the teaching experience and outlines the work of the students when approaching the sales forecast problem. The analysis of the experience in terms of the 'media and milieu dialectics' is detailed in the third section, before concluding about the importance of considering the 'mathematisation' of a mathematical system – that is, 'intra-mathematical modelling' – as a step of the modelling process analogue to those included in the modelling cycle.

2. A MODELLING WORKSHOP ON 'FORECASTING SALES'

2.1. Conditions of the experience

The didactical experimentation we present here corresponds to a first course of mathematics in Economics Studies during the academic year 2006/07. It is important to underline that the teaching conditions of this course do not correspond to a traditional one. First, the university we refer to is a private university that organizes teaching in not very large groups (between 30 and 60 students) where every student has a personal laptop computer. Second, the course has been designed by a researcher in mathematics education and the experimentation was carried out by four teachers, three of whom are also researchers in didactics.

The course was designed drawing special attention to modelling activities. Its main goal, as it explicitly appears in the syllabus, is 'to get students learn to elaborate and use mathematical models for the description, analysis and resolution of problematic situations that can be found in business, economy, finance or daily life. […] Students should be able to analyze problematic situation in terms of dependence between variables, pointing out the relevant information needed to construct a mathematical model of this situation. And they should know how to use the mathematical model proposed and how to synthesize the results obtained with these models in order to generate new knowledge and new questions about problematic situations considered.'

The programme is divided into three blocks that correspond to the three term periods of an academic year: linear algebra, calculus in one variable, and calculus in several variables/optimization. The course is structured in two weekly sessions of two hours:

the first one is a lecture (teachers' explanations and problem resolutions on the blackboard) and the second one is used to carry out a 'mathematical modelling workshop', centred on the study of a problematic question connected to the field of economy or business. The work here presented corresponds to the workshop experimented during the second term, within the domain of 'one variable calculus', which lasted 5 sessions.

The work at the workshop was organised in the following way: The students work in groups of 3 or 4 and have to write and present a weekly report about the partial results obtained at each session. At the end of the term, an individual final report has to be presented at the moment of the evaluation (a written exam which includes two different problems and a question related to the workshop). This exam represents 50% of the qualification; the written reports 40%, and the remaining 10% corresponds to the individual resolution of problems during some of the lectures.

2.2. The question of 'forecasting sales': analysis of its generative power

The initial question of the workshop was formulated as follows:

> A firm registers the term sales of its 7 main products during 3 years. They ask us the following questions:
> → What amount of sales can be forecasted for the next terms? Can we get a formula to estimate the forecasts? Which are its limitations and guarantees? How to explain them?
> → What products sales are increasing more than 10% a term? Less than 12% a term?

The data were 'prepared' by the teachers so that they correspond to seven elementary functions of different types (quadratic, cubic, rational, exponential) with an error term added.[1] The values of each function were slightly changed with the aim of distorting them, but without losing the general "tendency" of the original function.

The workshop's aim was to give students a problem close to a real situation where functions appear as a suitable model. Both the use of Excel in the first term of the course and the students' familiarity with elementary functions (it was the theme of the sessions just preceding the workshop) allowed them to initially detect a tendency in the sales (for example from a graphic representation of the data) and look for a function that fitted this tendency. The firm question proposed also included the idea of percent variation, which we expected would make the study of function variations appear during the workshop. Given that the workshop was run in parallel with the lectures on function derivatives, it was also expected that, at any time, the study of the sales' variation could be connected with them.

[1] The concrete functions were: $0{,}5(x-6)^3+2000$; $2{,}5(x+5)^2+100$; $5500/(x+4)$; $1300\cdot 085^x$; $1500-1200/(x+1)$; $2{,}5(x+5)^2+100$; $1300\cdot 085^x$). The second experimentation in 2007/08 was carried out with 'real' data taken from some macroeconomic magnitudes of different countries: population, oil production, traffic crashes, unemployment rate, etc. The main difference between the two workshops appears in the study of the variations, because the real data have stronger fluctuations and do not always present a clear tendency.

The election of a sales forecast situation was mainly motivated by the fact that it enables to clearly distinguish between the economic system (sales) and the models used (functions). Moreover, working with different products needs to consider different models, raising the problem of the fitting between the model and the modelled system. In other words, the aim of the workshop was to make students use functions as a model of a simple economic system and quickly raise the question of the election of the model and its validation.

2.3. General organisation of the modelling workshop

We here report the four workshops experienced, corresponding to four classes of a (the) first-year course of mathematics for economics and business led by four different teachers working in team. Each group has a teacher, the same one for the lectures and the workshop sessions. All classes were prepared by the team and all sessions were discussed personally or by mail before and after being carried out. Each teacher, at the end of each workshop session, wrote a report in which he/she explained the development of the session, and sent it by mail to the other teachers.

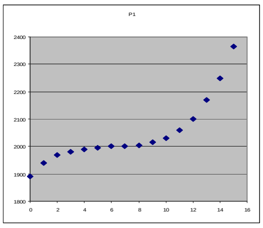
Before the workshop started, the students had four lectures dedicated to introducing the elementary families of functions, from straight lines to exponential functions. The students learned how to use the general expression of every family of functions and to associate them with different graphics. In other words, the students were taught how to assign an algebraic expression to the graphic of a function, among a set of given families. They saw how to deduce the graphic of $y = af(x – b) + c$, from the 'basic function' $y = f(x)$ and, reciprocally, how to deduce the expression of any function $y = af(x - b) + c$ given its graphic and knowing the original 'basic function' $y = f(x)$. The lectures given in parallel with the workshop introduced the notion of absolute and relative variation of a function between two points, the notion of the derivative's function, the notion of straight line tangent, etc. within the general problem of the study of variations. The functions considered were always related to economical situations, such as the *incomes* depending on the *sales*, the *cost* depending on the *production*, the *demand* depending on the *price*, etc.

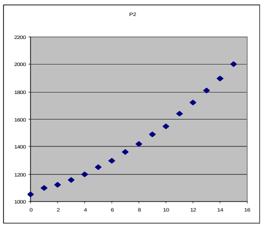
2.4. Description of the workshop sessions

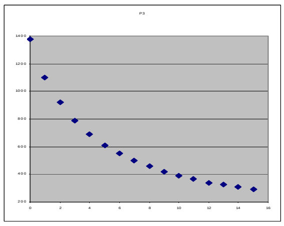
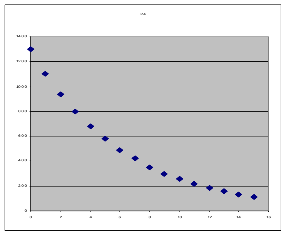
We are now presenting a brief summary of the workshop sessions based on the teachers' reports, the students' weekly summaries of the workshop and the students' individual summaries at the end of the term.

<u>Session 1: Considering the initial question and first exploration of data</u>

The first session is dedicated to present the generative question and the data. Each group is assigned two products from the list. During some time, the students can explore the question and propose a first forecast for the next three-month period. Most of the groups decided to introduce the data in an Excel sheet so as to represent them graphically. Most groups were able to associate the graphic representation with some of the families of functions previously studied. Some of the graphs obtained were:





Depending on the product considered, different types of functions can be associated with the graphic. The case of product 1 is different because the form of the data clearly suggests a cubic function. In this case, the students easily found an analytic expression $y = a(x - b)^3 + c$ fitting the data, first detecting the inflexion point (b;c) and then testing different values for parameter a. At the end of the session, the teacher asked some of the groups to present their procedure used and results to the whole group. A structure for the Excel sheet was agreed upon and the teams were asked to bring in a possible model with its corresponding forecasts for the next session.

Session 2: Finding different models and comparing them

Each group presented the analytic expression obtained for the products assigned. As each product was assigned to different groups, different possible models appeared for the same set of data. Hence the problem of deciding which forecast was "better" quickly appeared. As it was impossible to decide on at first sight, the teacher introduced a possible criterion to 'measure how different each model was from the data'. It consists in computing the difference (in absolute value) between the values of the function and the data of the product. A new column was added to the Excel sheet (with) which, at the end, mentioned the arithmetic average of the differences. It was called the 'average error'.

Then the session work consisted in finding, for each product and within a given family of functions, the model that gives the minimum average error. The first procedure was to modify the parameters of each function to find the best model by trial and error. In the middle of the session, the teacher introduced the Excel tool 'SOLVER' that gives the parameter combination that minimizes the average error, when initial values are close to the solution. The Solver function allows finding the best approximation to data when models are considered within the same family of functions, but it is not an effective tool to decide between two models belonging to different families (a parabola and an exponential function, for instance). Besides given two sales forecasts done with functions of a different type, the fact that one of them gave a lower average error than the other, did not always seem a good criterion to determine that it was a better forecast (it is not always so clear graphically, for example). The session concluded by asking the students to bring in 'the best model' for each product and the corresponding forecast. In a sense, the first question of the initial problem was almost answered.

Session 3: Study of the average term variations

The session started by sharing the expressions provided by each group. The problem of finding a criterion to select the best model was raised in the case of different models for the same product with a similar average error. At this moment, in one of the four groups, the teacher took advantage of the work done by a team that initially, during the first session, used the term variation of the sales. They found out the rate of the previous terms' variation and then took an average to do the forecast. This idea was introduced to the rest of the teams and also to the other class groups.

Therefore, besides the data of term sales and its possible models, appear a new set of data, the term variations of the sales, which can be modelled in turn. The students were thus asked to proceed with this new data in the same way they did before: doing the graphic representation, deciding which family of functions seems to correspond to the visual tendency, finding the concrete function that gives the lower average error.

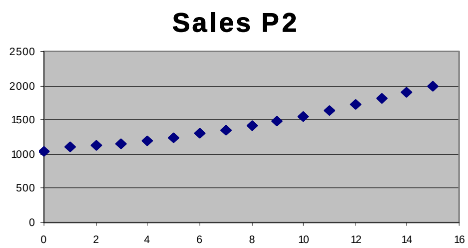
In the case of product 1 (cubic function), the new data appeared as having a quadratic tendency. In the case of products given by a quadratic function, the term variations seemed to correspond to a straight line, in the case of a rational or an exponential function, to another rational or exponential function respectively.

Sessions 4 & 5: Comparing the model of the variations to the variation of the model

When the different groups presented their models for the sales forecast and for the sales variation forecast, the teacher asked for a possible relation between the two models corresponding to the same product. In the case of the products with only one 'good model' (such as product 1 with a 'cubic tendency') the conclusion was quite complicated. With those products accepting more than one model, the variation study led to a better conclusion: the graphic that best fitted the term variations of sales was similar to the graphic of the derivative of the function that best modelled the product.

