THE MATHEMATICAL PREPARATION OF TEACHERS: A FOCUS ON TASKS

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In this article we elaborate a conceptualization of mathematics for teaching as a form of applied mathematics (building on Bass's idea of characterizing mathematics education as a form of applied mathematics) and we examine implications of this conceptualization for the mathematical preparation of teachers. Specifically, we discuss issues of design and implementation of a special kind of mathematics tasks whose use in teacher education is intended to promote mathematics for teaching.

The notion of *Mathematics for Teaching* (*MfT*) (Ball & Bass, 2000) describes the mathematical content that is important for teachers to know and be able to use in order to manage successfully the mathematical issues that arise in their practice. According to Ball and Bass (2000), this specialized kind of mathematical knowledge, referred to as *Mathematical Knowledge for Teaching* (*MKfT*), is important for solving the barrage of "mathematical problems of teaching" that teachers face as they teach mathematics: offering mathematically accurate explanations that are understandable to students of particular ages, validating student assertions, etc.

In this article, we focus on the following research question: What kind of learning opportunities might mathematics teacher education programs design to effectively support the development of prospective teachers' MKfT? To address this question, we elaborate a conceptualization of MfT as a form of applied mathematics and probe the implications of this conceptualization for the mathematical preparation of teachers, with particular attention to the nature of mathematics tasks that might be important for use in mathematics (content) courses for prospective teachers. To exemplify the constructs we discuss in the article, we use data from a research-based mathematics course for prospective elementary teachers in the United States.

CONCEPTUALIZING MATHEMATICS FOR TEACHING AS A FORM OF APPLIED MATHEMATICS

In thinking about the problem of teachers' mathematical preparation, we found useful Bass's (2005) suggestion of viewing mathematics education as a form of applied mathematics: "[Mathematics education] is a domain of professional work that makes fundamental use of highly specialized kinds of mathematical knowledge, and in that sense it can […] be usefully viewed as a kind of applied mathematics" (p. 418). Given that mathematics education makes use of specialized knowledge from several other fields in addition to mathematics (psychology, sociology, linguistics, etc.), we propose that the characterization "form of applied mathematics" be used to refer specifically to the mathematical component of mathematics education, notably MfT.

The conceptualization of MfT as a form of applied mathematics calls attention to the domain of application of MfT (i.e., the work of mathematics teaching) and the specialized nature of "mathematical problems of teaching" (Ball & Bass, 2000). In particular, the conceptualization has two important and interrelated implications for the mathematical preparation of teachers, which are aligned with existing research and theoretical accounts in the area of MKfT.

First, the conceptualization implies that the mathematical preparation of teachers should take seriously the idea that "there is a specificity to the mathematics that teachers need to know and know how to use" (Adler & Davis, 2006, p. 271). This idea relates to broader epistemological issues about the situativity of knowledge (e.g., Perressini et al., 2004) and to research findings that different workplaces require specialized mathematical knowledge by their practitioners (e.g., Hoyles et al., 2001).

Second, the conceptualization implies that the mathematical preparation of teachers should aim to "create opportunities for learning subject matter that would enable teachers not only to know, but to learn to use what they know in the varied contexts of practice" (Ball & Bass, 2000, p. 99). In other words, it underscores the importance of the development of a "pedagogically functional mathematical knowledge" (ibid, p. 95), which can support teachers to solve successfully mathematical problems that arise in their work. The characterization of MKfT as "pedagogically functional" helps clarify further the meaning we assign to the term "applied mathematics" in the proposed conceptualization of MfT. Specifically, our use of this term refers to mathematics that is (or can be) *useful for* and *usable in* mathematics teaching (the domain of application), and thus, important for teachers to know and be able to use when they teach mathematics (i.e., when they function in the domain of application).

Acceptance of the conceptualization of MfT as a form of applied mathematics necessitates that mathematics courses in teacher education design opportunities for prospective teachers to learn and use mathematics from the perspective of a *teacher of mathematics*. How might these opportunities be designed in teacher education?

Given the central role that mathematics tasks can play in individuals' learning experience in classrooms, we considered fruitful to begin to address the question above (which is a reformulation of our research question) by conceptualizing a special kind of mathematics tasks that we call *Pedagogy-Related mathematics tasks* (*P-R mathematics tasks*). These tasks are intended to embody essential elements of MfT as a form of applied mathematics and support mathematical activity that can enhance the development of prospective teachers' MKfT.

