TEACHERS AND TRIANGLES

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During a workshop about triangles designed for in- and pre-service basic-school teachers, a diagnostic test was applied. The results are analysed in terms of several variables: the teachers' sex, the level at which they work, their occupation (namely, in- or pre-service teachers), and their professional experience. An important impact of the latter was found in the decrease of incorrect answers obtained.

FRAMEWORK

Shulman (1986) characterised the types of knowledge that he considered enabled teachers to carry out their practice. He proposed three categories: mathematical content knowledge (MCK), curriculum knowledge (CK), and pedagogical content knowledge (PCK).

There have been several discussions about Shulman's categories. We want to mention two in particular. The first one is a discussion both about the exact meaning of MCK. Some researchers stress that within MCK there is a difference between the knowledge of the formal academical discipline and the scholar subject (see e.g. Bromme, 1994). The former is the knowledge that professional mathematicians develop, and the latter is the mathematics that teachers must teach.

The second discussion is about how much MCK is a valid variable in understanding teachers' practices and designing teachers' education. There has been a variety of researches that show that "teachers' mathematics knowledge is generally problematic in terms of what teachers know, and how they hold this knowledge of mathematics concepts or processes, including fundamental concepts from the school mathematics curriculum. They do not always possess a deep, broad, and thorough understanding of the content they are to teach" (da Ponte & Chapman, 2006, p. 484). According to some authors, these researches are important because of several reasons. On the one hand, they allow to understand how "elementary teachers' understanding of subject matter influences presentation and formulation as well as the instructional representations that the teacher uses" (Sánchez & llinares, 1992, quoted by da Ponte & Chapman, 2006, p. 434). On the second hand, they have prompted "studies centred on describing student teachers' beliefs and knowledge as determining factors in their learning processes [... and have also] provided information used to prepare research-based material for use in teacher education and to develop research-based teacher education programmes." (Llinares and Krainer, 2006, p. 430). On the other extreme, some authors question MCK's importance, because "the academic mathematical knowledge may not be 'naturally' a helpful instrument for the teacher in the school practice, since some of its values and forms of conceptualizing objects conflict with the demands of that practice". (Moreira & David, 2007, p. 38). They stress that to help students to think mathematically, teachers need to understand student thinking, and thus the comprehension about the cognitive processes of the students becomes more important than MCK itself

While we are aware that many variables may qualify the importance of MCK, such as teachers' beliefs and practices, the cognitive processes of students, etc., we sustain that teachers should at least have a solid understanding of the contents they must teach. This does not always happen in Mexico, and in order to explain why, we must make a brief exposition of the Mexican situation about teacher training. Teachers receive their training not in universities but in "Escuelas Normales", which they attend after 6+3+3 years of regular schooling. There are "Escuelas Normales" (ENP) for student teachers who will become Primary school teachers (i.e, grades 1-6), and other "Escuelas Normales" (ENS) for those who will teach at the Secondary level (grades 7-9). At the ENP it is taken for granted that during those 12 years of previous schooling they have learnt all the mathematics they will ever need to teach, and that all they need to know about teaching mathematics is PCK; at the ENS student teachers have some courses focused on MCK. (Another situation in Mexico is the fact that there is not an assessment or a diagnostic about teachers' MCK with results widely spread). Thus, if teachers enter the ENP with misconceptions or deficiencies, these are not solved there, and the dragging of misconceptions and deficiencies becomes, through teachers' practice, a vicious circle. One of the well-known consequences of this process is that Mexico is always among the countries that obtain the lowest results in international assessments of students' performance, like PISA and TIMSS.

While other countries do not share the extremely low results in PISA and TIMSS, teachers' misconceptions and deficiencies are not exclusive of ours. For example, Hershkowitz & Vinner (1984, quoted in da Ponte & Chapman) investigated the processes of concept formation in children, through the comparison of students' learning and elementary teachers' knowledge of the same concepts; they found that one of the factors that affects the students' learning is the teachers' conceptions.