For example, if we consider product 2, we find:

Term	t	Sales
March-03	0	1050
June-03	1	1100
Sept- 03	2	1120
Dec-03	3	1160
………	…	…



The graphic representation shows a tendency that can be modelled by a linear, a quadratic or an exponential function. The corresponding average errors are:

a =	60,92
b =	973,03

t	P2	$y=ax+b$	ERROR
0	1050	973,03	76,98
1	1100	1033,95	66,05
...	...	...	...
		AVERAGE	**40,625**

a =	326,96
b =	1,10
c =	732,97

t	P2	$y=ab^x+c$	ERROR
0	1050	1059,92	9,92
1	1100	1091,30	8,70
...	...	...	...
		AVERAGE	**7,16**

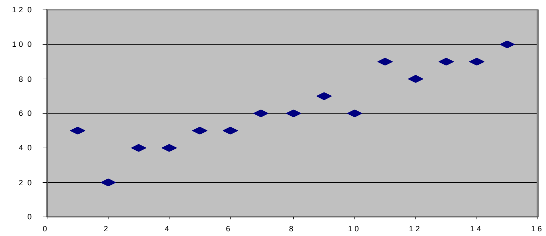
a =	2,46
b =	-5,18
c =	995,01

t	P2	$y=a(x-b)^2+c$	ERROR
0	1050	1061,00	11,00
1	1100	1088,95	11,05
...	...	...	...
		AVERAGE	**3,63**

The study of the average error rules the linear model out, but does not provide a good criterion to exclude the exponential function or the parabola. If we consider the term variation of sales and model the new data, we obtain the following:

Time	P2	Term Var.
0	1050	
1	1100	50

Sales term average



a =	5,0000
b =	24,9999

t	P2	T.V.	$y=ax+b$	Error Abs
0	1050			
1	1100	50	30,00	20
2	1120	20	35,00	15
3	1160	40	40,00	0
4	1200	40	45,00	5
...	...	...	...	...
			AVERAGE	5,67

a =	-0,9915
b =	-1,2772
c =	61,0730

t	P2	T.V.	$y=ab^x+c$	Error Abs
0	1050			
1	1100	50	62,34	12,34
2	1120	20	59,46	39,46
3	1160	40	63,14	23,14
4	1200	40	58,43	18,43
...	...	...	...	...
			AVERAGE	16,93

Looking at the two corresponding term variation models, it clearly appears that the linear model has a lower average error than the exponential one. To summarize, we have found two models that fit the initial data in a similar way. Their analytic expressions, using the Excel tool 'Solver', are:

OPTION 1: $y = 326{,}96\,(1{,}09)^x + 732{,}96 \rightarrow$ *average error*: 7,16

OPTION 2: $y = 2{,}46\,(x + 5{,}18)^2 + 995{,}01 \rightarrow$ *average error*: 3,63

The lower error corresponds to the parabola, but both are similar (comparing to other considered possible models). When considering the term average of the sales, the model that fits better is: $y = 5x + 25$. Finally, if we take the first model expression $y = 2{,}46\,(x + 5{,}18)^2 + 995{,}01$ and derivate it, we get an expression very similar to the model found: $y' = 2{,}46\cdot 2\cdot(x + 5{,}18) = 5{,}2x + 26{,}936 \approx 5x + 25$

Therefore, we have a new criterion to decide between two models: studying both the tendency of the sales and of their term variation, and choosing as 'best model' *the* *function that has a derivative that fits the model of the term variation*. At this moment, further work on the mathematical model can follow, looking at the derivative as a model of the term variation $\Delta f(x) = f(x) - f(x - 1)$. The use of a symbolic calculator was an important tool for this final step of the modelling process, which was left to the students as a complementary theoretical analysis of the whole work done in the workshop. After these five sessions, students were able to use all the information to present a forecast for the sales and report a complete answer to the initial question.

3. THE 'MATHEMATISING STEP': CONTRASTING MODELS TO DATA

3.1. First part of the workshop: the problem of choosing the best model

The process of mathematising or assigning an appropriate mathematical model to a given system can be done in a simple way by directly choosing a previously available model given by an external source (a 'media'). However, the productivity of the model, that is, the fact that it produces new knowledge about the system, requires a certain 'fit' or 'adaptation' to the system. This process is rarely done once and for all. It requires a forth and back movement between the model and the system, in a sort of questions-answers or trial-error dynamics. We will now see the details of this process in the concrete modelling process of the workshop presented below.

In the first part of workshop, the aim is to look for a function that accurately reproduces the sales dynamic. The first decision to take is to fix the family function that seems to reproduce the observed dynamic in the data. The students' first gesture was to represent the data in a calculus sheet and determine a priori which type of function would be chosen[2]. In terms of the 'media and milieu dialectics', we can consider that the Excel graphic works as a *milieu*: when representing the chosen function, it allows to visually contrast the 'proximity' between the model and the data.

The problem about how to construct a criterion to determine the best fit is the crucial question that drives the study process. Except in one or two cases, the only visual comparison between different sales models becomes an early limited *milieu*. The necessity of establishing a 'measure of the fit' comes up, and enriches the initial *milieu* given by the numeric data series and its graphic representation. The option chosen –a new message (*media*) given by teacher– is to calculate the average of the differences (in absolute value) between the data and the values of the considered function. The incorporation of the Solver function –that works as a black box for the students– provides another *milieu* that makes the search of the function that minimizes the error more dynamic. However, this new enriched milieu can also show its limitations when the errors between different 'competitive' models are similar.

[2] The fact that students work with a small group of a pre-established family of functions does not have to be considered as a didactic limitation. It reproduces the usual situation of the genuine modelling work.

3.2. Second part: the model of the variations and the variations of the model

In the case of having different models with similar errors, the *milieu* made up of numeric values and the graphics of both sales and models is newly enriched by the introduction of a new variable: the sales variation. A new modelling process starts, similar to the previous one. The derivative function, as an approximation of the variation, soon becomes a new element of the *milieu* brought by the teacher acting as a *media*. It will contribute as a new criterion of validation: if a model fits the sales, the derivative of the model should be a good fit of the sales variation. For example, if sales seem to follow a parabolic growth, it is expected that the sales variation will follow a straight line growth. In this case, the *milieu* is all the work done during the first part of the workshop, that is, the construction of different models to each data series.

The teacher is who introduces the relation between the term variations and the derivative of the pre-established model (*media*). Besides, as students had a symbolic calculator that allowed them to easily calculate the algebraic expression of the average value $f(x + 1) - f(x)$ of any function, it was also possible to compare the derivative value of the model with this average value and confirm the approximation. It is important to underline that the increase of the '*milieu*'s complexity' made the development of this second part of the workshop more difficult, the 'system' that was to be modelled being less known and 'unstable' for the students. However, the work done represents an exemplary case of the functionality of the derivative as a simple way to calculate the average variation of a function between two points.

4. CONCLUSIONS

Using the modelling cycle proposed by Blum & Leiβ (2006), the whole process can be described in the following way. The problem of forecasting sales given a time-series of data constitutes the initial extra-mathematical situation, that we will call the 'system' (as opposed to the 'model'). At this stage, the system considered was a 'real one' (extra-mathematical). The first step of the modelling process consists in representing the data graphically to make a first hypothesis about the tendency of the time series. This first graphical model helps to decide on the type of functional model that best fits the data, giving rise to a mathematising process aimed to decide on the parameters of the chosen concrete function by a trial and error procedure using Excel, going forth and back from the model to the system. A new question arises when different types of functions are used to fit the data and one has to decide which model is best. The search for a criterion needs to consider a new 'real system' formed by the data and the possible models, with the problematic question of how to determine the 'best fit', that is, how to mathematically model the 'fitness' of a model. This new system is in turn mathematised by the average error of the fit. Again, the insufficiencies of this new model lead to the consideration of a new enriched 'system': the one formed by the original data and the term variation of the sales. A possible criterion is set up by considering the double modelling of the sales and the term variation of the sales. Finally, considering the derivative as a model of the term variation constitutes the last mathematisation step that leads to a final conclusion for the forecast problem.

It is important to note that, in this entire process, the successive 'systems' that are modelled are more and more mathematised, and that the successive 'models' constructed progressively integrate the previous systems, creating new problems and, thus, generating the need to go on with the modelling process. We have interpreted these successive mathematising processes using the 'milieu and media dialectics' introduced by Chevallard (2004), which has helped us provide a detailed analysis of the mathematising step of the modelling process, showing how being a 'system' to be modelled or a 'model' of the system is more related to the *function* assigned to a given object during the modelling process than to its very 'nature' (it being mathematical or extra-mathematical). The example here described shows how the development of a modelling activity, even if initiated with an extra-mathematical situation, leads to consider, not only a sequence of new models, but also new and enriched systems more and more mathematised. Hence, extra-mathematical and intra-mathematical modellings appear as strongly intertwined.

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WHAT ROLES CAN MODELLING PLAY IN MULTIDISCIPLINARY TEACHING

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This paper presents a research- and development project about mathematics in multidisciplinary teaching, running as a pilot in 2008-2009 and planned to run in full scale in 2009-2010. Its aim is to inquire how learning potentials in mathematics are realised in a number of cases of good practice and, besides, to prepare materials for such teaching. The issue of this paper is to report on potentials and drawbacks experienced so far in the project and to discuss how to avoid the major drawbacks. The discussion takes as its starting point one example of modelling from the project, which invites critical discussions in the classroom about the use of mathematical models in statistics.

NEW CHALLENGES TO THE SCHOOL SUBJECT MATHEMATICS

As one consequence of a reform in 2006 of upper secondary school in Denmark, there is a need for examples of good teaching throwing light on and demonstrating what works for the learning of mathematics in multidisciplinary contexts. Furthermore, the reform's introduction of multi disciplinarity draws attention to the role of mathematics in different types of collaborations: It is not uncommon that multidisciplinary projects involve cultural, historical or philosophical aspects which are important but not at the heart of mathematics taught in schools. To balance this tendency, there is a need for advice and ideas about how to empower the learning of what one might call 'core mathematics' within a multidisciplinary teaching context.

THE DASG – NAVIMAT COLLABORATION PROJECT

This paper presents a research- and development project, which is running as a pilot (15 teachers in 4 schools) in 2008-2009 and planned to run in full scale (about 20 classes) in 2009-2010. The project deals with mathematics in multidisciplinary teaching projects. Its aim is to inquire how learning potentials in mathematics are realised in a number of cases of good practice and, besides, to prepare materials for such teaching.

The project is conducted in collaboration between Danish Science Gymnasiums (DASG)[1] and Nat. Knowledge Centre for Math. Ed. (NAVIMAT)[2]. DASG is a network[3], incorporating about 36 Danish Upper Secondary Schools (out of 200[4]). Membership implies an obligation for the school to spend resources, in the form of teachers' working hours, on participation in at least one of the 5 – 8 sub-projects, which are formulated and announced every year. The sub-projects run for two or three years and each one involves about 25 teachers. The collaboration between DASG and NAVIMAT encompasses a two-stage project. During the first year, three different types of teaching materials will be produced and tried out in a pilot; each of these materials represents the interplay between mathematics and one of the three participating faculties Human Sciences, Social Sciences and Natural Sciences. During the next stage, the following year, trials and evaluations of revised teaching materials from the pilot will be offered to the DASG schools as sub-projects. The revised versions of these materials hopefully will be published by NAVIMAT to provide inspiration for teachers at the conclusion of the trials. Teams of two to four teachers, a researcher in mathematics education and a professional specialist prepare the materials. The teams autonomously plan and make arrangements for their work during the first year of the project. DASG organises joint seminars for all the teams during this stage, for the exchange of ideas and experiences so far.

The professional specialist's are picked out depending of the mathematics teachers' choice of subject. The professional specialist's role in the team is to provide inspiration and expertise with regard to the content of the teaching materials. The mathematics education researcher provides inspiration and expertise with regard to the design of the materials and observes and evaluates the teaching experiment. The researcher is responsible for development and formulation of guidelines for good practice in multidisciplinary teaching. The team's teachers design and produce the teaching materials and carry out the teaching sequences. The teachers participate in the evaluation and discuss the results with the researchers.