"PEDAGOGY-RELATED MATHEMATICS TASKS": A VEHICLE TO PROMOTING MATHEMATICAL KNOWLEDGE FOR TEACHING

Feature 1: A primary mathematical object

Like all other kinds of mathematics tasks, P-R mathematics tasks have a *primary mathematical object.* This is intended to be the main focus of prospective teachers' attention and to engage them in activity that is primarily mathematical (as opposed to pedagogical). The mathematical object of a P-R mathematics task can take different forms such as validation of a conjecture or description of the mathematical relationship between two methods for obtaining the same mathematical result.

Feature 2: A focus on important aspects of MKfT

Like most other kinds of mathematics tasks used in mathematics courses for prospective teachers, the mathematical object of a P-R mathematics task relates to one or more mathematical ideas that have been suggested by theory or research on MKfT as being important for teachers to know (see, e.g., Stylianides & Ball, 2008). In our work with prospective teachers we pay special attention to such ideas that are also *fundamental* (Ma, 1999) and *hard-to-learn* for both students and teachers.

Feature 3: A secondary but substantial pedagogical object and a corresponding pedagogical space

The defining feature of P-R mathematics tasks is that they have a *secondary pedagogical object.* This object is substantial (i.e., it is an integral part of the task and important for its solution) and situates the mathematical object of the task in a particular *pedagogical space* that relates to school mathematics and, ideally, derives from actual classroom records. The pedagogical object and the corresponding pedagogical space of a P-R mathematics task help engage prospective teachers in mathematical activity from the perspective of a *teacher of mathematics*.

Consider for example a P-R mathematics task whose mathematical object is the development of a proof for a conjecture. The pedagogical object of this task could be a teacher's need that the proof be appropriate for the students in his/her class. The corresponding pedagogical space could be a description (scenario) of what the solvers of the P-R mathematics task might assume the students in the class to know in relation to mathematical content that is relevant to the task. Thus the solution of the task cannot be sought in a purely mathematical space, but rather in a space that intertwines content and pedagogy. As a result, the task can generate mathematical activity that is attuned to particular mathematical demands of mathematics teaching.

Next we discuss four points related to feature 3 of P-R mathematics tasks. First, the pedagogical object/space of a P-R mathematics task, and especially its connection to (actual) classroom records, can embody the ideas of "situativity of knowledge" and "pedagogical functionality" that we discussed earlier in relation to MfT as a form of applied mathematics. Specifically, the pedagogical object can support development of mathematical knowledge that is applicable in a particular context (pedagogical space) within the broader work of mathematics teaching.

Second, the pedagogical space of a P-R mathematics task determines to great extent what counts as an acceptable/appropriate solution to the task, because it provides a set of conditions with which a possible solution to the task needs to comply. This is important, because, almost always in teaching, a purely mathematical approach to a "mathematical problem of teaching" does not address adequately the different aspects of the pedagogical space in which the problem is embedded.

Third, given the complexities of any pedagogical situation, it is often impractical (if not impossible) to specify all the parameters of the situation that can be relevant to the mathematical object of a P-R mathematics task. This lack of specificity can be useful for teacher educators who implement P-R mathematics tasks with their prospective teachers: teacher educators can use the endemic ambiguity surrounding the pedagogical space in order to vary some of its conditions and create opportunities for prospective teachers to engage in related mathematical activities within the particular pedagogical space. The variation of conditions of the pedagogical space (and the mathematical activities that can result from this variation) can offer prospective teachers practice with grappling with the barrage of mathematical issues that arise (often unexpectedly) in almost every instance of a teacher's practice.

Fourth, the pedagogical object/space of a P-R mathematics task have the potential to motivate prospective teachers' engagement in the task by helping them see and appreciate why the mathematical ideas in the task are or might be important for their future work as teachers of mathematics. According to Harel (1998), "[s]tudents are most likely to learn when they see a need for what we intend to teach them, where by 'need' is meant intellectual need, as opposed to social or economic need" (p. 501; the original was in italics). In the case of prospective teachers, a "need" for learning mathematics may be defined in terms of developing mathematical knowledge that is useful for and usable in the work of teaching. By helping prospective teachers see a need for, and thus develop an interest in, the material that teacher educators engage them with, teacher educators increase the likelihood that prospective teachers will learn this material. This is particularly useful in relation to material that prospective teachers tend to have difficulty to see as relevant to their future teaching practices.

