MODELLING IN ENVIRONMENTS WITHOUT NUMBERS – A CASE STUDY

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In order to study how students are mathematising in modelling situations, students' work on problems having no obvious mathematical character is investigated. The task design aims at preventing students from concentrating on calculations, but challenges them to get involved in social interactions, where they argue and defend their ideas. The students' approaches to these mathematisation tasks are analysed; in particular it is discussed to what extent the students work mathematically. The concept of fundamental mathematical ideas is used in order to structure the way mathematics occurs in the students' works.

Keywords: modelling, mathematising, fundamental ideas, approximation, measuring

INTRODUCTION

In mathematics education, word problems are regarded as that type of mathematical exercises where information is provided in narrative, descriptive form, rather than in terms of numbers, variables, and so on. In extension, modelling problems are word problem solving activities, which involve not only handling data or calculating, but also observing patterns, testing conjectures and estimations of results (Schoenfeld, 1992). Tightly connected with modelling is the process of mathematising, i.e. the structuring of reality by mathematical means (Freudenthal, 1991). The aim of this case study is to understand and identify how mathematising emerges while students work on certain tasks of non-obvious mathematical nature.

THEORETICAL FRAMEWORK

Mathematising and modelling

Modelling can be viewed as linking the two sides of mathematics, namely its grounding in aspects of reality - and the development of abstract formal structures (Greer, 1997). In the modelling cycle described by Maaß (2006) (originating from Blum) reality and mathematics are regarded as distinct environments, and the process of modelling includes a number of phases between and within these 'worlds'. The 'step' in which the real-world model is translated into mathematics, leading to a mathematical model of the original situation is regarded as mathematising (Kaiser, 2006).

As working definition, mathematising is denoted here as the activity or process of representing and structuring real world artefacts and/or situations by mathematical means. The overall aim is to enable a logical, traceable and rational treatment of the given artefacts and situations with the help of mathematical knowledge and tools.

Modelling asks for certain cognitive demands, being determined by competencies like designing and applying problem solving strategies, arguing or representing, but it involves also communication skills, as well as real life knowledge (Blum and Borromeo-Ferri, 2007, Kaiser, 2006). Unlike the majority of problem situations,

> modeling activities are inherently social experiences, where students work in small teams to develop a product that is explicitly shareable. Numerous questions, issues, conflicts resolutions, and revisions arise as students develop, assess, and prepare to communicate their products. (English and Doerr, 2004, p. 3)

At the same time, mathematising is part of the modelling process and it is surely not possible to define neatly a border between mathematics and reality. They are interfering and depend on the contextual situation.

> The role of context is very important in mathematical modeling, since modeling requires a context in which to 'frame' the problem and 'develop' the mathematics. (Mousoulides, Sriraman and Christou 2007, p. 29)

According to Freudenthal, mathematising is the human activity consisting in organising matters from reality or mathematical matters, and “there is no mathematics without mathematising”. Later on, Treffers (1987) treated, in an educational context, the idea of two ways of mathematising, which led to a reformulation by Freudenthal in terms of 'horizontal' and 'vertical' mathematisation. In the horizontal mathematisation, mathematical tools are promoted and used to structure and solve a real-life problem, whereas vertical mathematisation supposes reorganisations and operations executed by students within mathematics. Adopting Freudenthal's (1991) formulation, mathematising horizontally means to go from the real world to the world of symbols, while mathematising vertically means to move within the symbols' world.

Maria van den Heuvel-Panhuizen studied the didactical use of models, which in Realistic Mathematics Education (RME) are

> seen as representations of problem situations, which necessarily reflect essential aspects of mathematical concepts and structures that are relevant for the problem situation, but that can have various manifestations. (Maria van den Heuvel-Panhuizen, 2003, p. 13)

Modelling always involves mathematising, which is regarded as the activity of observing, structuring and interpreting the world by means of mathematical models. Since the promotion of critical thinking by students represents one of the main pedagogical aims, “reflexive discussions amongst the students within the modelling process are seen as an indispensable part of the modelling process” (Kaiser and Sriraman, 2006).

Fundamental mathematical ideas

Often, it is no question what "mathematisation" is. If students work in a modelling framework, it cannot be expected that they develop a mathematical idea themselves if they are novices. Since they do not have formed a clear picture of mathematics, it is likely to see elements of different mathematical cultures in their modelling framework: mathematics in every day life or social practice, mathematics as a toolbox for applications, mathematics in school, and mathematics as a science.