THE POTENTIAL OF MULTI DISCIPLINARITY

Some of the potentials of multidisciplinary mathematics teaching were discussed in (Andresen and Lindenskov 2008). We see potentials achieving a number of different goals.

i) <u>Students' motivation and interest</u>. Multi-disciplinary projects can stimulate the students' interest and engagement in mathematics because the usefulness of the mathematics taught, and its links with the students' own, experienced world are in constant request in Danish school. Multi-disciplinary projects set the stage for the teaching of useful applications of mathematics in authentic, daily life settings. Hence, such projects can serve to meet the students' requests and to improve their desired understanding of connections between subjects and the world outside school. This is in accordance with Michelsen, Glargaard and Dejgaard (2005 p 33) who point to an alternative approach that stresses the importance of modelling activities in an inter-disciplinary context between the two school subjects physics and mathematics. Similarly, R. Filo and M. Yarkoni (2005) reported on a project, which integrated geometry and art, aiming at inter-disciplinary learning of parallel concepts. Filo and Yarkoni's assumption in this case was that an enriched concept formation was supplied by an advanced status of both subjects in the students' minds.

ii) <u>Transfer.</u> The authors report on their observations of the classroom that indicated

- Students' awareness of the possibilities to transfer concepts and results between subjects

- Students' consciousness about benefits, traps and misunderstandings caused by such transfer
- Students' reflections upon the relations between the project's subjects

The observations were interpreted in accordance with an interactionist's perspective like Heinrich Bauersfeld presents it in (Bauersfeld 1994 p 137-139). Hence, we looked for indications of a classroom culture where, for example, arguments from one subject (mathematics) were used and accepted in discussions within another subject (chemistry or physics) or used to convince other members of the group in discussions of problem-solving strategies etc. Besides, we evaluated signs of the students' formation of conceptions. The students seemed to build relations between the subjects in parallel with their formation of concepts and new skills belonging to the single subject.

iii) Implementation. Multi-disciplinarity can be seen as a means to revise the role of school mathematics and, thereby, to embed students' mathematical competence into a broad and reflected view of math and science. Compared to cross (or inter)-disciplinarity or to trans-disciplinarity, multi-disciplinarity has better odds for successful implementation because it resonates with the following four main elements of Fullan and Hargreaves's (1992 p 5) framework for understanding teacher development; 1) the teachers' purpose, 2) the teachers as a person, 3) the real world context for the teacher's work and 4) the culture of teaching

Hence, we argue (Andresen and Lindenskov 2008) that multidisciplinary teaching has important potentials for improving students' motivation and interest and for an enhanced transfer between subjects. We expect multidisciplinarity to be successfully implemented, and we expect it to serve as a means in the future to support the embedding of the students' competencies into broad and reflected view on mathematics.

MODELLING FOR CONCEPT FORMATION

In addition to this, the didactical potentials of a multi-disciplinary project rest on the role of mathematical modelling and reflections for concept formation. Mathematical models in multidisciplinary projects play a double role: on the one hand, the model can serve as the link between subjects and daily life, authentic problems etc., dealt with above. On the other hand, modelling plays an important role for concept formation. The role of modelling for concept formation in learning mathematics is described in the domain-specific instruction theory for realistic mathematics education, RME. (Gravemeijer and Stephan 2002 p 147ff). From this point of view, all mathematical activity concerns modelling, and it gives little meaning to try to discern theoretically between to learn, to apply or to develop new mathematics. Strict borderlines between the three are not to be drawn. In general, the use of the term 'modelling', therefore, has to be specified, since it depends on the context. (In this paper, though, we still use the terms in the 'common way' sense unless stated otherwise.)

POTENTIAL DRAWBACKS AND HOW TO AVOID THEM

Teaching multi-disciplinary projects in accordance with the Danish 2006 reform, hence, is a promising prospect. We also see some potential drawbacks. In some aspects, the impact of multi-disciplinarity on the students' view on mathematics is comparable to the impact of use of computers. The 2006-reform also imposed the introduction of compulsory use of computer algebra systems (CAS) in mathematics. Obviously, CAS has the potential for a huge extension and development of the teaching of models and technical modelling in the sense of comparing a number of models and fitting them with a set of data (Andresen 2007a p5). It also has potentials to support students' model recognition and capability to understand and criticize authentic use of ready-made models in different contexts.

Results from our previous research, however, show that in general, the use of CAS tends to change focus of attention to the technical and practical aspects of upper secondary school mathematics. In general, teaching with a computer is centred upon solving tasks, whereas the reading of proofs and theoretical treatments in general are carried out without use of computer (Andresen 2006 p 28).

There is a potential danger that the same trend might direct the multi-disciplinary teaching into a skills based view of mathematics by the students, at the expense of giving the students a more profound insight into mathematical activities, theory and knowledge. To avoid this, I suggest that the students' more technical and practical view on models and modelling, should be balanced by explicit reflections upon the use of models and upon the modelling process, that is, upon horizontal and vertical mathematizing.

MODELLING AND MATHEMATICAL REFLECTIONS

Reflection is the driving force for the process of mathematical modelling in the sense of progressive mathematizing (Gravemeijer and Stephan 2002 p 147 ff). Hence, Andresen and Froelund (2008) discuss how to make the students' philosophical reflections explicit, as a tool for mathematical reasoning and, thereby, to strengthening the students' consciousness of the art of reflection and of the relationship between reflection and learning. In line with the idea that awareness and consciousness about one's own learning support learning outcome, Andresen and Froelund suggest the explication of mathematical reflections as a tool for learning. The use of philosophical reflections as a tool for mathematical reasoning was recently discussed (Prediger 2007). Prediger's discussion was based on the stratification (Neubrand, 2000) of reflective practice in mathematics into four levels:

1. <u>Questions at the level of the mathematician</u> concern isolated, mathematical details. The questions are meant to deepen the students' understanding of the rise from a situational to a referential model which means that a preliminary or

emergent model is to be constructed. At this level objects still are marked by the context and, for example, referred to as ‘people’, ‘number of heart attacks’ etc.

2. Questions at the level of the deliberately working mathematician concern conscious use of mathematical objects and processes. The questions set focus of attention on generalisation of entities and their relations and, thereby, on the construction of a model for the case based on the model of the contextualised problem. The same type of questions could start discussion after the rise from referential level to general level; in the actual case by introduction of several distributions etc. The later discussion could lead to the next level of questions:
3. Questions at the level of the philosopher of mathematics concern mathematical methods and applications. Rise from general to formal level tends to happen over time, sometimes in a somehow subtle way. In the actual case discussions about the range of applicability and validity of hypothesis-test methods etc. serves to support the rise and make it more explicit to the students.
4. Questions at the level of the epistemologist concern the characteristics of mathematics compared to and delineated from other sciences. These questions relate to activities at the formal level which may be widened by further reflections. In the actual case, the multidisciplinary setting itself may lead to questions and discussions of the intended type.

Andresen and Froelund (2008) argue for the teaching of mathematics based on the use of a reflection guide containing thought-provoking questions at these four levels. A short analysis of the modelling process is prerequisite for the design of a reflection guide. The aim of this analysis is to identify potential levels of mathematical activity, referring to Gravemeijer’s model which includes four levels: situational, referential, general and formal. (Gravemeijer, K. & Stephan, M. (2002). p 159)

Teaching in a multi-disciplinary setting like in the example, provides a design that particularly favours explication of mathematical reflections. The didactical potential of such multi-disciplinary teaching, though, depends on its design: the design has to ensure that the project’s modelling processes are visible to the students as well as providing the opportunity to make students’ mathematical reflections explicit during classroom discussion etc.

ONE EXAMPLE OF THE PILOT’S TOPICS: THE VIOXX CASE

The materials presented in the following example takes the Vioxx case, described below, as its starting point and concentrate on probability theory and statistics in mathematics. Preparation of the materials is still ongoing (autumn 2008), based on experiences and notes from a pre-pilot teaching experiment carried out in 2007-2008. In the pre-pilot, all the project’s lessons were spent in mathematics, although the envisioned partner subject was the school subject social science. Philosophical ethics or chemistry might also have been appropriate. The teacher with his teaching experiences referred to below are from this pre-pilot.

The VIOXX case

Vioxx was a pain-reducing drug produced by Merck, and the case was about the statistical estimation of its long-term effects. In such cases it is impossible to carry out large-scale trials to determine the serious or long-term effects of drugs such as Vioxx. Therefore, when the drug is approved, such trials may be substituted by statistical inquiry of the population of users. For such inquiries, though, statistical models suitable for large-scale trials have to be modified and in particular, the criteria for the acceptance or rejection of hypotheses must be changed. Hence, the Vioxx case served as a context for the students in mathematics to study probability value (p-value), statistical significance and confidence intervals.

Vioxx, which was withdrawn from the U.S. market in 2004, is part of the class of drugs known as nonsteroidal anti-inflammatory drugs (NSAIDs). Vioxx was used to reduce pain, inflammation and stiffness caused by osteoarthritis; to manage acute pain in adults; to treat migraines and to treat menstrual pain. Merck, the manufacturer of Vioxx, announced a voluntary withdrawal of the drug from the U. S. and worldwide market, due to safety concerns of an increased risk of cardiovascular events (including heart attack and stroke) in patients taking Vioxx.

According to the U. S. Food and Drug Administration (FDA)'s website[5], FDA originally approved Vioxx in May 1999. The original safety database included approximately 5000 patients on Vioxx and did not show an increased risk of heart attack or stroke. A second study was primarily designed to look at the side effects of Vioxx such as stomach ulcers and bleeding and was submitted to the FDA in June 2000. The second study showed that patients taking Vioxx had fewer stomach ulcers and bleeding than patients taking naproxen, another NSAID, however, the study also showed a greater number of heart attacks in patients taking Vioxx. This second study was discussed at a February 2001 Arthritis Advisory Committee and the new safety information from this study were added to the labelling for Vioxx in April 2002. Merck then began to conduct longer-term trials to obtain more data on the risk of heart attack and stroke with long term users of Vioxx.

Merck's decision to withdraw Vioxx from the market was based on new data from this, later, trial in which Vioxx was compared to placebo (sugar-pill). The purpose of the trial was to see if Vioxx 25 mg was effective in preventing the recurrence of colon polyps. This trial was stopped early because there was an increased risk for serious cardiovascular events, such as heart attacks and strokes, first observed after 18 months of continuous treatment with Vioxx compared with placebo.

The Vioxx case attracted public attention since a large number of people had been taking Vioxx and amongst them, some had heart attacks. Heart attack victims and surviving relatives had taken legal action and were, in a number of cases, rewarded. For example, John McDarby, 77, and his wife were rewarded a $4.5 million dollar verdict and $9 million in punitive damages from a New Jersey jury in one of the first Vioxx trial cases against Merck[6]. The controversial question for judgement about

Merck's responsibility was to determine, whether data were sufficient to validate any hypothesis about correlation between Vioxx and the heart attacks.

Role and content of mathematics lessons

From the viewpoint of mathematics, Binomial distribution, Poisson distribution and Normal distribution were sufficiently strong tools to deal with these issues. Data from the original and from the later trials are available on Merck's website and then, the determination rests on decisions about level of significance and the confidence intervals. More profound model discussions may concern standards for comparison, compatibility and transfer of results etc.

In the pre-pilot, the teacher designed a sequence of about twenty lessons. This teacher had economy as his minor, so he agreed to spend some time and efforts on the inclusion of societal economics aspects in his teaching. The design was based on preceding discussions at a two-day seminar on authentic mathematics in upper secondary school and, subsequently, in a team with another mathematics teacher; a bio statistician and a researcher in mathematics education. This small group gathered twice during the semester where the experiment took place, for inspiration, exchange of ideas and evaluation.

The students had no prior experiences with probability or statistics. Consequently, the major part of the time was spent on the introduction and training of basic terms and relations within these branches. This introduction and training was based on the textbook with additional tasks collected from the web. In addition, the team prepared a spreadsheet for the students to experiment with distribution, confidence intervals and correlation coefficients.