With respect to MCK, Llinares and Krainer (2006) acknowledge the importance of detecting student teachers' misconceptions but propose that it be done within the frame of student teacher's learning. They suggest that it is important to study the relationship between student teachers' conceptual and procedural knowledge, and for this teachers should know about children's mathematical thinking. One method they propose for the study of the mentioned relationship is the use of open-ended questions based on vignettes describing hypothetical classroom situations where students propose alternative solutions to some mathematical problems. This kind of tasks have also been used by Empson & Junk (2004), who suggest that some of the teachers' answers are influenced by a disconnection between teachers' MCK and their understanding of children's thought, with the consequence that they precipitate to correct mistakes without establishing a contact with what the student is thinking.

Presently, there is not a unified theoretical perspective on the researches about MCK and its relation to teachers' training and professional development. It has been suggested that "future work should include a focus on understanding the knowledge the teachers hold in terms of their sense making and in relation to practice [… and that there is a] need to pursue the theorization of teachers' mathematical knowledge, framing appropriate concepts to describe its features and processes, and to establish clear criteria of levels of proficiency of mathematics teachers and instruments to assess it." (da Ponte & Chapman, 2006, p. 467).

The work we are presenting here fits da Ponte and Chapman (2006) and Llinares and Krainer (2006) characterisations, a difference with the last ones being that we investigate not only pre-service teachers but in-service teachers as well. Our principal goal is to study in- and pre-service teachers' mathematical content knowledge, but not in an isolated manner. As other researchers (see for example Prestage and Perks, 2001), we are also interested in understanding how teachers obtain, maintain and organise their mathematical content knowledge. It is worth mentioning that we are aware that mathematical content knowledge should not be separated from the other two kinds of knowledge. With this in mind, we designed some workshops that will be described below.

METHODOLOGY

TAMBA: Workshops on Basic Mathematics for in- and pre-service teachers

Within a broader project that combines research with professional development, we designed a set of workshops called TAMBA (Talleres de Matemáticas Básicas). The workshops are offered as modules that can work independently or as a set. Each one is centred on one specific mathematical content linked to the elementary school curriculum in mathematics. They all have a duration of 2-4 hours, and a common structure: they start with a short paper-and-pencil diagnosis, which is immediately commented with the participants, followed by an activity designed to raise a cognitive conflict, which takes most of the workshop's time. After it, several issues are discussed in the group: the mathematical topics and the pedagogical difficulties, including the children's most frequent misconceptions. The workshops are video taped. The design of both the diagnosis and the activity is based on our previous knowledge of the population to which each workshop is directed, and on the specialised literature.

Geometry in TAMBA

One of TAMBA's workshops is called "coloured triangles". After the diagnosis, which will be described below, the activity is centred on the unicity of the triangle's area whatever the side used as "base" (this topic follows from the item 3 of the diagnosis). Depending on the teachers' cognitive level on the subject, a demonstration is presented, and then the MCK and PCK issues of item 3 are discussed in the group.

The diagnostic evaluation has three items. In Item 1, four sets of three measures are given, and the participants are asked to say if a triangle can be built with them and, if not, why (two are possible and in the remaining two the triangle inequality is not accomplished). In Item 2, three triangles are given with measures for the sides and heights, and the participants are asked to say if the measures are possible or not, and why (two of the figures are not possible, because some heights are larger than a side from the same vertex). In Item 3, a hypothetical conversation between three girls who must calculate a triangle's area is presented, where they all make different mistakes and do not agree on the calculation, and the teacher is asked to write what s/he would say to the girls.

The teachers' answers to the written evaluation were analysed and classified according to their correctness and the kind of geometrical criteria used. The results, focused from a geometrical point of view, are being presented elsewhere. Here only the broad categories are briefly described. Teachers' ideas were classified as correct or incorrect; in the second case, several misconceptions were identified: about the triangle inequality, the base and/or height, the Pythagorean theorem, or other geometrical misconceptions. Within each of these broad categories, some finer subcategories were identified. In addition, the amount of items answered by each of the participants was registered, as well as the amount of ideas that s/he expressed clearly.