EXEMPLIFYING THE USE OF P-R MATHEMATICS TASKS IN A MATHEMATICS COURSE FOR PROSPECTIVE TEACHERS

General description of the course

The course was the context of a *design experiment* (see, e.g., Cobb et al., 2003) that we conducted over a period of four years and that aimed to develop practical and theoretical knowledge about ways to promote prospective teachers' MKfT. It was a three-credit undergraduate-level mathematics course for prospective elementary teachers, prerequisite for admission to the masters-level elementary teaching certification program at a large state university in the United States. It was the only mathematics content course in the admission requirements for the program,[1] and so it was designed to cover a wide range of mathematical topics. The students in the course pursued undergraduate majors in different fields and tended to have weak mathematical backgrounds. Also, given that the students were not yet in the teaching certification program, they had limited or no background in pedagogy.

[1] The students who are admitted to the teaching certification program take also a mathematics pedagogy course, but the focus of this course is on teaching methods.

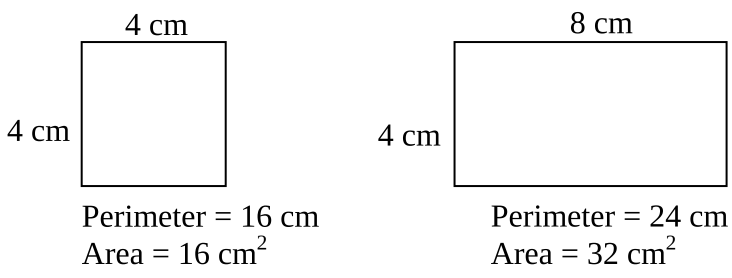
The most relevant aspect to this article of the approach we took in the course to promote MKfT is the design and implementation of task sequences that included both P-R mathematics tasks and *typical mathematics tasks*, which embody only features 1 and 2 of P-R mathematics tasks. A common task sequence in the course began with a typical mathematics task that engaged prospective teachers in mathematical activity from an adult's point of view. The P-R mathematics task that followed described some pedagogical factors that prospective teachers needed to consider in their mathematical activity. To satisfy feature 3 of P-R mathematics tasks about situating prospective teachers' mathematical activity in a pedagogical space, we used a range of actual classroom records such as video records or written descriptions (as in scholarly publications) of classroom episodes, excerpts from student interviews or textbooks, etc. Less frequently and when actual classroom records were unavailable, we used (similar to Biza et al., 2007) fictional but plausible classroom records.

An example of a task sequence and its implementation in the course

We illustrate the use of P-R mathematics tasks in the course with a task sequence that included a typical and a P-R mathematics task. To develop this and other task sequences in the course we followed a series of five research cycles of implementation, analysis, and refinement over the years of our design experiment. In this article we use data from the last research cycle that involved enactment of the course in two sections; these sections were attended by a total of 39 prospective teachers and were taught by the first author. Specifically, the data come from one of the two sections and include video and audio records of relevant classroom episodes, and fieldnotes that focused on prospective teachers' small group work.

The focal task sequence aimed to promote prospective teachers' knowledge about a possible relation between the area and perimeter of rectangles, with special attention to the ideas of generalization and proof by counterexample, which are considered important for elementary mathematics teaching (see feature 2 of P-R mathematics tasks in relation to Stylianides and Ball, 2008). The task sequence is an adaptation of an interview task used by Ma (1999) and developed originally by Ball (1988).

> Imagine that one of your students comes to class very excited. She tells you that she has figured out a theory that you never told the class. She explains that she has discovered that <u>as the perimeter of a rectangle increases, the area also increases</u>. She shows you this picture to prove what she is doing:



1. Evaluate mathematically the student statement? (underlined)

2. How would you respond to this student?

Although question 1 refers to a student statement, it is essentially a typical mathematics task because the prompt asks prospective teachers to evaluate mathematically the statement, without asking (or expecting) them to take account of the fact that the statement was produced by a student. Question 2, on the other hand, is a P-R mathematics task because it introduces a student consideration that prospective teachers need to consider in their mathematical activity. The *mathematical object* of this P-R mathematics task is to evaluate mathematically the underlined statement, which is essentially what the prospective teachers were asked to do in question 1 (a teacher would need to know about the correctness of the statement before deciding how to respond to the student who produced it). The *pedagogical object* of the task is the teacher's need to respond to the student who produced the statement. The *pedagogical space* is the (fictional) scenario in the task with a student announcing enthusiastically to the teacher a mathematical "discovery," which was supported by a single example in the domain of the corresponding statement. Although an appropriate response to question 1 could say that the statement is false and provide a counterexample to it, an appropriate response to question 2 would need to include more than that. Specifically, from a pedagogical standpoint, it would be useful and important for the student's learning if the teacher did not just prove her statement false, but also helped her understand why the statement is false and the mathematical conditions under which the statement is true.