In parallel, the students learned about the Vioxx case. Different aspects of the case were discussed in the class; in particular, the weighting between ethical and economical aspects and the role of mathematising in such cases were examined and debated. This part of the teaching might have taken place in the lessons on social science as well.

The challenge for the teacher was to combine the following three elements:

i) The mathematical content: introduction and basic training of terms and relations in probability and statistics. The content was taught in line with common practice in this class, based on the same textbook.

ii) The role of mathematics in the Vioxx case. In the Vioxx case the process of mathematising, obviously, was an issue of debate because of its implications for clients, the Merck Company etc. Thus, the case did not serve as a bare illustration of a 'neutral application' of mathematics. On the contrary, the case intended to draw attention to the modelling process itself.

iii) A look from outside at the societal role of mathematics. Development, test and application of medical treatments are based on the use of bio statistics and play an important role for healthcare at individual and societal levels. Though, it implies ethical and economical perspectives. Besides, public discussion of these issues may be seen as one element of democracy.

RESULTS AND OUTCOMES OF THE PRE - PILOT

The design of revised teaching materials and plans in the pilot will be based on the following summary of resulting outcomes related to the bullets i) – iii) above:

Mathematical Content: During the teaching experiment, the students showed large interest in the subject and in the Vioxx case. According to the teacher, the students were so eager to understand and to feel comfortable with the mathematical terms and relations and as a consequence, the class had to spend more time than expected on the technical-mathematical part of the course. For example, they spent six lessons just on working with level of significance. The teacher noted that this part of the sequence worked very well for the students.

Mathematical Modelling: The teacher indicated that the discussions stayed at the level of ready-made models. No attempts were made to modify the binomial distribution or to critically sort out the website's data. Modelling as such appeared not to be self-explanatory; on the contrary, every step had to be pointed out explicitly if the students were expected to be aware of it. For example, it was complicated for the students to make mathematical decisions, such as stating the level of significance. The teaching experiment, evidently, intended to demonstrate exactly that point to the students; so the stage was set to go deeper into the – complex - questions. The class spent time on changing the levels of significance and studied the consequences and effects. But they did not have time to follow these studies up.

Societal Role of mathematics: The teacher had the impression that the envisioned 'look from outside' on mathematics and its role could give input to very interesting and fruitful lessons on societal economy, law and on issues about democracy, public opinion and politics. It could be great, according to the teacher, to arrange a replay of one of the big hearings as a game with students arguing for and against. To complete this, teachers from both subjects should collaborate. The instructional materials for such a replay could be found on the various web sites but it should, preferably, be prepared in the – enlarged – team, including a teacher of social sciences.

GUIDELINES FOR THE PILOT

The big challenge for the pilot project will be to make the three elements, listed above as a coherent and convincing whole. In the pre- pilot part i), the mathematics content, was marked by its status at the introduction. In the pilot the sequence, consequently, will start only after the students' introduction to basic probability and statistics. They will then be able to concentrate on the role of mathematics and to work deliberately with the involved models and modelling. Connections, then, should be established more easily between the mathematical activities and the other elements of the Vioxx case. Such connections will be established, based on the teacher's guiding questions and sub tasks, focusing on specific aspects of the distribution, the level of significance or the parameters influencing the basic probabilities etc. combined with special points of interest from a social science point of view.

To sum up, the teacher recommended that part i) precede the proper multidisciplinary part which should then combine parts ii) and iii). The complete project should build up to a student role play of one of the big hearings, with arguments and a final verdict in the form of a verdict.

Further, preparation of a reflection guide should be included in the design of the pilot. The reflection guide should contain thought- provoking questions, which aim to stimulate the students' mathematical reflections and put them in focus of attention. The guide should be tailored to fit with the teaching materials, not vice versa. Preparation of one example of a guide is outlined in the following – a more detailed example may be found in (Andresen 2008).

CONCLUSION

In this actual case, the reflection guide's questions to the students can give rise to reflections upon the modelling process as a whole, as well as reflections upon the single parameters and how they are related, what they stand for etc. (level one and two) and, besides, to reflections upon smaller parts of the modelling process picked out to be studied separately. So, the scene is successfully set for reflections at all four levels in Prediger/Neubrand's model. Hence, it may be concluded that even if the teacher chooses a design where the technical mathematical part of the sequence precedes the other parts, and even if the VIOXX case in itself attracts the students' attention, it is still possible to choose a design that 1) makes the mathematical content, the role of mathematics in the Vioxx case and the societal role of mathematics as a coherent and convincing whole and 2) gives the students a profound insight into mathematical activities, theory and knowledge.

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1 http://www.emu.dk/gym/tvaers/sciencegym/english.html (loc. 22 May 2008)
2 www.navimat.dk (in Danish, loc. 22 May 2008)
[3] The Lundbeck Foundation in 2006 provided a five year sponsorship for the DASG networks activities[3] and each gymnasium in DASG has applied for its membership on a voluntary basis.
4 155 general and 42 technical upper secondary schools (gymnasiums)
5 http://www.fda.gov/cder/drug/infopage/vioxx/vioxxQA.htm (loc. June 2008)
6 http://www.weitzlux.com/vioxx/trial_402790.html (loc. June 2008)

MODELLING IN ENVIRONMENTS WITHOUT NUMBERS – A CASE STUDY

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In order to study how students are mathematising in modelling situations, students' work on problems having no obvious mathematical character is investigated. The task design aims at preventing students from concentrating on calculations, but challenges them to get involved in social interactions, where they argue and defend their ideas. The students' approaches to these mathematisation tasks are analysed; in particular it is discussed to what extent the students work mathematically. The concept of fundamental mathematical ideas is used in order to structure the way mathematics occurs in the students' works.

Keywords: modelling, mathematising, fundamental ideas, approximation, measuring

INTRODUCTION

In mathematics education, word problems are regarded as that type of mathematical exercises where information is provided in narrative, descriptive form, rather than in terms of numbers, variables, and so on. In extension, modelling problems are word problem solving activities, which involve not only handling data or calculating, but also observing patterns, testing conjectures and estimations of results (Schoenfeld, 1992). Tightly connected with modelling is the process of mathematising, i.e. the structuring of reality by mathematical means (Freudenthal, 1991). The aim of this case study is to understand and identify how mathematising emerges while students work on certain tasks of non-obvious mathematical nature.

THEORETICAL FRAMEWORK

Mathematising and modelling

Modelling can be viewed as linking the two sides of mathematics, namely its grounding in aspects of reality - and the development of abstract formal structures (Greer, 1997). In the modelling cycle described by Maaß (2006) (originating from Blum) reality and mathematics are regarded as distinct environments, and the process of modelling includes a number of phases between and within these 'worlds'. The 'step' in which the real-world model is translated into mathematics, leading to a mathematical model of the original situation is regarded as mathematising (Kaiser, 2006).

As working definition, mathematising is denoted here as the activity or process of representing and structuring real world artefacts and/or situations by mathematical means. The overall aim is to enable a logical, traceable and rational treatment of the given artefacts and situations with the help of mathematical knowledge and tools.

Modelling asks for certain cognitive demands, being determined by competencies like designing and applying problem solving strategies, arguing or representing, but it involves also communication skills, as well as real life knowledge (Blum and Borromeo-Ferri, 2007, Kaiser, 2006). Unlike the majority of problem situations,

> modeling activities are inherently social experiences, where students work in small teams to develop a product that is explicitly shareable. Numerous questions, issues, conflicts resolutions, and revisions arise as students develop, assess, and prepare to communicate their products. (English and Doerr, 2004, p. 3)

At the same time, mathematising is part of the modelling process and it is surely not possible to define neatly a border between mathematics and reality. They are interfering and depend on the contextual situation.

> The role of context is very important in mathematical modeling, since modeling requires a context in which to 'frame' the problem and 'develop' the mathematics. (Mousoulides, Sriraman and Christou 2007, p. 29)

According to Freudenthal, mathematising is the human activity consisting in organising matters from reality or mathematical matters, and “there is no mathematics without mathematising”. Later on, Treffers (1987) treated, in an educational context, the idea of two ways of mathematising, which led to a reformulation by Freudenthal in terms of 'horizontal' and 'vertical' mathematisation. In the horizontal mathematisation, mathematical tools are promoted and used to structure and solve a real-life problem, whereas vertical mathematisation supposes reorganisations and operations executed by students within mathematics. Adopting Freudenthal's (1991) formulation, mathematising horizontally means to go from the real world to the world of symbols, while mathematising vertically means to move within the symbols' world.

Maria van den Heuvel-Panhuizen studied the didactical use of models, which in Realistic Mathematics Education (RME) are

> seen as representations of problem situations, which necessarily reflect essential aspects of mathematical concepts and structures that are relevant for the problem situation, but that can have various manifestations. (Maria van den Heuvel-Panhuizen, 2003, p. 13)

Modelling always involves mathematising, which is regarded as the activity of observing, structuring and interpreting the world by means of mathematical models. Since the promotion of critical thinking by students represents one of the main pedagogical aims, “reflexive discussions amongst the students within the modelling process are seen as an indispensable part of the modelling process” (Kaiser and Sriraman, 2006).

Fundamental mathematical ideas

Often, it is no question what "mathematisation" is. If students work in a modelling framework, it cannot be expected that they develop a mathematical idea themselves if they are novices. Since they do not have formed a clear picture of mathematics, it is likely to see elements of different mathematical cultures in their modelling framework: mathematics in every day life or social practice, mathematics as a toolbox for applications, mathematics in school, and mathematics as a science.

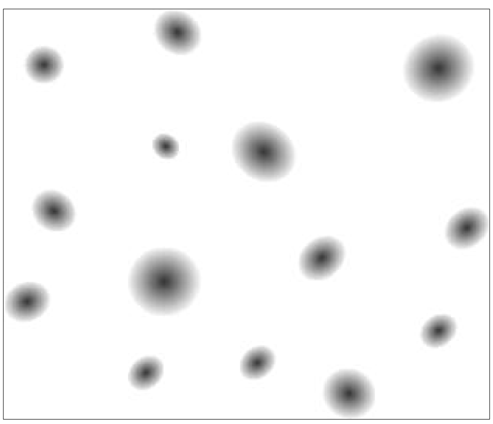
Fundamental ideas in mathematics may serve as a framework in this setting because they connect different mathematical cultures (Schweiger, 2006). Fundamental ideas recur in four dimensions: the historical development of mathematics (time dimension), in different areas of mathematics (horizontal dimension), at different levels (vertical dimension), in everyday activities (human dimension). Schweiger lists a synopsis of fundamental mathematical ideas from different sources: *algorithm, characterisation, combining, designing, exhaustion/approximation, explaining, function, geometrisation, infinity, invariance, linearisation, locating, measuring, modelling, number/counting, optimality, playing, probability, shaping*. Since modelling is discussed in detail and consists of the worked out tasks, it will not be considered in the sequel.

The main aim of this investigation is to see to what extent these fundamental ideas can be recognised in the answers to the rather open-ended Mars task (see next section). The overall pedagogical aim is to design such tasks that students are led to the consideration of fundamental mathematical ideas in a natural way.

EMPIRICAL SETTING

The task of non-obvious mathematical character that students have been given to work out is as follows:

"Imagine you are a scientist at NASA and you have a picture of the planet Mars. This picture shows different spots which indicate craters. These craters were obviously generated by impacts of several meteorites. It is possible that such an impact



generates more craters.