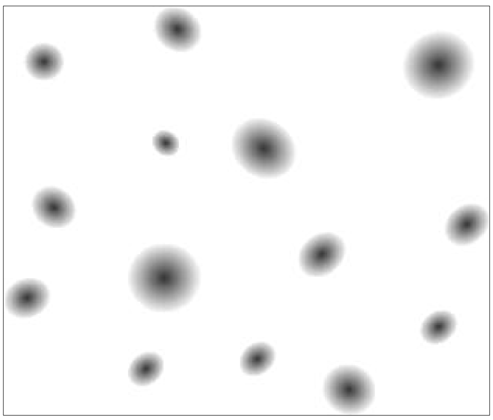
Fundamental ideas in mathematics may serve as a framework in this setting because they connect different mathematical cultures (Schweiger, 2006). Fundamental ideas recur in four dimensions: the historical development of mathematics (time dimension), in different areas of mathematics (horizontal dimension), at different levels (vertical dimension), in everyday activities (human dimension). Schweiger lists a synopsis of fundamental mathematical ideas from different sources: *algorithm, characterisation, combining, designing, exhaustion/approximation, explaining, function, geometrisation, infinity, invariance, linearisation, locating, measuring, modelling, number/counting, optimality, playing, probability, shaping*. Since modelling is discussed in detail and consists of the worked out tasks, it will not be considered in the sequel.

The main aim of this investigation is to see to what extent these fundamental ideas can be recognised in the answers to the rather open-ended Mars task (see next section). The overall pedagogical aim is to design such tasks that students are led to the consideration of fundamental mathematical ideas in a natural way.

EMPIRICAL SETTING

The task of non-obvious mathematical character that students have been given to work out is as follows:

"Imagine you are a scientist at NASA and you have a picture of the planet Mars. This picture shows different spots which indicate craters. These craters were obviously generated by impacts of several meteorites. It is possible that such an impact



generates more craters.

Fig. 1: Picture of the planet Mars depicting a crater

1. Write to a colleague a half-page report about the spots in Figure 1.
2. Describe, respectively label the spots.
3. Find out how the position of each spot could be described.
4. How would you specify the relationships between spots?
5. Could you order the spots by means of mathematical criteria? How?"

The task was given to 13-14 - aged students - in group-work in the classroom environment. Teams of three students were video-taped while working. The present study focuses only on one working group. No intervention from the teacher's side took place, unless students wanted to clarify the formulation of the task.

Data analysis

In the following excerpt, one can see a typical mathematical debate (see Figure 3).

32 J	So, which points are farthest away from each other?... K13 and K2...
33 A	K10 and K11... come, we measure them!
34 J	K13 and K2 are farther away from each other...
35 A	We take the middle point of the crater.
36 F	This is 8...
37 A	7.5...
38 J	They are both 8.
39 F	Where from, do you mean?
40 A	We start from the middle point.
41 F	Yes, I mean... which one do you mean?
42 A	K10 and K11.
43 J	K2 and K13 are a bit more...
44 F	Yes, 0.6cm

The students formulated themselves a small task, generated by the idea of finding 'extreme' points. This yielded the need to measure (line 33, as verifying action), which was not really unproblematic, since the 'spots' are of irregular form. J raised then the idea of comparing, which brought student A to the decision of taking the middle point. That means implicitly that the spots were seen as circles (or even ellipse, though they most probably did not meet it so far as subject in school). The idea of considering the middle point was proposed (line 35), but apparently no attitude was taken by the other two team-colleagues. Nevertheless, the idea was somehow tacitly adopted and they measured (lines 36, 37, 38) distances between points, which involves the assumption that the middle point was taken. In line 40, student A reminded of the middle point, but again no certain remark in this sense was made by his colleagues. However, the idea was carried out, and after some approximation trials (lines 36, 37), they obtained a very small result, namely that K2 and K13 were the searched points; thus, their initial claim was checked.

Another mathematical idea arose when mentioning 'coordinate system' in line 69.

67 J	This is a brilliant idea!
68 A	What?
69 J	This with the coordinate system... It came from me...
70 A	It came from me!!!
71 J	So, if all the points have now to be mapped there... We can write, yes, Z-point, G-point... and then somehow one-two maps... or so
72 A	Do we now want to mark all the points?
73 F&J	No!
74 J	We do just an example. K11 is simple..
75 F	No, also K10... K10 is also simple... that is 1... ehh... 8... 1-8

The students became quite enthusiastic about the idea. They appreciated it as being 'brilliant' and two of the students were almost simultaneously claiming it. They saw this as a mathematical criterion for describing the position of the spots, but the idea offered them an expanded perspective and view, which six minutes later (see next excerpt) brought A (who had the idea, in fact) to the vision of a virtual map.