The prospective teachers in the course worked on the two questions first individually, then in small groups, and later in the whole class. The whole class discussion started with the teacher educator asking different small groups to report their work on the task, beginning with question 1 (all prospective teacher names are pseudonyms).

Andria: We said that it [the student statement] was mathematically sound because as you increase the size of the figure, the area is going to increase as well.

Tiffany: We thought the same, because as the sides are getting bigger… [inaudible]

Stylianides: Does anybody disagree? [no group expressed a disagreement]

Evans: I agree. [Evans was in a different small group than both Andria and Tiffany]

Stylianides: And how would you respond to the student?

Melissa: I think it's true but they haven't proved it for all numbers so it's not really a proof.

Andria: I think that you don't have to try every number [she means every possible case in the domain of the statement] to be able to prove it because if the student can explain why it works like we just did, like if you increase the length then the area increases. [pause]

Stylianides: Yeah, so it's impossible to check all possible cases [of different rectangles].

Meredith: I'd say that it's an interesting idea, and I'd see if they can explain why it works.

As the excerpt shows, all small groups believed that the student statement was true, but at the same time they realized that the evidence the student provided for her claim was not a proof (see, e.g., Melissa's comment). As a result, the prospective teachers started to think how they could prove the statement and what they could respond to the student. For example, Andria observed that it would be impossible to check every possible case. Also, both Andria and Meredith pointed out that the student needed to explain why (i.e., prove that) the area of a rectangle increases as its perimeter increases. Yet, the teacher educator knew that the statement was false, and so he probed the prospective teachers to check more cases and see whether they could find an example where the student statement failed. All small groups found quickly at least one counterexample to the statement and concluded that it was actually false.[2]

The prospective teachers did not expect this intuitively "obvious" statement to be false, so they became motivated to work further on question 2. The teacher educator gave them more time to think about this question in their small groups. The excerpt below is from the whole class discussion that followed the small group work.

Natasha: We said that the way that they [the students] are doing it, where they're just increasing the length of one side, it's always going to work for them but if they try examples where they change the length on both sides that's the only way it's going to prove that it doesn't work all the time. So you should try examples by changing both sides.

Stylianides: What do you think about Natasha's response? Does it make sense? [the class nodded in agreement] So what else? What else do you think about this?

Evans: You can kind of ask them to restructure the proof so that it would work.

Stylianides: What do you mean by "restructure the proof"?

Evans: Like once they figure out that it doesn't work for all cases they could say it's still like… if they saw it and if they revise it like the wording or just add a statement in there that if they can come up with a mathematically correct statement…

Stylianides: Anything else? [no response from the class]
I think [that] both ideas [mentioned earlier] are really important. So when you have something [a statement] that doesn't work, then it's clear that this student would be interested to know more. For example, why it doesn't work or under what conditions does it work because, obviously, some of the examples that the student checked worked. […]

Natasha and Evans proposed two related issues that the elementary teacher in the task scenario could address when responding to the student: why the statement is false and the conditions under which the statement would be true. Based on our planning for the implementation of the task, the teacher educator would raise these issues anyway, because, as we explained earlier, a teacher response to the student that would consist only of a counterexample to the statement would be mathematically sufficient but pedagogically inconsiderate. The fact that the two issues were raised by prospective

[2] The prospective teachers had opportunities earlier in the course to discuss the idea that one counterexample suffices to show that a general statement is false.

teachers instead of the teacher educator is noteworthy, because Natasha and Evans had no teaching experience and also the issues they raised were requiring further mathematical work for themselves and the teacher education class. Take for example Evans's contribution, which raised essentially the following new mathematical question: Under what conditions would the statement be true? It is hard to explain what provoked Natasha and Evans's contributions, but we hypothesize that the pedagogical object/space of the P-R mathematics task played an important role in this. Specifically, we hypothesize that the need to respond to a false but plausible student statement made the prospective teachers think hard about related mathematical issues and how to "unpack" them in pedagogically meaningful ways (Ball & Bass, 2000; see also Adler & Davis, 2006).