Fig. 1: Picture of the planet Mars depicting a crater

1. Write to a colleague a half-page report about the spots in Figure 1.
2. Describe, respectively label the spots.
3. Find out how the position of each spot could be described.
4. How would you specify the relationships between spots?
5. Could you order the spots by means of mathematical criteria? How?”

The task was given to 13-14 - aged students - in group-work in the classroom environment. Teams of three students were video-taped while working. The present study focuses only on one working group. No intervention from the teacher's side took place, unless students wanted to clarify the formulation of the task.

Data analysis

In the following excerpt, one can see a typical mathematical debate (see Figure 3).

32 J	So, which points are farthest away from each other?... K13 and K2...
33 A	K10 and K11... come, we measure them!
34 J	K13 and K2 are farther away from each other...
35 A	We take the middle point of the crater.
36 F	This is 8...
37 A	7.5...
38 J	They are both 8.
39 F	Where from, do you mean?
40 A	We start from the middle point.
41 F	Yes, I mean... which one do you mean?
42 A	K10 and K11.
43 J	K2 and K13 are a bit more...
44 F	Yes, 0.6cm

The students formulated themselves a small task, generated by the idea of finding 'extreme' points. This yielded the need to measure (line 33, as verifying action), which was not really unproblematic, since the 'spots' are of irregular form. J raised then the idea of comparing, which brought student A to the decision of taking the middle point. That means implicitly that the spots were seen as circles (or even ellipse, though they most probably did not meet it so far as subject in school). The idea of considering the middle point was proposed (line 35), but apparently no attitude was taken by the other two team-colleagues. Nevertheless, the idea was somehow tacitly adopted and they measured (lines 36, 37, 38) distances between points, which involves the assumption that the middle point was taken. In line 40, student A reminded of the middle point, but again no certain remark in this sense was made by his colleagues. However, the idea was carried out, and after some approximation trials (lines 36, 37), they obtained a very small result, namely that K2 and K13 were the searched points; thus, their initial claim was checked.

Another mathematical idea arose when mentioning 'coordinate system' in line 69.

67 J	This is a brilliant idea!
68 A	What?
69 J	This with the coordinate system... It came from me...
70 A	It came from me!!!
71 J	So, if all the points have now to be mapped there... We can write, yes, Z-point, G-point... and then somehow one-two maps... or so
72 A	Do we now want to mark all the points?
73 F&J	No!
74 J	We do just an example. K11 is simple..
75 F	No, also K10... K10 is also simple... that is 1... ehh... 8... 1-8

The students became quite enthusiastic about the idea. They appreciated it as being 'brilliant' and two of the students were almost simultaneously claiming it. They saw this as a mathematical criterion for describing the position of the spots, but the idea offered them an expanded perspective and view, which six minutes later (see next excerpt) brought A (who had the idea, in fact) to the vision of a virtual map.

Further on students wanted to perform measurements, but it was not really clear for them how, probably because the exact corresponding geometrical figure associated to the 'spots' was not explicitly debated and agreed on.

93 A	... Measure the dimension...
94 J	The dimension...
95 F	The dimension...
96 A	We cannot measure that, a crater goes also up... doesn't it? ... and also down...
97 J	We cannot do that, because we have no photo.
98 F	You now want to position this somehow like this (placing the set square perpendicularly on a crater) and measure the dimension?... Now tell me shortly how should we get this?
99 A	... in order to build a virtual map!... Measure the volume and building a virtual map.

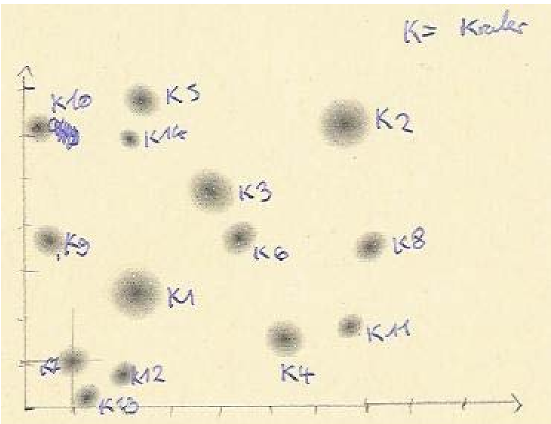
Fig. 2: Student's transfer from 2-D to 3-D

In fact, A had something indefinite in mind, following from the idea of coordinate system, an idea to construct a virtual map, which would have also allowed calculations for the volume of the crater. His colleagues showed themselves quite sceptical with respect to the idea, they faced a lack of understanding of A's mental representation, then the idea vanished, and A did not bring further arguments for sustaining it. The basis of A's idea of coordinate system seemed to be the experience he probably had with the geographical atlas or the moving of chess pieces, which is expressed in the form of horizontal coordinate and its corresponding vertical one. It might also be that A's idea of virtual map is inspired by computer games. Student A realised a mental transfer from 2-D to 3-D, unfortunately without a further development.

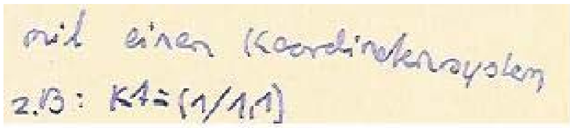
Findings on used mathematical ideas

As main fundamental ideas, approximation, geometrisation, locating, measuring, number/counting, and optimality were observed several times. The students *approximated* the craters with a circle. They did not state it explicitly, but this assumption was carried out during their work. The middle point was only roughly marked. *Geometric* shapes were developed, which helped students to describe the situation. These were not named explicitly, but the concepts of distance, area and volume were used by students. A coordinate system was used to *locate* the craters. Some groups described the location of the craters absolutely, some relative to a coordinate system or somehow relative to other craters in a fancy way. *Measuring* appeared several times and was discussed intensively. The idea was adopted for measuring distances between the craters, as well as the size of the craters. One of the very first actions of the groups was to *count*/order the craters. In the “relations” (see task), *optimality* occurs, e.g. as the closest/furthest distances between craters.

Some students chose to label the spots in a rather mathematical way, as seen in Figure 3. They also had the idea to give the name 'N250i' to a probe, as being the instrument used to observe the given task phenomena, i.e. the generation of the craters.



for describing positions of craters



The task was formulated in such a way that it was not clear to the students what the actual aim of the task was. In fact, one might investigate the given data with different goals in mind. For example, one might want to have an estimate of the number of meteorites creating the craters (on falling apart), which leads to a clustering problem, or one might just want to have a precise map of the craters, yielding a position measuring mainly. It seems that the students interpreted the task as having this latter aim. Basically, there is no 'ideal' solution. The students had to come up with their own interpretation of the goal. The quality of their answers could be judged by the 'depth' of their analysis. The task allows an analysis on different levels of sophistication.

The theoretical framework of Freudenthal, in particular horizontal mathematisation, could be recognised in students' answers, as is apparent from the coordinates they introduced. However, vertical mathematisation, relating these coordinates, for example in a clustering procedure, did not take place. This is probably due to the fact that no specific goal was mentioned, and therefore students were not guided to mathematise in a vertical direction. Their considerations stayed on the level of description.

As characteristic of the modelling processes, the frequent moving between environments, see also (Grigoraş and Halverscheid, 2008), seemed to happen not randomly, but generated by certain 'needs' (e.g. additional data demands). During the discussions towards finding a solution for the problem in a systematic way, students posed questions and set themselves small tasks.

Besides the initial idea of naming the spots, which might not necessarily be a mathematical act, but rather seen as usual labelling (see Figure 3), some students proposed a coordinate system as idea of describing the position of the 'spots', which is a mathematising action. Further on, they started to calculate positions of several 'spots', but finally they decided to give just examples, e.g. 'spot' K10 having 1 as horizontal coordinate and 8 as vertical one (see second excerpt of the previous section). The measuring idea was also found in their talks, and students debated on it for some time, while trying to find out which spots are in extremal position. These acts count as at least two mathematisation achievements.

In this case, but also in several previous surveys carried out on tasks without numbers, it was seen that many fundamental ideas occurred as mathematisation acts. However, not all of these ideas lead to intensive mathematical modelling activities. As for the task discussed here, deeper mathematical activities were started concerning measuring and optimality. The students proceeded by taking the set square or ruler and measured the distance between the 'spots', whereas for the other mentioned fundamental ideas no mathematical activities were performed. Students also handled the approximation of spots by circles in a mathematical manner, and measured distances between them by taking the middle points of the 'spots'. Ideas like radius, circumference, volume (even unclear whether applicable in this case) completed the mathematical 'picture' of what students built up around the 'spots'.

A finer look at the last excerpt

The situation may seem at a first sight somehow simplistic, such as students using mathematics when working out tasks without numbers. But it is interesting to examine the subtle aspects and reasons behind the usage of mathematics. This is done here with respect to the question whether and how the need of mathematising occurs.

It should be remarked that simplification in various forms (schema, drawings, etc.) is a characteristic of modelling itself. For students in their age, modelling rests on the principle of representing a situation type in a simplified, general manner which allows extended applications.

While tackling the task of finding mathematical criteria, the students approached ideas one by one like finding distances (they coped well with planning and measuring distances between extremal 'spots'), coordinate system, then finally the volume, then they stopped doing further things, since their 'tools' for calculating things were not sufficient. Mathematical concepts used by students - *distance, area* (implicit, through the coordinate system and middle point of a plane figure), and *volume* originated from a need of simplification of the initially given problem. Therefore the approximation of spots by circles was done, though never explicitly stated.

The idea of finding distances between extremal 'spots' was conducted through measuring, since students found something they could do. Somewhat further, they came to the idea of coordinate system, by which they were quite absorbed and dealt successfully with in the 2-D situation. Then the *dimension* was mentioned, but the students faced up to some problems with the data, that seemed not to suffice (line 97 in the transcript). Once they met this data demand, students were confronted with an unclear situation of the model, since they did not know which mathematical object would fit to that stage, where a virtual map was proposed. At that point, their debate stopped, hence no simplification was achieved. Therefore the 3-D situation failed, because of a lack of tools and/or data.

DISCUSSION

It is intriguing in this case to study how fundamental mathematical ideas occurred through mathematising. There were fundamental ideas leading to the model (biggest, smallest, extreme, measure). But how did students build a modelling idea? It seems they looked in mathematics for 'tools' which would allow them to work out a model.

When discussing the real situation, students emitted sometimes mathematical ideas, e.g. the idea of measuring, which means that a transfer to mathematics took place. There then were two possibilities: either they remained within mathematics and performed further on, or they turned back to the real situation. Such frequent forth-back transitions are analysed in (Borromeo-Ferri, 2007). The decision (often unconsciously taken) whether to stay or not within mathematics seemed to be influenced and caused by a number of factors, as described in the following. We refer to the real situation as being *data situation*, as the existing task formulation students have at their disposal. When being situated in mathematics, there could be *tools*, *knowledge*, *experience*, *motivation*, among others, all of these determining whether students stayed and worked with and within mathematics, or they turned again to data and tried to handle them and searched for next steps. If one or more of these items were missing and students were facing a dilemma in performing further on with and within mathematics, then they came back to the data.

The analysis of the present study showed that while acquiring a real modelling experience, students produced many fundamental mathematical ideas, but when confronted with a lack of tools, knowledge, or even experience, their activities stopped at the level of ideas' supply.

As for a future analysis, a first hypothesis is that a modelling task developes through fundamental ideas. A second hypothesis is that mathematics is reached by means of assumptions, which students proposed and agreed on while taking decisions during solving a modelling problem. It would be interesting to examine how mathematising differs according to the mathematical nature of the task formulation.