Further on students wanted to perform measurements, but it was not really clear for them how, probably because the exact corresponding geometrical figure associated to the 'spots' was not explicitly debated and agreed on.

93 A	... Measure the dimension...
94 J	The dimension...
95 F	The dimension...
96 A	We cannot measure that, a crater goes also up... doesn't it? ... and also down...
97 J	We cannot do that, because we have no photo.
98 F	You now want to position this somehow like this (placing the set square perpendicularly on a crater) and measure the dimension?... Now tell me shortly how should we get this?
99 A	... in order to build a virtual map!... Measure the volume and building a virtual map.

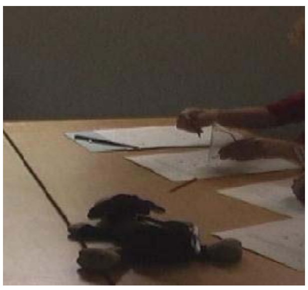


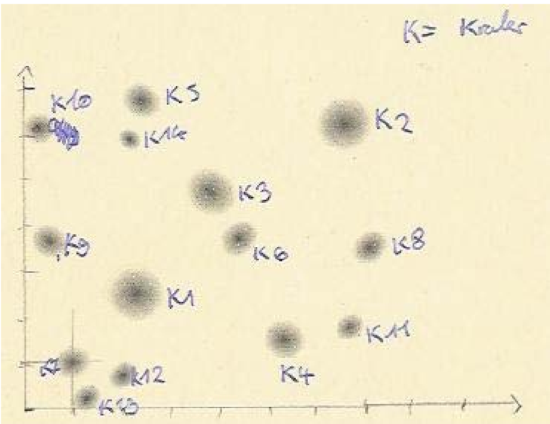
Fig. 2: Student's transfer from 2-D to 3-D

In fact, A had something indefinite in mind, following from the idea of coordinate system, an idea to construct a virtual map, which would have also allowed calculations for the volume of the crater. His colleagues showed themselves quite sceptical with respect to the idea, they faced a lack of understanding of A's mental representation, then the idea vanished, and A did not bring further arguments for sustaining it. The basis of A's idea of coordinate system seemed to be the experience he probably had with the geographical atlas or the moving of chess pieces, which is expressed in the form of horizontal coordinate and its corresponding vertical one. It might also be that A's idea of virtual map is inspired by computer games. Student A realised a mental transfer from 2-D to 3-D, unfortunately without a further development.

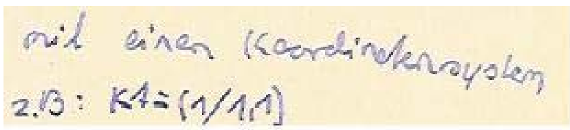
Findings on used mathematical ideas

As main fundamental ideas, approximation, geometrisation, locating, measuring, number/counting, and optimality were observed several times. The students *approximated* the craters with a circle. They did not state it explicitly, but this assumption was carried out during their work. The middle point was only roughly marked. *Geometric* shapes were developed, which helped students to describe the situation. These were not named explicitly, but the concepts of distance, area and volume were used by students. A coordinate system was used to *locate* the craters. Some groups described the location of the craters absolutely, some relative to a coordinate system or somehow relative to other craters in a fancy way. *Measuring* appeared several times and was discussed intensively. The idea was adopted for measuring distances between the craters, as well as the size of the craters. One of the very first actions of the groups was to *count*/order the craters. In the "relations" (see task), *optimality* occurs, e.g. as the closest/furthest distances between craters.

Some students chose to label the spots in a rather mathematical way, as seen in Figure 3. They also had the idea to give the name 'N250i' to a probe, as being the instrument used to observe the given task phenomena, i.e. the generation of the craters.



for describing positions of craters



The task was formulated in such a way that it was not clear to the students what the actual aim of the task was. In fact, one might investigate the given data with different goals in mind. For example, one might want to have an estimate of the number of meteorites creating the craters (on falling apart), which leads to a clustering problem, or one might just want to have a precise map of the craters, yielding a position measuring mainly. It seems that the students interpreted the task as having this latter aim. Basically, there is no 'ideal' solution. The students had to come up with their own interpretation of the goal. The quality of their answers could be judged by the 'depth' of their analysis. The task allows an analysis on different levels of sophistication.

The theoretical framework of Freudenthal, in particular horizontal mathematisation, could be recognised in students' answers, as is apparent from the coordinates they introduced. However, vertical mathematisation, relating these coordinates, for example in a clustering procedure, did not take place. This is probably due to the fact that no specific goal was mentioned, and therefore students were not guided to mathematise in a vertical direction. Their considerations stayed on the level of description.