Following the summary of the two issues as in the previous excerpt, the teacher educator engaged the prospective teachers in an examination of the conditions under which the student statement would be true. A more detailed discussion of the prospective teachers' work on the task sequence is beyond the scope of this article.

To conclude, our discussion in this section exemplified the idea that the application of mathematical knowledge in contextualized teaching situations can be different than its application in similar but purely mathematical contexts. Although the mathematical objects of the typical and P-R mathematics tasks in the sequence were the same (namely, the mathematical evaluation of a statement about a possible relation between the area and perimeter of rectangles), the pedagogical space in which the P-R mathematics task was embedded changed what could count as an appropriate solution to it, thereby generating mathematical activity in a combined mathematical and pedagogical space.

CONCLUDING REMARKS

Although the primary object of P-R mathematics tasks is mathematical, their design, implementation, and solution require some knowledge of pedagogy. This requirement derives primarily from the pedagogical objects of P-R mathematics tasks, which, although secondary to the tasks, determine to great extent what counts as acceptable/appropriate solutions to the tasks and influence the mathematical activity (to be) generated by the primary objects of the tasks. For example, the design of the P-R mathematics task that we discussed earlier used knowledge about a common student misconception regarding the relation between the area and perimeter of rectangles. Furthermore, successful implementation and solution of this task required appreciation of the pedagogical idea that a mere counterexample might be a limited teacher response to a flawed but plausible student statement.

The pedagogical demands implicated by the design, implementation, and solution of P-R mathematics tasks make it reasonable to say that instructors of mathematics courses for prospective teachers need to have, in addition to good knowledge of mathematics, knowledge of some important pedagogical ideas. This requirement might be hard to fulfill in contexts such as the North American where mathematics courses for prospective teachers are typically offered by mathematics departments and are taught by (research) mathematicians. However, if such knowledge is agreed to be essential for teaching MfT to prospective teachers, then the field of mathematics teacher education needs to find ways to support the work of instructors of mathematics courses for prospective teachers. One way might be to offer instructors access to what we may call *educative teacher education curriculum materials*. This is the teacher education equivalent of the notion of educative curriculum materials, i.e., curriculum materials that aim to promote teacher learning in addition to student learning at the school level (see, e.g., Davis & Krajcik, 2005).

The pedagogical aspects of P-R mathematics tasks raise also the following question: Would it make sense to promote MKfT in mathematics courses designed specifically for prospective teachers, or would it make more sense to promote it in combined mathematics/pedagogy courses, which, by definition, pay attention to both pedagogical and mathematical issues? The idea of promoting MKfT in combined mathematics/pedagogy courses may be attractive to some given the potential of P-R mathematics tasks to intertwine mathematics and pedagogy. Yet a possible decision to eliminate mathematics courses designed specifically for teachers in favor of combined mathematics/pedagogy courses might create different kinds of problems. In their examination of different types of tasks in formal assessments used across a range of mathematics teacher education courses in South Africa, Adler and Davis (2006) reported the concern that in combined mathematics/pedagogy courses the mathematical and pedagogical objects lose their clarity and that evaluation in these courses tends to condense meaning toward pedagogy.

The conceptualization of MfT as a form of applied mathematics that we elaborated in this article highlights the idea that, irrespectively of whether MfT is promoted in specialized mathematics courses or combined mathematics/pedagogy courses, prospective teachers' learning of MfT should not happen in isolation from pedagogy. P-R mathematics tasks can facilitate the integration of mathematics and pedagogy in prospective teachers' learning: although these tasks make mathematics the focus of prospective teachers' activity, they situate this activity in a substantial pedagogical space that shapes and influences the activity. Future research may explore ways in which to facilitate the integration of mathematics and pedagogy from the opposite direction, i.e., by making pedagogy the focus of prospective teachers' activity and having mathematics play a secondary but substantial role in this activity. Towards this end, one can reverse the relative importance of mathematical and pedagogical objects in P-R mathematics tasks to coin the twin notion of *Mathematics-Related pedagogy tasks*. Specifically, these tasks can be defined to have a primary pedagogical object (with a corresponding pedagogical space) and a secondary but substantial mathematical object, and can be used to generate activity that is predominantly pedagogical (as opposed to mathematical in P-R mathematics tasks).

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