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MODELLING ACTIVITIES WHILE DOING EXPERIMENTS TO DISCOVER THE CONCEPT OF VARIABLE

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Physical experiments have a great potential in math lessons. Students discover the aspects of the concept of variable and while doing that run through the whole modelling cycle. In this paper we show how physical experiments can contribute to the modelling activities and the concept of variable and how scientific issues influence the students' conceptions based on interviews with them.

MODELLING AND PHYSICAL EXPERIMENTS

In the PISA framework the authors emphasize the functional use of mathematics. Students should discover problems, formulate them and should then be able to solve and interpret them. While doing that different mathematical contents and competencies are activated. One of these competencies is modelling, which has a central place within the framework:

> This involves structuring the field or situation to be modelled; translating reality into mathematical structures; interpreting mathematical models in terms of reality; working with a mathematical model; validating the model; reflecting, analysing and offering a critique of a model and its results; communicating about the model and its results (including the limitations of such results); and monitoring and controlling the modelling process.

Mathematics is a tool often used in real world and in Science. The role of mathematics is predominantly brought through the building, employment and assessment of mathematical models (Michelsen, 2006).

How can physical experiments contribute to modelling activities in math lessons? If you look at the different steps mentioned above, physical experiments have a great potential. The experiments are derived from a phenomenon of everyday life and represent an idealized setting, considering certain factors only. Students doing physical experiments work with concrete terms. These terms are in a functional relationship with each other. If you want to describe them in a quantitative way you have to translate that relationship into mathematical structures. All kind of representations (graph, tables, etc.) can be applied. Students have to communicate about the phenomenon and the correspondent formula. A modelled formula can be checked directly through the measuring values and by new measurements. Because of measurement errors the formula is never correct. So it is natural to talk about the correctness and the limitations of the model and its results. If one slightly changes the setting of the experiment, the formula might change. Hence there is a strong emphasis on the validation process which often plays a minor role in the modelling process.

THE CONCEPT OF VARIABLE – ASPECTS

Malle (1986) differentiates three different aspects of variables:

- Variable as an object (Gegenstandsaspekt)

  A variable stands for an unknown item or an unknown object.

- Placeholder aspect (Einsetzaspekt)

  A variable means a placeholder, which you can substitute through a number.

- Calculational aspect (Kalkülaspekt)

  A variable stands for a meaningless symbol, with which you can apply certain rules.

He differentiates variable as an object into single number aspect and interval aspect. Single number aspect means an arbitrary but fixed number within a given domain. Variables which match to the interval aspect represent the whole domain. Within that interval aspect it can be differentiated between simultaneous aspect (representation at the same time) or changing aspect (representation in succession).

On the other hand variable as an object can be classified by a dynamic and a static component. Dynamic component means a changing number and static component means a specific unknown, i.e. it might change in another context.

If you compare the decomposition of the concept of variable according to Trigueros et. al (1996) into generalized number (representing a general entity, which can assume any value), specific constant (representing a constant value, which might change in another situation) and variable in a functional relationship, generalized number can be attributed to variable as an object, which can be represented at the same time or in succession. Specific constant is equivalent to the static component. To conceptualize variables in a functional relationship, knowledge of dynamic and static components is needed.

Malle demands among other things that in the beginning emphasis should be put to variable as an object and to the conception and interpretation of formulas.

THE CONCEPT OF VARIABLE AND PHYSICAL EXPERIMENTS

Michelsen (2006) proposes that by expanding the domain, mathematical concepts can be developed in a more practical and coherent structure, since

> the student's conceptions of a mathematical concept is determined by the set of specific domains in which that concept has been introduced for the student.

If students do physical experiments they can identify variables with concrete terms. That's why these variables can be classified to the aspect variable as an object. Students' major problem seeing variables as symbols to be manipulated (Schoenfeld & Arcavi, 1988) can therefore be diminished. Both dynamic and static components are touched since the values of the measurands change with each new measurement and the (anti-)proportional constant is constant in the same context. The possible values of the measurands determine the domain of the corresponding variable. The (anti-)proportional constants mostly are representatives of a discrete set.

By experimenting, students can discover the aspects of the concept of variable *before* they are properly defined in class. This is in accordance to Freudenthal's philosophy that context problems and real life problems are used to constitute and apply mathematical concepts. The aspects don't have to be touched in the abstract level at once. If they are touched on a descriptive level that can be enough. One example is the functional relationship of two measurands. While doing the experiment, students actively discover that change of one measurand causes a change of the other measurand. Especially, weaker students have problems to interpret this into a formula. But if you present a formula and give further explanations after the experiments the formula will not seem that abstract anymore because they can identify the formula with their experiences made while doing the experiment.

PHYSICAL EXPERIMENTS IN MATHEMATICS LESSON

In the above sections a few advantages and commonalities have been shown. But there are also subject specific characteristics, which have to be taken into consideration. In physics math is mostly seen as a tool for describing phenomena in a quantitative way. On the other hand mathematicians don't care how data was gained in detail; their only interest is the correctness of that data. Algebra is a correct theory. Experiments are never exact, because measurement errors always occur, even if they are very small. If you want to find relationships those measurement errors have to be kept in mind. School physics shows that all the time; school math only in a few fields. Therefore students have to be prepared to handle measurement errors.

If one wants to use experiments for mathematical concepts, emphasis should be given to the common and mathematical aspects. That means

- Experiments should have an easy setting

  Mathematics' interest is data and not how to get data. Therefore the experiment should be done with few materials and measured quickly allowing students to concentrate more on the math.

- Intervals of measurement errors should be small

  To find the relationship between the measurands quickly, there should be (if possible) no chances for systematic measurement errors and small intervals for random ones.

- The physical terms should be familiar to the students

This doesn't mean that physical terms not covered in physics class are forbidden. It is legitimate to use terms which are familiar in every-day life, like pressure, volume, temperature and so on.

- The interval of the measurands should be suitable

  Especially in experiments which contain an antiproportional relationship intervals should be chosen where the constant product stands out. Otherwise students may see (with consideration of measurement errors) a proportional relationship.

Physical experiments can be used within interdisciplinary lessons. This can be in separate classes, i.e. each class covers subject specific aspects; or for a short period in a common class in which all aspects are covered. On overview of different forms of cooperation can be found in Beckmann (2003, p. 9ff).

CONCEPTUALISATION IN SCHOOL

The use of experiments to introduce the concept of variable has been tested on 90 students of 7th grade in three different schools. They were required to do three out of five physical experiments. After the experiments, the concepts of variable and term were introduced formally. This was done to see if physical experiments can be applied in class and which experiments are appropriate. To get a deeper insight of the concept of variable and of reflection and validation of their modelled formula, another examination was done in spring 2008. 18 Students of 6th grade attending a German Gymnasium were required to do one experiment out of three working in groups of two. These 18 students knew the placeholder aspect, i.e. variables can be substituted by numbers, and that variables stand for a number which is unknown and changes continuously. Theoretical knowledge of the concept of variable concerning the object aspect hasn't taught yet. While doing the experiments, they were observed by students of the University of Education Schwaebisch Gmuend. After the experiments the 6th graders were interviewed by the students. The main research questions covered the aspects of the concept of variable touched by the experiment and of how convinced the students were of the formula found. The second question is to determine students' abilities to reflect and validate their results. Problem oriented interviews were chosen, so students could talk freely and were only slightly guided by the interviewers through open questions. The interviews were transcribed. Emotional factors like emphasizing words etc. were not considered during the process of transcription. Students' answers were categorized in the different aspects of variable and how they reflected the validity of their modelled formula.

The following experiments were done by the students:

- Buoyancy

The students measure the force of different masses in air and in water and conceptualize a formula which describes a proportional relationship between the forces in air and in water.

- Thermal expansion of a liquid

  The students measure the heights of an uncalibrated thermometer at different temperatures. Then they conceptualize a formula which describes a proportional relationship between difference of heights and difference of temperatures.

- Law of Boyle-Mariotte

  The students measure the pressure as well as position or volume of a piston. Then they conceptualize a formula which describes an antiproportional relationship between pressure respective to position or volume.

The design of the instructional sheets allows students to work by themselves. Assistance is only given, if students are at a loss and if tasks are essential for the following tasks. In that case hints were given and written down for consideration of students' results. No solution of tasks was given to the students.

The instructional sheets start with an impulse from real life. It shall motivate the students towards the experiment and shall put the experiment in a real setting. Through measuring different measurands students shall qualitatively experience the functional relationship of the two measurands. After measuring at least six different values, students are asked to describe the relationship first in their own words and then through a formula. This formula shall then be used to calculate measurands. These values shall be checked by looking at the values they measured before. This is to reflect their formula found. After that there follow questions concerning the domain of the variables and their properties. To touch the specific constant and change of formula in different contexts they were asked how the formula changes if one alters the setting of the experiment followed by a question for a more general formula. In the three classes students had to write a protocol containing the most important aspects. The 6th graders didn't have to write a protocol since they were interviewed after the experiment.

RESULTS

Concept of variable

Variable as an object according to Malle is touched. Students can identify the measurands with their chosen variables. A few examples:

Buoyancy experiment:

I2: Can you tell how you recognize (the experiment in your formula)?

S6: yes, you see the statement for air and for water. And yes the result, yes…

Here the group chose word variables. If they didn't choose words they chose the units of the measurands.

Boyle-Mariotte experiment

I1: what are those cm? What do they stand for?

S1: mmh here at that strip for example 6cm

I1: mmhmm

S1: so for the respective number

I1: and the x?

S1: for the respective pressure

Here the student chose the units of the physical terms as the name of his variables. Since he didn't know the unit of pressure, he chose x.

The functional relationship between the two variables has been recognized by the students both statically and dynamically.

Buoyancy experiment

S4: Then we agreed that if you divide air by water, the result is always the same. It doesn't matter, if there are 1, 2,3,4,5 cylinders. The result is always 1.2.

Boyle-Mariotte experiment

I1: What have you found out?

S1: yes, that device. If you turn further that thing moves forward and the further it moves the measuring number gets smaller and the pressure gets higher.

Thermal expansion experiment

S17: We had to find formulas. These were height times x is difference of temperature and difference of temperature divided by x is then height and difference of temperature divided by height is then x.

[…]

I8: and what changes in general in your formula?

S17: temperature and the head of liquid there, both get higher the more water you add.

Modelling process

Students went through the first part of the modelling cycle by examining the phenomenon and structuring it in a formula. That has been done on different levels. Weaker students could only explain in their own words and the strongest students have even presented three equivalent formulas.

To check how they reflected their formula students were asked if their findings are valid, since their measured values and the corresponding quotients/products weren't constant. They differed in small intervals due to random measurement errors. Before the experiments began, the instructor told the students that one could never measure exactly and that they had to keep that in mind. That is not easy as the following example shows:

Buoyancy experiment

S6: In the beginning I thought that I had to take the numbers which we had measured and then I thought for a longer time, if that was right.

After they accepted the influence of measurement errors, they rounded the quotients and then all but one were constant. Then they were convinced about the constant quotient.

I2: Did you notice anything about your result? In the case of normal water and air?

S5: Yes, the result was always 1.1; always the same.

They are convinced of the correctness of their formula because they have actively experienced that their results weren't always correct, but close to the "correct" answer.

Boyle-Mariotte experiment

S2: they aren't that correct.

I1: But the formula, that you have written down, is exact, isn't it?

S2: Well not that exact. It is… It could be also 7.1 instead of seven.

I1: Would you say your relationship is valid or your relationship is wrong?

S2: I would say, the relationship is valid, because with this device you can't determine that number that exactly. And the numbers I have written down, are actually as exact as possibly can be done with this device.