As characteristic of the modelling processes, the frequent moving between environments, see also (Grigoraş and Halverscheid, 2008), seemed to happen not randomly, but generated by certain 'needs' (e.g. additional data demands). During the discussions towards finding a solution for the problem in a systematic way, students posed questions and set themselves small tasks.

Besides the initial idea of naming the spots, which might not necessarily be a mathematical act, but rather seen as usual labelling (see Figure 3), some students proposed a coordinate system as idea of describing the position of the 'spots', which is a mathematising action. Further on, they started to calculate positions of several 'spots', but finally they decided to give just examples, e.g. 'spot' K10 having 1 as horizontal coordinate and 8 as vertical one (see second excerpt of the previous section). The measuring idea was also found in their talks, and students debated on it for some time, while trying to find out which spots are in extremal position. These acts count as at least two mathematisation achievements.

In this case, but also in several previous surveys carried out on tasks without numbers, it was seen that many fundamental ideas occurred as mathematisation acts. However, not all of these ideas lead to intensive mathematical modelling activities. As for the task discussed here, deeper mathematical activities were started concerning measuring and optimality. The students proceeded by taking the set square or ruler and measured the distance between the 'spots', whereas for the other mentioned fundamental ideas no mathematical activities were performed. Students also handled the approximation of spots by circles in a mathematical manner, and measured distances between them by taking the middle points of the 'spots'. Ideas like radius, circumference, volume (even unclear whether applicable in this case) completed the mathematical 'picture' of what students built up around the 'spots'.

A finer look at the last excerpt

The situation may seem at a first sight somehow simplistic, such as students using mathematics when working out tasks without numbers. But it is interesting to examine the subtle aspects and reasons behind the usage of mathematics. This is done here with respect to the question whether and how the need of mathematising occurs.

It should be remarked that simplification in various forms (schema, drawings, etc.) is a characteristic of modelling itself. For students in their age, modelling rests on the principle of representing a situation type in a simplified, general manner which allows extended applications.

While tackling the task of finding mathematical criteria, the students approached ideas one by one like finding distances (they coped well with planning and measuring distances between extremal 'spots'), coordinate system, then finally the volume, then they stopped doing further things, since their 'tools' for calculating things were not sufficient. Mathematical concepts used by students - *distance*, *area* (implicit, through the coordinate system and middle point of a plane figure), and *volume* originated from a need of simplification of the initially given problem. Therefore the approximation of spots by circles was done, though never explicitly stated.

The idea of finding distances between extremal 'spots' was conducted through measuring, since students found something they could do. Somewhat further, they came to the idea of coordinate system, by which they were quite absorbed and dealt successfully with in the 2-D situation. Then the *dimension* was mentioned, but the students faced up to some problems with the data, that seemed not to suffice (line 97 in the transcript). Once they met this data demand, students were confronted with an unclear situation of the model, since they did not know which mathematical object would fit to that stage, where a virtual map was proposed. At that point, their debate stopped, hence no simplification was achieved. Therefore the 3-D situation failed, because of a lack of tools and/or data.

DISCUSSION

It is intriguing in this case to study how fundamental mathematical ideas occurred through mathematising. There were fundamental ideas leading to the model (biggest, smallest, extreme, measure). But how did students build a modelling idea? It seems they looked in mathematics for 'tools' which would allow them to work out a model.

When discussing the real situation, students emitted sometimes mathematical ideas, e.g. the idea of measuring, which means that a transfer to mathematics took place. There then were two possibilities: either they remained within mathematics and performed further on, or they turned back to the real situation. Such frequent forth-back transitions are analysed in (Borromeo-Ferri, 2007). The decision (often unconsciously taken) whether to stay or not within mathematics seemed to be influenced and caused by a number of factors, as described in the following. We refer to the real situation as being *data situation*, as the existing task formulation students have at their disposal. When being situated in mathematics, there could be *tools*, *knowledge*, *experience*, *motivation*, among others, all of these determining whether students stayed and worked with and within mathematics, or they turned again to data and tried to handle them and searched for next steps. If one or more of these items were missing and students were facing a dilemma in performing further on with and within mathematics, then they came back to the data.

The analysis of the present study showed that while acquiring a real modelling experience, students produced many fundamental mathematical ideas, but when confronted with a lack of tools, knowledge, or even experience, their activities stopped at the level of ideas' supply.

As for a future analysis, a first hypothesis is that a modelling task developes through fundamental ideas. A second hypothesis is that mathematics is reached by means of assumptions, which students proposed and agreed on while taking decisions during solving a modelling problem. It would be interesting to examine how mathematising differs according to the mathematical nature of the task formulation.

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