MODELLING ACTIVITIES WHILE DOING EXPERIMENTS TO DISCOVER THE CONCEPT OF VARIABLE

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Physical experiments have a great potential in math lessons. Students discover the aspects of the concept of variable and while doing that run through the whole modelling cycle. In this paper we show how physical experiments can contribute to the modelling activities and the concept of variable and how scientific issues influence the students' conceptions based on interviews with them.

MODELLING AND PHYSICAL EXPERIMENTS

In the PISA framework the authors emphasize the functional use of mathematics. Students should discover problems, formulate them and should then be able to solve and interpret them. While doing that different mathematical contents and competencies are activated. One of these competencies is modelling, which has a central place within the framework:

> This involves structuring the field or situation to be modelled; translating reality into mathematical structures; interpreting mathematical models in terms of reality; working with a mathematical model; validating the model; reflecting, analysing and offering a critique of a model and its results; communicating about the model and its results (including the limitations of such results); and monitoring and controlling the modelling process.

Mathematics is a tool often used in real world and in Science. The role of mathematics is predominantly brought through the building, employment and assessment of mathematical models (Michelsen, 2006).

How can physical experiments contribute to modelling activities in math lessons? If you look at the different steps mentioned above, physical experiments have a great potential. The experiments are derived from a phenomenon of everyday life and represent an idealized setting, considering certain factors only. Students doing physical experiments work with concrete terms. These terms are in a functional relationship with each other. If you want to describe them in a quantitative way you have to translate that relationship into mathematical structures. All kind of representations (graph, tables, etc.) can be applied. Students have to communicate about the phenomenon and the correspondent formula. A modelled formula can be checked directly through the measuring values and by new measurements. Because of measurement errors the formula is never correct. So it is natural to talk about the correctness and the limitations of the model and its results. If one slightly changes the setting of the experiment, the formula might change. Hence there is a strong emphasis on the validation process which often plays a minor role in the modelling process.

THE CONCEPT OF VARIABLE – ASPECTS

Malle (1986) differentiates three different aspects of variables:

- Variable as an object (Gegenstandsaspekt)

  A variable stands for an unknown item or an unknown object.
- Placeholder aspect (Einsetzaspekt)

  A variable means a placeholder, which you can substitute through a number.
- Calculational aspect (Kalkülaspekt)

  A variable stands for a meaningless symbol, with which you can apply certain rules.

He differentiates variable as an object into single number aspect and interval aspect. Single number aspect means an arbitrary but fixed number within a given domain. Variables which match to the interval aspect represent the whole domain. Within that interval aspect it can be differentiated between simultaneous aspect (representation at the same time) or changing aspect (representation in succession).

On the other hand variable as an object can be classified by a dynamic and a static component. Dynamic component means a changing number and static component means a specific unknown, i.e. it might change in another context.

If you compare the decomposition of the concept of variable according to Trigueros et. al (1996) into generalized number (representing a general entity, which can assume any value), specific constant (representing a constant value, which might change in another situation) and variable in a functional relationship, generalized number can be attributed to variable as an object, which can be represented at the same time or in succession. Specific constant is equivalent to the static component. To conceptualize variables in a functional relationship, knowledge of dynamic and static components is needed.

Malle demands among other things that in the beginning emphasis should be put to variable as an object and to the conception and interpretation of formulas.

THE CONCEPT OF VARIABLE AND PHYSICAL EXPERIMENTS

Michelsen (2006) proposes that by expanding the domain, mathematical concepts can be developed in a more practical and coherent structure, since

> the student's conceptions of a mathematical concept is determined by the set of specific domains in which that concept has been introduced for the student.

If students do physical experiments they can identify variables with concrete terms. That's why these variables can be classified to the aspect variable as an object. Students' major problem seeing variables as symbols to be manipulated (Schoenfeld & Arcavi, 1988) can therefore be diminished. Both dynamic and static components are touched since the values of the measurands change with each new measurement and the (anti-)proportional constant is constant in the same context. The possible values of the measurands determine the domain of the corresponding variable. The (anti-)proportional constants mostly are representatives of a discrete set.

By experimenting, students can discover the aspects of the concept of variable *before* they are properly defined in class. This is in accordance to Freudenthal's philosophy that context problems and real life problems are used to constitute and apply mathematical concepts. The aspects don't have to be touched in the abstract level at once. If they are touched on a descriptive level that can be enough. One example is the functional relationship of two measurands. While doing the experiment, students actively discover that change of one measurand causes a change of the other measurand. Especially, weaker students have problems to interpret this into a formula. But if you present a formula and give further explanations after the experiments the formula will not seem that abstract anymore because they can identify the formula with their experiences made while doing the experiment.