The use of experiments stimulates one to critically review the results and actively discuss the validity of the formulas found. Some students tend to extrapolate their formula after measuring a few values.

Buoyancy experiment

S3: (constant quotient) It is actually with all numbers! With six it is the same.

S4: That I don't know. You can't say … You don't know, what is with six. We haven't done that.

S3: Yes, but with 1 and 2 it is same, too.

But student 3 would not be convinced anymore, when one measures other values.

S3: and if we were to repeat that and would get other results, then we would be in a fix and wouldn't know what would be right.

But the more students measure the more convinced they are about their formula.

Thermal expansion experiment:

I8: You have found a formula, if a teacher comes to you […] and says your formula is wrong, would you say your formula is wrong or your formula is right?

S17: Yes, I think it is right, because of the different experiments we have done. Well with the different degrees and with the table at the beginning. We have measured the head of liquid and the difference of temperature six times and that was true all the time.

Hence physical experiments stimulate discussion about the formula. Reflection and validation of their formula is promoted.

Static component of variable & limit of modelled formula

If you ask students about the specific constant, you implicitly ask about the limit of their formula found. The (anti-)proportional constant is only constant in the same context. Changes of the context might cause a change of those constants. In the experiments, students were asked if the formula changes when you change the setting. In the buoyancy experiment, they were shown a man reading newspaper in the Dead Sea and asked if and how their formula would change, if they did the same experiment with salt water. Students doing the thermal expansion experiment were asked, if they were to change the thermometer, would that cause a change of their formula. In the Boyle-Mariotte experiment, students were asked if changes in the environment would cause changes of the formula.

Most of the students say that the formula changes and explain it on a descriptive level. Stronger students can tell which part changes while the strongest students set up a general formula.

Thermal expansion experiment

S10: We have found out that, if the glass tube is thicker, then it raises slower and if the glass tube is thinner, the liquid raises faster.

Buoyancy experiment

I3: Good. Is there a term, which doesn't change? Or changes everything?

S3: I think, if you stay in normal water, then it is always 1.2.

S4: Well, once you add a liquid, it will be heavier

S3: Yes, salt water or – then

S4: is, I think, heavier.

S3: Then 1.2 will be

S4: bigger.

Boyle-Mariotte experiment

I1: If I change anything on this device, how would your position and pressure change?

S1: mmhmm. Well, I think. Well, position will be the same but pressure will change.

Thermal expansion experiment

S17: If you change the glass tube, that means making them wider or yes thicker or thinner, then the constant changes. Otherwise it stays constant with the same glass tube.

I8: How does it change if you have a wider or thinner …?

S17: There it changes, well with a thinner, when it gets thinner, then the constant will get higher and when it gets thicker, then it will get lower. […]

That x is the constant, well, in our experiment it was six und it can change when the glass tube gets thicker or thinner.

As you have seen, questions about the specific constant have a great potential for discussion about the limits of a given model.

CONCLUSION

The use of physical experiments to introduce the concept of variable is as well a good way to promote the modelling process.

All of the aspects of variable as an object according to Malle are touched, especially within the functional relationship of two measurands. Formulas make sense to them, because they can identify variables with concrete objects. Not everybody touches the aspects on an abstract level but most do on a descriptive level. In the lessons afterwards those students will have fewer problems to understand abstract formulas because they can make connections to those experiments.

Like Maass (2006) found out, that students of lower secondary level were able to develop modelling competencies. Physical experiments can contribute to those competencies since the complete modelling cycle is covered. Especially “reflecting, analysing and offering a critique of a model and its results” has a main role in that concept. That is mainly through the appearance of measurement errors. Students learn that the modelled formula is an idealization, but still a good representation of the phenomenon. They can discover the limit of the formula found by scenarios they can imagine. Experimenting in groups stimulates the discussion about the model.

Students are motivated to do experiments, but finding a formula is a cognitive challenge. That's why students might get frustrated. A working sheet covering all aspects on the concept of variable and modelling on a descriptive level, i.e. without students coming up with a formula by themselves, would be better in cognitively weaker classes. If one stays on the descriptive level major phases of the modelling process are still touched. Then emphasis goes even more to analysing and criticizing the model.

This sequence is also a good basis for interdisciplinary teaching to see the same phenomenon with "subject driven" eyes. An overview gives the framework "Math and Science under one roof" which can be found on the homepage of the EU ScienceMath Project http://www.sciencemath.ph-gmuend.de.

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MODELING WITH TECHNOLOGY IN ELEMENTARY CLASSROOMS

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In this study we report on an analysis of the mathematical developments of twenty two 11 year old students as they worked on a complex environmental modeling problem. The activity required students to analyse a real-world situation based on the water shortage problem in Cyprus using Google Earth and spreadsheet software, to pose and test conjectures, to compare alternatives, and to construct models that are generalizable and re-usable. Results provide evidence that students successfully used the available tools in constructing models for solving the environmental problem. Students' mathematical developments included creating models for selecting the best place to supply Cyprus with water, finding and relating variant and invariant measures such as tanker capacity, oil consumption, and water price. Finally, implications for further research are discussed.

Keywords: Modeling, technological tools, environmental modeling problem.

INTRODUCTION

The importance of modeling and applications has been well documented and a significant number of researchers discussed the impact of modeling in the teaching and learning of mathematics (Pollak, 1970; Blum & Niss, 1991; Lesh & Doerr, 2003). Additionally, professional organizations, like the National Council of Teachers Mathematics (NCTM, 2000), recommended that the inclusion of real world based problems in the curriculum can capture students' interest and students will gain mathematical problem solving skills, as well as an appreciation of the power of mathematics and some essential mathematical concepts and skills (NCTM, 2000).

Students, even at the elementary school level, need to be able to successfully work with complex systems that daily appear to the mass media (English, 2006). More than ever before, the nature of the mathematical problem-solving experiences has to be changed, if we want to prepare students to adequately deal with the complexity of the rapidly changing world (English, 2006; Lesh & Zawojewski, 2007). Traditional forms of problem solving constrain opportunities for students to explore complex, messy, real-world data and to generate their own constructs and processes for solving authentic problems (Kaiser & Sriraman, 2006). In contrast, mathematical modeling provides rich opportunities for students to experience complex data within challenging, yet meaningful contexts. Students' interactions within these experiences can assist them in building mathematical understandings and in developing their problem solving skills (Mousoulides & English, 2008).

In this attempt, given the potential value of technology for enhancing learning, it is imperative that students undertake realistic modeling problems and appropriately use technological tools for developing their ideas about and their understandings of related mathematical concepts (Mousoulides, Sriraman, & Lesh, 2008; Mousoulides, 2007). Although the increased interest on modeling and applications, even at the elementary school level, only a limited number of researchers focused their agendas on investigating the role of technology in mathematical modeling, on exploring how spreadsheets are used in constructing models (Blomhøj, 1993; Mousoulides et al., 2008), and on identifying how dynamic geometry software features might influence the modeling process (Christou et al., 2005).

This paper reports on the mathematical developments of one class of eleven year old students, as they worked on an environmental modeling problem that involved interpreting a real world situation and dealing with digital maps, tracing ship routes, working with tables of data, exploring relationships among data, and representing findings in visual and written forms. We were particularly interested in exploring the ways in which the students used the available tools (Google Earth and spreadsheets) in constructing the necessary mathematical developments for solving the problem.

MATHEMATICAL MODELING AND TECHNOLOGY IN THE ELEMENTARY SCHOOL

Mathematical models and modeling have been defined variously in the literature (e.g., Greer, 1997; Lesh & Doerr, 2003). In this paper, models are defined as "systems of elements, operations, relationships, and rules that can be used to describe, explain, or predict the behavior of some other familiar system" (Doerr & English, 2003, p.112). A definition of modeling, as a problem solving approach, is presented in Lesh and Zawojewski (2007): "A task, or goal-directed activity, becomes a problem (or problematic) when the "problem solver" (which may be a collaborating group of specialists) needs to develop a more productive way of thinking about the given situation" (p. 782).

Research studies have shown that mathematical modeling can be considered as an effective medium to improve students' problem solving abilities in working with unfamiliar complex real world situations by thinking flexibly and creatively (Haines, Galbraith, Blum, & Khan, 2007; English, 2006). One approach to having students solve complex problems is through team oriented activities, called model eliciting activities (MEAs). These activities are based upon the models and modeling perspective (Lesh & Doerr, 2003), and they are designed to document students' thinking. MEAs, therefore, provide an ideal setting to assess the knowledge and the abilities that students express during the modeling process (Lesh & Doerr, 2003). MEAs usually consist of three sessions. The first session provides the problem statement and introduces students to the modeling activity. Students define for themselves the problem, assess the problem situation and create a plan of action to successfully solve the problem. During the problem solving session of the modeling problem students work in small groups and go through multiple iterations of testing and revising their solution(s) to ensure that their solution(s) is the best possible for the problem situation. In the third session of the modeling activity each group of students present their solution(s) to the rest of the class for constructive feedback and discussion of the mathematical ideas presented in the modeling activity (Mousoulides, 2007; Lesh & Doerr, 2003).

Modeling activities, set within authentic contexts, engage students in mathematical thinking that extends beyond the traditional curriculum, as they embed the important mathematical processes within the problem context and students elicit these as they work the problem (English, 2006). Problems presented in modeling activities are not carefully mathematized for the students, and therefore students have to unmask the mathematics by mapping the problem information in such a way as to produce an answer using familiar quantities and basic operations (English, 2006). The problems necessitate the use of important, yet underrepresented in traditional mathematical curriculum, mathematical processes such as constructing, describing, explaining, predicting, and representing, together with quantifying, coordinating, and organizing data (Mousoulides, 2007). Key mathematical ideas that appear in the modeling problems can be accessed at different levels of sophistication and therefore all students through questions, revisions and communication can have access to the important modeling and mathematical content. This can result in improving competencies in using mathematics to solve problems beyond the classroom (English, 2006; Kaiser & Sriraman, 2006; Mousoulides et al., 2008).

Recent research studies focusing on mathematical modeling at the elementary school level indicated that students can build on their existing knowledge and develop their mathematical ideas and modeling competencies that they would not meet in the traditional school curriculum (English, 2006; Mousoulides & English, 2008). Students' informal knowledge and ideas assist students in understanding the problem presented in the modeling activity, in identifying variables and constrains, and in building mathematical models for solving the modeling problem (Mousoulides, 2007). The framework of modeling activities does not narrow students' work in only performing calculations or working with ready made models; on the contrary, students need to construct models in a meaningful way for solving a real problem and this approach can lead to conceptual understanding and mathematization (Greer, 1997; Mousoulides et al., 2008; Mousoulides & English, 2008). Conceptual understanding was also reported as students worked in modeling activities in exploring quantitative relationships and in comparing varying rates of change (Doerr & English, 2003), in probabilistic reasoning (English, 2006), and in geometric reasoning and spatial abilities (Mousoulides et al., 2006).

The availability of technological tools is one factor that might influence students' work and outcomes in working with modeling activities (Mousoulides et al., 2006). Recent research studies indicate that appropriate use of technological tools can enhance students' work and therefore result in better models and solutions. In Blomhøj's (1993) research, students successfully used a specially designed spreadsheet for setting models and for expressing relations between variables in spreadsheet notation. More recently, Mousoulides (2007) reported that school and undergraduate students successfully used spreadsheets in developing simple and more complex models for connecting the real world problem with the mathematical world. The contribution of technological tools in modeling problems was also examined in the areas of geometry and spatial geometry. Christou and colleagues (2005) reported that students, using a dynamic geometry package, modelled and mathematized a real world problem, and utilized the dragging features of the software for verifying and documenting their results. In line with previous findings, Mousoulides and colleagues (2007) reported that students' work with a spatial geometry software broadened students' explorations and visualization skills through the process of constructing visual images and these explorations assisted students in reaching models and solutions that they could not probably do without using the software. As a concluding point, it is important to underline that the inclusion of appropriate software in modeling activities can provide a pathway in better understanding how students approach a real world problem and how they might develop technology-based solutions for these problems.