Implementation

The described workshop has been given twice. In 2007 it was offered to 36 teachers at the Conference of the Mexican Mathematical Society in the city of Monterrey (MR), and in 2008 it was offered to 31 teachers in a Teachers' Centre in Mexico City (MC). Table 1 summarises the characteristics of the participants in both workshops:

	SEX			LEVEL			OCCUPATION				EXPERIENCE		
	F	M	N/A	Primary	Secondary	N/A	In-service	Pre-service	Other*	N/A	n	Mean ± SD in years	N/A
MR	22	9	5	20	9	7	16	7	4	9	22	17.9 ± 10.6	0
MC	29	2	0	20	3	8	5	24	1	1	7	17.7 ± 9.6	7
Total	51	11	5	40	12	15	21	31	5	10	29	17.9±10.2	7

* "Other" occupations are pedagogical consultants (PC) and experts in Special-Education Teachers (SET).

Table 1

The main difference between both groups is that there were more in-service teachers in Monterrey and more pre-service ones in Mexico City. In addition, all of the pre-service teachers in Monterrey were of the secondary level, whereas in Mexico City 15 of the pre-service were of the primary level and 2 of the secondary level (7 more did not answer that question). Another difference is that in Monterrey the participant teachers were highly interested in Mathematics Education, and had applied for and obtained funding to participate in the Conference (which was given for teachers with high scores in a national assessment), whereas in Mexico City the participants were regular attendants to a Teachers Centre located in a low-income zone.

RESULTS

For each participant, the percentage of items answered was calculated, as well as the percentage of those that had clear arguments. Then the total amount of ideas expressed was figured, each idea was classified according to one or several of the categories above mentioned, and the quantity thus obtained for each participant in each category was expressed as a percentage of the total amount of ideas expressed. Finally, for each category averages were calculated taking all of the participants (see Table 2) or diverse groups of them.

	Items answered	With argument	Correct ideas	Incorrect ideas	Misconceptions				
					Triangle inequality	Base	Height	Pythago-rean th.	Other
All participants	80.0%	71.6%	27.8%	62.0%	15.0%	6.5%	8.6%	5.5%	8.1%

Table 2 [1]

As Table 2 shows, the average participants answered most of the items, and, when they did, mostly expressed their ideas with clear arguments. However, only a small percentage of these ideas were correct. Among the misconceptions, those about the triangle inequality were the most frequent.

In the following sections, these results will be analysed according to the recorded experimental variables: venue, sex, level, occupation, and teaching experience. Each time the arithmetic means are reported and analysed, although no statistical inferential analysis is carried out, the samples being neither representative nor large enough.

Venue

The 36 participants of the workshop held in Monterrey (MR) and the 31 of Mexico City (MC) differed in all of the variables considered. Table 3 shows the results obtained by teachers in both venues.

	Items answered	With argument	Correct ideas	Incorrect ideas	Misconceptions				
					Triangle inequality	Base	Height	Pythagorean th.	Other
MR	91.9%	77.3%	45.2%	42.2%	9.2%	6.0%	5.8%	8.4%	2.7%
MC	66.2%	64.9%	7.6%	85.1%	21.8%	7.0%	12.0%	2.2%	14.3%

Table 3

The teachers in MR obtained better results from all points of view: they answered more items, and expressed better their reasoning (more answers with argument). They had six times as many correct ideas and about half of the incorrect ideas expressed by their counterparts in MC; also, MR teachers had fewer responses classified in all but one of the different detected misconceptions. The largest differences were in the misconceptions about the triangle inequality, where MC teachers more than doubled their MR counterparts, and "other" geometrical misconceptions, where MC teachers made five times as many mistakes as MR participants. The one exception is the incorrect uses of the Pythagorean theorem, where MR teachers had in average 8.4% answers as opposed to only 2.2% of MC teachers. All this, as will be shown later, is related to the different characteristics of the participants in both venues.

Gender

There were also differences among the 62 teachers who reported their sex: In general, the 11 male respondents had better results than the 51 female participants did. Table 4 shows this.

	Items answered	With argument	Correct ideas	Incorrect ideas	Misconceptions				
					Triangle inequality	Base	Height	Pythagorean th.	Other
F	77.6%	66.5%	8.4%	84.1%	21.4%	7.8%	11.8%	2.2%	14.8%
M	86.9%	75.0%	41.7%	46.7%	10.4%	5.0%	5.9%	9.5%	2.5%

Table 4

The male teachers answered more questions in average than the female, and were slightly better in expressing their reasoning. Men had more of the correct ideas and fewer incorrect ones, and scored lower in all of the misconceptions, again with the exception of misuses of the Pythagorean theorem. This apparent gender effect will be commented later on.