PHYSICAL EXPERIMENTS IN MATHEMATICS LESSON

In the above sections a few advantages and commonalities have been shown. But there are also subject specific characteristics, which have to be taken into consideration. In physics math is mostly seen as a tool for describing phenomena in a quantitative way. On the other hand mathematicians don't care how data was gained in detail; their only interest is the correctness of that data. Algebra is a correct theory. Experiments are never exact, because measurement errors always occur, even if they are very small. If you want to find relationships those measurement errors have to be kept in mind. School physics shows that all the time; school math only in a few fields. Therefore students have to be prepared to handle measurement errors.

If one wants to use experiments for mathematical concepts, emphasis should be given to the common and mathematical aspects. That means

- Experiments should have an easy setting

  Mathematics' interest is data and not how to get data. Therefore the experiment should be done with few materials and measured quickly allowing students to concentrate more on the math.

- Intervals of measurement errors should be small

  To find the relationship between the measurands quickly, there should be (if possible) no chances for systematic measurement errors and small intervals for random ones.

- The physical terms should be familiar to the students

This doesn't mean that physical terms not covered in physics class are forbidden. It is legitimate to use terms which are familiar in every-day life, like pressure, volume, temperature and so on.

- The interval of the measurands should be suitable

  Especially in experiments which contain an antiproportional relationship intervals should be chosen where the constant product stands out. Otherwise students may see (with consideration of measurement errors) a proportional relationship.

Physical experiments can be used within interdisciplinary lessons. This can be in separate classes, i.e. each class covers subject specific aspects; or for a short period in a common class in which all aspects are covered. On overview of different forms of cooperation can be found in Beckmann (2003, p. 9ff).

CONCEPTUALISATION IN SCHOOL

The use of experiments to introduce the concept of variable has been tested on 90 students of 7th grade in three different schools. They were required to do three out of five physical experiments. After the experiments, the concepts of variable and term were introduced formally. This was done to see if physical experiments can be applied in class and which experiments are appropriate. To get a deeper insight of the concept of variable and of reflection and validation of their modelled formula, another examination was done in spring 2008. 18 Students of 6th grade attending a German Gymnasium were required to do one experiment out of three working in groups of two. These 18 students knew the placeholder aspect, i.e. variables can be substituted by numbers, and that variables stand for a number which is unknown and changes continuously. Theoretical knowledge of the concept of variable concerning the object aspect hasn't taught yet. While doing the experiments, they were observed by students of the University of Education Schwaebisch Gmuend. After the experiments the 6th graders were interviewed by the students. The main research questions covered the aspects of the concept of variable touched by the experiment and of how convinced the students were of the formula found. The second question is to determine students' abilities to reflect and validate their results. Problem oriented interviews were chosen, so students could talk freely and were only slightly guided by the interviewers through open questions. The interviews were transcribed. Emotional factors like emphasizing words etc. were not considered during the process of transcription. Students' answers were categorized in the different aspects of variable and how they reflected the validity of their modelled formula.

The following experiments were done by the students:

- Buoyancy

The students measure the force of different masses in air and in water and conceptualize a formula which describes a proportional relationship between the forces in air and in water.

- Thermal expansion of a liquid

  The students measure the heights of an uncalibrated thermometer at different temperatures. Then they conceptualize a formula which describes a proportional relationship between difference of heights and difference of temperatures.

- Law of Boyle-Mariotte

  The students measure the pressure as well as position or volume of a piston. Then they conceptualize a formula which describes an antiproportional relationship between pressure respective to position or volume.

The design of the instructional sheets allows students to work by themselves. Assistance is only given, if students are at a loss and if tasks are essential for the following tasks. In that case hints were given and written down for consideration of students' results. No solution of tasks was given to the students.

The instructional sheets start with an impulse from real life. It shall motivate the students towards the experiment and shall put the experiment in a real setting. Through measuring different measurands students shall qualitatively experience the functional relationship of the two measurands. After measuring at least six different values, students are asked to describe the relationship first in their own words and then through a formula. This formula shall then be used to calculate measurands. These values shall be checked by looking at the values they measured before. This is to reflect their formula found. After that there follow questions concerning the domain of the variables and their properties. To touch the specific constant and change of formula in different contexts they were asked how the formula changes if one alters the setting of the experiment followed by a question for a more general formula. In the three classes students had to write a protocol containing the most important aspects. The 6th graders didn't have to write a protocol since they were interviewed after the experiment.