THE PRESENT STUDY

Participants and Procedures

One class of 22 eleven year olds and their teacher worked on an environmental modeling problem as part of a longitudinal study, which focuses on exploring students' development of models and processes in working with modeling problems. The students are from a public K-6 elementary school in the urban area of a major city in Cyprus. The students only met such modeling problems before during their participation in the current project, as the mathematics curriculum in Cyprus rarely includes any modelling activities. Students were quite familiar in working in groups for solving more complex problems than those appear in their mathematics textbooks. However, this was the first time students had the opportunity to work with spreadsheets and Google Earth for solving a real world modeling problem.

The data reported here are drawn from the problem activities the students completed during the first year of the project. The *Water Shortage* modeling problem (appears in the appendix) entails: (a) a warm-up task comprising a mathematically rich "newspaper article" designed to familiarize the students with the context of the modeling activity, (b) "readiness" questions to be answered about the article, and (c) the problem to be solved, including the tables of data (see Table 1). This environmental modeling problem presented in the activity asked from students to help the local authorities in finding the best country for supplying Cyprus with water. Water shortage is one of the biggest problems Cyprus face these days. As a result, students were very familiar with the problem, since almost everyday there are discussions on TV about the possible solutions to the problem.

Table 1: The Water Shortage Problem Data

Country	Water Supply per week (metric tons)	Water Price (metric ton)	Tanker Capacity	Oil cost per 100 km	Port Facilities for Tankers
Egypt	3 000 000	€ 3.50	30 000	€ 20 000	Good
Greece	4 000 000	€ 2.00	50 000	€ 25 000	Very Good
Lebanon	2 000 000	€ 4.00	50 000	€ 25 000	Average

The problem was implemented by the authors and the classroom teacher. Working in groups of three to four, the children spent five 40-minute sessions on the activity. During the first two sessions the children worked on the newspaper article and the readiness questions and familiarize themselves with the Google Earth and spreadsheet software. Introduction to Google Earth focused on the following commands: “Fly to” for visiting a place, “Add Placemark” and “Ruler” for calculating the distance between two points, and “Path” for drawing a path between two points. In contrast to regular maps, Google Earth can help students in making accurate calculations, being more precise in drawing the tanker routes, in “visiting” the different countries for exploring their major ports and finally in observing country’s landscape. In the next three sessions the children developed their models, wrote letters to local authorities, explaining and documenting their models/solutions, and presented their work to the class for questioning and constructive feedback. A class discussion followed that focused on the key mathematical ideas and relationships students had generated.

Data Sources and Analysis

The data sources were collected through audio- and video-tapes of the students’ responses to the modeling activity, together with the Google Earth and spreadsheet files, student worksheets and researchers’ field notes. Data were analysed using interpretative techniques (Miles & Huberman, 1994) to identify developments in the model creations with respect to the ways in which the students: (a) interpreted and understood the problem, (b) used and interacted with the software capabilities and features in solving the environmental problem, and (c) selected and categorized the data sets, used digital maps and applied mathematical operations in transforming data. In the next section we summarize the model creations of the student groups in solving the *Water Shortage* activity.

RESULTS AND DISCUSSION

Group A Model Creations

Group A started their exploration by visiting Lebanon, a nearby country, using the “Fly to” command. This approach helped students in identifying that there were many mountains and therefore Lebanon could supply Cyprus with water. In their final report, students documented that: “Lebanon has a high percentage of precipitation, because there are many mountains there. So, they will probably sell water to Cyprus”. They then “zoom in” for finding a port. They decided that Tripoli was a major port and their next step was to add a placemark to Tripoli. Students then “zoom out” from Lebanon and gradually moved to the west for finding Cyprus. Students in group A directly focused on Limassol, the major port in Cyprus and added a second placemark. Group A then used the “ruler” feature of the software for calculating the distance between Tripoli and Limassol.

Students followed the same approach for placing placemarks in Pireus (in Greece) and Cairo (Egypt), and for finding the distances between Cyprus and the other three countries. Since the data table (see Appendix) was supplied in spreadsheet software, students added one column presenting the distances between the three different countries and Cyprus. Students explicitly discussed about oil price, and they reached the conclusion that buying water from Greece would be more expensive than buying water from Lebanon or Egypt due to the greater distance between Greece and Cyprus. Students, however, failed to successfully use the provided data and they finally based their choice (Lebanon) partly on the provided data and on their calculations, without providing a coherent model.

Group B Model Creations

Similar to the work of Group A, students in this group quite easily visited the three countries and added placemarks in their major ports. They drew precise paths between each country’s port and Limassol and used ruler to calculate the distances (see Figure 1). They reported that: “It is not easy to decide from which country Cyprus should buy water. Lebanon for example is closer than Greece, but water from Greece is much cheaper than water from Lebanon. After calculating the distances between the countries using Google Earth, they moved into the spreadsheet software and added one column in the provided table, presenting the distances. They, however, failed to incorporate into their model the provided data about oil cost, tanker capacity and water price.

Group C Model Creations

This group commenced the problem by finding a major port in each one of the three countries and by drawing paths from these ports to Limassol. Students in this group then calculated the distances between the ports and continued in calculating oil and water cost for *each* tanker trip. In contrast to Groups A and B, students in this group incorporated within their model one more factor; instead of calculating the total cost for each trip and then ranking the three countries, they decided to calculate the cost

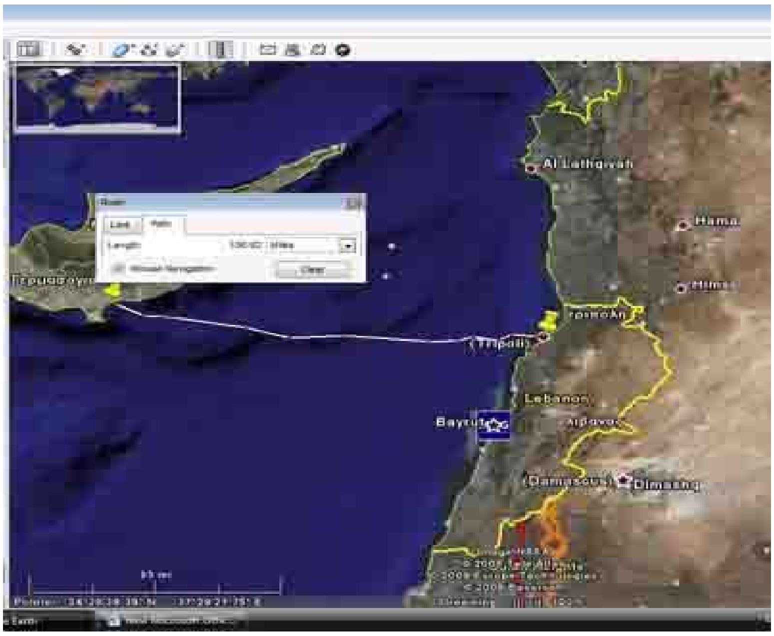

Figure 1: Finding the distance between Tripoli and Limassol.

per water metric ton and based their ranking on this factor. As a result, this model ranked Lebanon as the best possible choice, since the average cost per water ton was only €4.20. On the contrary, the average costs for Egypt and Greece were €6.70 and €7.00 respectively. Student calculations and final selection are presented in Table 2.

Table 2: Group C calculations and final model

Country	Distance	Oil cost	Water cost per tanker	Total cost	Average water cost per ton
Egypt	480	€ 96000	€ 105000	€ 201000	€ 6.70
Greece	1100	€ 275000	€ 75000	€ 350000	€ 7.00
Lebanon	240	€ 60000	€ 150000	€ 210000	€ 4.20

Although this group differed from other groups in that they used a more refined model, they also failed to apply in their model factors such as port facilities for tankers and each country's resources for supplying water to Cyprus. Students in this group, similar to group A and B did not use in their calculations round trips but they rather based their calculations on single trips.

Remaining Groups' Model Creations

Students in the remaining four groups faced a number of difficulties in ranking the different countries. In the first component of the problem, using Google Earth for finding appropriate ports and calculating the distances between Cyprus and the three countries, two groups focused their efforts only on Greece, by finding the distance between Pireus and Limassol. Some other groups faced a number of difficulties in using the software itself.

In the second component of the problem, transferring the distance measurements in the spreadsheet software and calculating the different costs, the students faced more difficulties. Most of their approaches to problem solution were not successful. Many students, for example, just made random calculations, using partially the provided data, and finally making a number of data misinterpretations. One group, for example reported that buying water from Greece is the best solution, since the water price per ton from Greece was only €2.00 (see Table 1).

CONCLUDING POINTS

There are a number of aspects of this study that have particular significance for the use of modeling in mathematical problem solving in elementary school mathematics. First, although a number of students in the present study experienced some difficulties in solving the problem, elementary school students can successfully participate and satisfactorily solve complex environmental modeling problems when presented as meaningful, real-world case studies. Second, our findings show that the available software broadened students' explorations and visualization skills through the process of constructing visual images to analyze the problem, and by using appropriately the spreadsheet's formulas they performed quite complex calculations.

The students' models varied in the number of problem factors they took into consideration. Interestingly, at least three groups succeeded in identifying dependent and independent variables for inclusion in an algebraic model and in representing elements mathematically so formulae can be applied. A number of groups of students made the relevant assumptions for simplifying the problem and ranking the three countries. Finally, the first three groups (as presented in the results session) successfully chose the technological tools/mathematical tables to make precise graphical models in Google Earth and to enable calculations in spreadsheets.

Substantial more research is clearly needed in the design and implementation of technology-based modeling problems and in studying the learning generated. Of interest are, for example, the developments in elementary school students' learning in solving technology-based modeling problems, the ways in which the features of the technological tools can assist students in broadening their explorations and in constructing better models for solving modeling problems, and the teacher professional development training programs that are needed to facilitate mathematical modeling as a problem solving. In concluding, using computer based learning environments for mathematical modeling, at the school level, are a seductive notion in mathematics education. However, further research towards the investigation of their role is needed, to promote both students' conceptual understandings and mathematical developments.

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APPENDIX

Water Shortage Problem: Cyprus will buy Water from Nearby Countries

Background Information: One of the biggest problems that Cyprus face nowadays is the water shortage problem. Instead of constructing new desalination plants, local authorities decided to use oil tankers for importing water from other countries. Lebanon, Greece and Egypt expressed their willingness to supply Cyprus with water. Local authorities have received information about the water price, how much water they can supply Cyprus with during summer, tanker oil cost and the port facilities.

Problem: The local authorities need to decide from which country Cyprus will import water for the next summer period. Using the information provided, assist the local authorities in making the best possible choice. Write a letter explaining the method you used to make your decision so that they can use your method for selecting the best available option (The following **table** was supplied).

Country	Water Supply per week (metric tons)	Water Price (metric ton)	Tanker Capacity	Oil cost per 100 km	Port Facilities for Tankers
Egypt	3 000 000	€ 3.50	30 000	€ 20000	Good
Greece	4 000 000	€ 2.00	50 000	€ 25000	Very Good
Lebanon	2 000 000	€ 4.00	50 000	€ 25000	Average