Level

Only 52 of the 67 participants declared in which level they work or study. Their results are shown in Table 5.

	Items answered	With argument	Correct ideas	Incorrect ideas	Misconceptions				
					Triangle inequality	Base	Height	Pythagorean th.	Other
P	78.1%	74.6%	25.3%	63.8%	20.2%	5.5%	9.4%	6.9%	7.3%
S	86.2%	77.4%	40.1%	52.7%	11.6%	8.1%	5.0%	8.0%	4.6%

Table 5

Generally speaking, the 12 teachers of the Secondary level had results that were only slightly better than those of the 40 of the Primary level: more items answered as an average, more responses with argument, more correct ideas, and fewer incorrect ones. However, it is noticeable that the distribution of misconceptions found is not homogenous: Secondary level teachers have fewer answers with misconceptions about the triangle inequality, the height and other errors, but have more answers with misconceptions about the triangle's base and the Pythagorean theorem.

Occupation

Of the 67 participants, 57 declared if they were in-service teachers (21), pre-service teachers (31), or if they had other occupation (5 were PC or SET). Table 6 shows the results for the first two categories.

	Items answered	With argument	Correct ideas	Incorrect ideas	Misconceptions				
					Triangle inequality	Base	Height	Pythagorean th.	Other
In-	86.9%	78.8%	31.5%	58.9%	15.4%	7.2%	10.1%	13.1%	1.5%
Pre-	68.3%	60.9%	17.1%	72.8%	17.9%	5.0%	5.7%	3.1%	12.1%

Table 6

In-service teachers had better results than the pre-service ones: more items answered, more answers with argument, more of the correct ideas, and fewer incorrect ones. However, in-service teachers scored higher than pre-service ones in three of the identified misconceptions: about the triangle's base and height, and about the Pythagorean theorem.

Experience

Of the 36 participants who were in-service teachers, PC, or SET, 22 declared their teaching experience. Their results are shown in Table 7.

	Items answered	With argument	Correct ideas	Incorrect ideas	Misconceptions				
					Triangle inequality	Base	Height	Pythago-rean th.	Other
1-10 yrs	81.9%	80.3%	16.2%	76.2%	18.2%	6.1%	11.7%	22.0%	6.2%
11-20 yrs	98.3%	86.5%	31.3%	60.3%	13.1%	13.1%	18.3%	5.6%	5.2%
>20 yrs	90.9%	78.1%	53.2%	34.1%	9.0%	5.1%	6.5%	0.0%	4.2%

Table 7

Teachers with more years of experience have a tendency towards better results, and teachers with less experience towards worse results, in almost all aspects. However, teachers with between 11 and 20 years of teaching experience have more answers classified as misconceptions on base and height than the other two groups.

Overall, the teaching experience does have a marked influence on a decrease in incorrect ideas, as the graph of Figure 1 shows (in it the value for 0 years is the average for all student teachers). The correlation coefficient between teaching experience and percentage of incorrect ideas is r = –0.51.

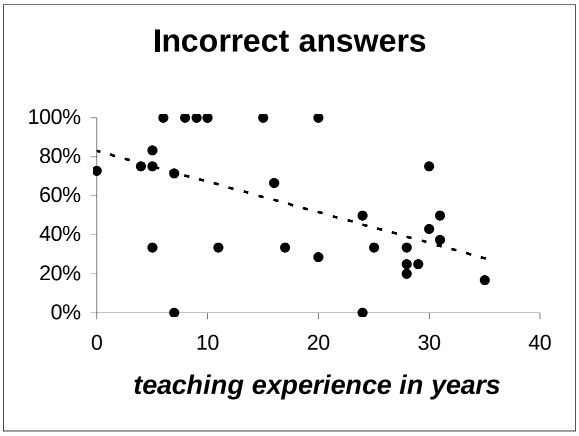

Figure 1

Language and didactical competence

Another characteristic of the responses to the diagnosis given by the participants is the quality of the language used and of the didactical explanations provided in the hypothetical situation of Item 3. Although we do not have here the space to show the analysis that we carried out, we want to state some of the findings. Many answers are based on orders or assessment, which reflect the disconnection described by Empson & Junk (2004) between MCK and the understanding of children's thought. It is also evident, as was stressed by Boero et al. (2002), that the natural language can provoke difficulties in the acquisition of concepts. Finally, some teachers, particularly of the Secondary level, have an attitude that could be expressed as "*I know so much that you cannot understand me*".