RESULTS

Concept of variable

Variable as an object according to Malle is touched. Students can identify the measurands with their chosen variables. A few examples:

Buoyancy experiment:

I2: Can you tell how you recognize (the experiment in your formula)?

S6: yes, you see the statement for air and for water. And yes the result, yes…

Here the group chose word variables. If they didn't choose words they chose the units of the measurands.

Boyle-Mariotte experiment

I1: what are those cm? What do they stand for?

S1: mmh here at that strip for example 6cm

I1: mmhmm

S1: so for the respective number

I1: and the x?

S1: for the respective pressure

Here the student chose the units of the physical terms as the name of his variables. Since he didn't know the unit of pressure, he chose x.

The functional relationship between the two variables has been recognized by the students both statically and dynamically.

Buoyancy experiment

S4: Then we agreed that if you divide air by water, the result is always the same. It doesn't matter, if there are 1, 2,3,4,5 cylinders. The result is always 1.2.

Boyle-Mariotte experiment

I1: What have you found out?

S1: yes, that device. If you turn further that thing moves forward and the further it moves the measuring number gets smaller and the pressure gets higher.

Thermal expansion experiment

S17: We had to find formulas. These were height times x is difference of temperature and difference of temperature divided by x is then height and difference of temperature divided by height is then x.

[…]

I8: and what changes in general in your formula?

S17: temperature and the head of liquid there, both get higher the more water you add.

Modelling process

Students went through the first part of the modelling cycle by examining the phenomenon and structuring it in a formula. That has been done on different levels. Weaker students could only explain in their own words and the strongest students have even presented three equivalent formulas.

To check how they reflected their formula students were asked if their findings are valid, since their measured values and the corresponding quotients/products weren't constant. They differed in small intervals due to random measurement errors. Before the experiments began, the instructor told the students that one could never measure exactly and that they had to keep that in mind. That is not easy as the following example shows:

Buoyancy experiment

S6: In the beginning I thought that I had to take the numbers which we had measured and then I thought for a longer time, if that was right.

After they accepted the influence of measurement errors, they rounded the quotients and then all but one were constant. Then they were convinced about the constant quotient.

I2: Did you notice anything about your result? In the case of normal water and air?

S5: Yes, the result was always 1.1; always the same.

They are convinced of the correctness of their formula because they have actively experienced that their results weren't always correct, but close to the "correct" answer.

Boyle-Mariotte experiment

S2: they aren't that correct.

I1: But the formula, that you have written down, is exact, isn't it?

S2: Well not that exact. It is… It could be also 7.1 instead of seven.

I1: Would you say your relationship is valid or your relationship is wrong?

S2: I would say, the relationship is valid, because with this device you can't determine that number that exactly. And the numbers I have written down, are actually as exact as possibly can be done with this device.

The use of experiments stimulates one to critically review the results and actively discuss the validity of the formulas found. Some students tend to extrapolate their formula after measuring a few values.

Buoyancy experiment

S3: (constant quotient) It is actually with all numbers! With six it is the same.

S4: That I don't know. You can't say … You don't know, what is with six. We haven't done that.

S3: Yes, but with 1 and 2 it is same, too.

But student 3 would not be convinced anymore, when one measures other values.

S3: and if we were to repeat that and would get other results, then we would be in a fix and wouldn't know what would be right.

But the more students measure the more convinced they are about their formula.

Thermal expansion experiment:

I8: You have found a formula, if a teacher comes to you […] and says your formula is wrong, would you say your formula is wrong or your formula is right?

S17: Yes, I think it is right, because of the different experiments we have done. Well with the different degrees and with the table at the beginning. We have measured the head of liquid and the difference of temperature six times and that was true all the time.

Hence physical experiments stimulate discussion about the formula. Reflection and validation of their formula is promoted.

Static component of variable & limit of modelled formula

If you ask students about the specific constant, you implicitly ask about the limit of their formula found. The (anti-)proportional constant is only constant in the same context. Changes of the context might cause a change of those constants. In the experiments, students were asked if the formula changes when you change the setting. In the buoyancy experiment, they were shown a man reading newspaper in the Dead Sea and asked if and how their formula would change, if they did the same experiment with salt water. Students doing the thermal expansion experiment were asked, if they were to change the thermometer, would that cause a change of their formula. In the Boyle-Mariotte experiment, students were asked if changes in the environment would cause changes of the formula.