ANALYSIS AND CONCLUSIONS

Two considerations must be taken into account. Firstly, we must stress that if a teacher does not manifest a misconception, this does not necessarily mean that s/he does not have it; it could also be that in his/her expression the misconception just did not show. Secondly, although no hard facts can be deduced of the information obtained from this study, the results we have shown can be interpreted in terms of possible tendencies that could be investigated in a next step of the research.

It would seem that, with respect to MCK relating triangles, male teachers, secondary school teachers, in-service teachers and highly experienced teachers obtain better results than their counterparts do.

The gender effect that we found in these results could make sexists happy. However, in the group of teachers that participated in the two workshops, 62% of the female teachers were pre-service ones, and among the male teachers the percentage was 20%; thus, the gender effect could be confounded with the variable “occupation”. The other groups with better results were to be expected: teachers of the Secondary level receive more mathematical training in ENS, and in-service teachers have dealt with the teaching (and are thus more in contact with the students’ way of thinking, in accordance with the findings of Empson and Junk’s, 2004), and even more so as their teaching experience increases.

As for the differences between the obtained results in the two venues, the better results of MR can be related to two factors. The first factor is that, as Table 1 shows, in MR there were more Secondary level teachers (25% vs 10%), and more in-service teachers (44% vs 16%): two of the three “better” groups (with no differences on the fourth variable, the teaching experience). The second factor, which could be of even more importance, is the difference in the ways that teachers arrived to the workshops. MR teachers were highly interested in mathematics and its teaching, and also had good scores in a national assessment, whereas MC teachers did not share this characteristics and were regular attendants of a teachers’ centre in a low-income part of the city.

It can be interesting to comment on the cases that stray from the reported tendencies, which relate to misconceptions about the triangle’s base and/or height, and about the Pythagorean theorem. We carried out an analysis using the fine-categories in addition to the broad ones about base and height described and used in this paper, which we do not have here the space to present. However, this analysis shows that some of the misconceptions can be linked to didactical strategies (where the informal and potentially incorrect use of mathematics serves a didactical purpose), and that modern teacher training is slowly (and partly!) fighting some misconceptions about base and height, through fewer prototypical examples in the textbooks for student teachers. As for the misuses of the Pythagorean theorem, there are more answers with this classification in two of the three “better” groups (Secondary, in-service). One possible interpretation of this is that the groups with a higher level in general also have some idea about the existence of the Pythagorean theorem and, approximately, what it is about. (It could also be that more recently trained teachers have heard about the theorem). However, all of the teachers who pretended to use this result did it in one of several incorrect ways; this relates to Hershkowitz (1990) characterisation of misconceptions that increase as the students advance throughout their schooling.

The effect that the teaching experience has in decreasing (but not nullifying!) the amount of incorrect answers is something that must be valued in professional development programs. When the teacher (and particularly the Primary school one) starts her/his practice, s/he must deal not only with the students' difficulties in the learning of mathematics, but also with her/his own deficiencies in MCK, which in turn have the effect of not only perpetuating but also aggravating their students' misconceptions. The professional practice can help in dealing with both the students' learning difficulties and the teacher difficulties in MCK, but if s/he had more support with MCK, the pedagogical difficulties would be easier to handle. Therefore, we coincide with Bromme (1994) in that MCK must be understood as the scholar subject, and we assert that it is something that must be attended to, diagnosed and solved, both in initial training and in professional development.

NOTE

1. The 71.6% of ideas with argument is 100% minus the answers without clear argument: 10.1% that were potentially correct and 18.3% that were incorrect. The 100% of ideas is formed by correct ones, plus those that were potentially correct but without clear argument, plus the incorrect ones, including those without argument. The same calculations were carried out for the other tables.

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