Most of the students say that the formula changes and explain it on a descriptive level. Stronger students can tell which part changes while the strongest students set up a general formula.

Thermal expansion experiment

S10: We have found out that, if the glass tube is thicker, then it raises slower and if the glass tube is thinner, the liquid raises faster.

Buoyancy experiment

I3: Good. Is there a term, which doesn't change? Or changes everything?

S3: I think, if you stay in normal water, then it is always 1.2.

S4: Well, once you add a liquid, it will be heavier

S3: Yes, salt water or – then

S4: is, I think, heavier.

S3: Then 1.2 will be

S4: bigger.

Boyle-Mariotte experiment

I1: If I change anything on this device, how would your position and pressure change?

S1: mmhmm. Well, I think. Well, position will be the same but pressure will change.

Thermal expansion experiment

S17: If you change the glass tube, that means making them wider or yes thicker or thinner, then the constant changes. Otherwise it stays constant with the same glass tube.

I8: How does it change if you have a wider or thinner …?

S17: There it changes, well with a thinner, when it gets thinner, then the constant will get higher and when it gets thicker, then it will get lower. […]

That x is the constant, well, in our experiment it was six und it can change when the glass tube gets thicker or thinner.

As you have seen, questions about the specific constant have a great potential for discussion about the limits of a given model.

CONCLUSION

The use of physical experiments to introduce the concept of variable is as well a good way to promote the modelling process.

All of the aspects of variable as an object according to Malle are touched, especially within the functional relationship of two measurands. Formulas make sense to them, because they can identify variables with concrete objects. Not everybody touches the aspects on an abstract level but most do on a descriptive level. In the lessons afterwards those students will have fewer problems to understand abstract formulas because they can make connections to those experiments.

Like Maass (2006) found out, that students of lower secondary level were able to develop modelling competencies. Physical experiments can contribute to those competencies since the complete modelling cycle is covered. Especially "reflecting, analysing and offering a critique of a model and its results" has a main role in that concept. That is mainly through the appearance of measurement errors. Students learn that the modelled formula is an idealization, but still a good representation of the phenomenon. They can discover the limit of the formula found by scenarios they can imagine. Experimenting in groups stimulates the discussion about the model.

Students are motivated to do experiments, but finding a formula is a cognitive challenge. That's why students might get frustrated. A working sheet covering all aspects on the concept of variable and modelling on a descriptive level, i.e. without students coming up with a formula by themselves, would be better in cognitively weaker classes. If one stays on the descriptive level major phases of the modelling process are still touched. Then emphasis goes even more to analysing and criticizing the model.

This sequence is also a good basis for interdisciplinary teaching to see the same phenomenon with "subject driven" eyes. An overview gives the framework "Math and Science under one roof" which can be found on the homepage of the EU ScienceMath Project http://www.sciencemath.ph-gmuend.de.

REFERENCES

Beckmann, A. (2003). *Fächerübergreifender Mathematikunterricht Teil 1: Ein Modell, Ziele und fachspezifische Diskussion.* Hildesheim, Berlin: Franzbecker.

Beckmann, A. (2003a). *Fächerübergreifender Mathematikunterricht Teil 2: Mathematikunterricht in Kooperation mit dem Fach Physik.* Hildesheim, Berlin: Franzbecker.

Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures.* Dordrecht: Kluwer Academic Publishers.

Malle, G. (1986). Variable – Basisartikel mit Überlegungen zur elementaren Algebra. *Mathematik Lehren, 15, 2-8.*

Malle, G. (1993). *Didaktische Probleme der elementaren Algebra.* Braunschweig, Wiesbaden: Vieweg.

Maaß, K. (2006). What are modelling competencies? *ZDM, 38, 113-142*

Michelsen, C. (2006). Functions: a modelling tool in mathematics and science. *ZDM, 38, 269-280.*

OECD (Ed.). *Assessing Scientific, Reading and Mathematical literacy – A framework for PISA 2006.*

Schoenfeld, A. & Arcavi, A. (1988). On the meaning of variable. *Mathematics Teacher, 81, 420-427.*

Trigueros, M. & Ursini, S. & Reyes A. (1996). College students‘ conceptions of variable; *in Proceedings of the 20th PME conference Vol.4, 315-322*

Zell, S. (2008). Erkunden des Variablenbegriffs durch physikalische Experimente. *Contribution to mathematics education 2008*