THE NATURE OF NUMBERS IN GRADE 10: A PROFESSIONAL PROBLEM

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Teachers who teach grade 10[1] in France have to ensure the continuity of the mathematics taught between Junior High and Senior High[2] without doing any systematic revision. It seems to be a difficult task as teachers have to elaborate on reprise gestures[3] (Larguier, 2005) to go over knowledge already taught in Junior High while also introducing new knowledge. It is thus this problem of the profession (Cirade, 2006), which we analyze through direct observations of classes and data collected, about the way teachers tackle this. This study has allowed us to show some characteristic elements of this teaching problem. For example, the determination of the nature of numbers is a type of tasks between the two institutions; it can also be gone over as a reprise in various niches of the syllabus throughout the year. However, we show that teachers do not seem to take advantage of these opportunities.

Keywords: reprise, professional gestures, the filter of the numeric

A PROBLEM IN THE PROFESSION OF TEACHING THE NUMERIC

Going into grade 10 in France is a threshold to be crossed between Junior High and Senior High; it is an important passage between the two institutions. The mathematics syllabus states that in grade 10 students have to master the knowledge and know-how that most of them have already been taught in Junior High. A question then becomes central: the relationship between the *professionnal reprise gestures* and the knowledge and know-how. It takes us to the broader question of *interweaving* (Bucheton, 2009). We analyze the kind of gestures about the synthesis of numbers encountered during Junior High which must be done thoroughly during Senior High. We update the problems for teachers even though this part of the syllabus does not seem to be problematic for them to teach.

THEORETICAL FRAME

To study the question of the construction of numeric space in grade 10, we essentially use the framework of the “anthropological theory of didactics” which has been developed by Chevallard (2007) and studies concerning the numeric and the algebraic

[1] In France, there are two distinct institutions after primary school: “collège” for students aged 11 to 15 and “lycée” for students aged 15 to 18. The first Senior High class is called “seconde” and corresponds to grade 10.

[2] Junior High school will be used for “collège” and Senior High school will refer to “lycée”.

[3] “Reprise” can mean to go over, to patch together, to interweave. We shall use the word “reprise” for reasons of economy.

in Alain Bronner's works (1997, 2007). Bronner developed a tool for the study of numeric space: the “filter of the numeric”. The function of this filter is to " pursue" the numeric, whether it is at a practical level or an institutional level. Various elements of a numeric space can be identified:

- **The objects**: the number systems, the set operators (taking the square root) and comparators (< …);
- **The types of practices** (exact calculation, approached calculation and mixed calculation) as well as the various institutional contracts of calculation;
- **The articulations and the dynamics** of the numeric domain with the other domains as well as the underlying contracts;
- **The rationales** (“raisons d’être” in French) of the numeric.

Analysis of the numeric domain is completed by the identification of the “mathematical organizations” of the numeric. Together they make up a numeric space. The observation of the numeric space also includes the “didactic organization” to say what is specifically numeric. We also take from Chevallard (1999) the notion of praxeology which is broken down into four elements: type of tasks, technique, technology, theory. It permits us to model a teaching task which we indicate by professional gestures. We also use the levels of didactic determination defined by this author (1999) to question the conditions and restrictions of various origins which weigh on the didactic choices of the teachers. These levels as defined by Chevallard are: civilization, society, school, pedagogy, discipline, domain, sector, theme, and subject.

The study of the *reprises* can be analyzed according to different criteria (Larguier, 2005). The principal criteria of all the *reprises* can be represented on an axis, the extremes of which are:

- on the one hand, systematic revisions which do not meet with the new knowledge required by the syllabus;

- on the other hand the *reprises* which link up with new knowledge. In other words, the new learning and knowledge are the continuation of the study which began in the previous classes. This first criterion can also vary between systematic revisions (a kind of repetition of the same), a form denounced by the official curriculum; and *reprises* in accordance with the syllabus which introduce something new.

The second criterion of analysis of the *reprises* concerns the mathematical contents institutionalized at the end of the learning experience. It involves the targeted mathematical praxeologies, in other words the mathematical organization. This establishes a connection with the objectives of the teacher with regard to the types of mathematical tasks which are given. These objectives are:

- techniques to be reproduced by imitation and without a justification, so that technologico-theoretical elements of the praxeology are missing;
- know-how only for action, legitimized only by explanations which do not allow for updating mathematical rationales. Technologico-theoretical elements of the

praxeology are then incorrect towards the epistemology of the discipline;
- knowledge constituted with complete praxeologies that supposes that four elements of the praxeology are present and based on mathematical rationales.

This second criterion is called completeness of the praxeologies. It identifies the degree of completeness between two extremes: they are complete, and it seems that they are mathematically valid; otherwise they are incomplete.

METHODOLOGY

Our research on the teaching praxeology concerning the *reprises* of the numeric leans on the study of grade 10 with a particular methodology. It differs from usual methods in the didactics of mathematics; in fact the analysis of the teaching practices in the classes is not conditioned by the objectives and the expected behavior of the researcher. This would have been clarified by an analysis a priori according to the research project. Here, observation in class comes first, permitting discovery and access to the knowledge taught, without any interaction between the teacher and the researcher. From elements revealed to the researcher in the dynamic of the teaching, an analysis a priori is elaborated. This is done by taking into account the previous experiences of the students, the didactic memory (so called by Brousseau) of the class and the requirements of the syllabus. It is then possible to make parallels between this analysis a priori and the project of the teacher reconstituted by the researcher after the session. In the same way, parallels can be drawn between this analysis a priori and the analysis a posteriori of the observed session. The collected data by observing sessions in a class throughout the school year are completed by interviews with teachers and with some students representing various levels, as well as by all the written traces of the year (exercises, lessons, homework …). Teachers and students only knew that the researcher was interested in the teaching of mathematics. They did not know about our interest for numerical domain. So the interviews with teachers and students were open and the focus of research was hidden. This condition was important to capture ordinary practices with the least possible influence of the researcher. Two experimented teachers (but not experts) agree to the researcher's presence in their classes, Mathieu in 2006 2007 and Clotilde in 2007 2008. This research follows a study in the framework of a Master 2 qualification (Larguier, 2005) which had made it possible to track down the difficulty of *reprises* at the beginning of the school year for novice teachers in grade 10, notably Rosalie.

THE PROBLEMATIC OF NUMERIC

In the document which accompanies the syllabus (June 2000) we found the following commentary concerning the sector “numbers” and the theme “nature and writing of numbers”: “We will make a summary of the knowledge encountered so far by the students and we will introduce the ordinary notations of the different sets. The students will have to know how to identify which numbers belong to which set”. So, the recognition of the nature of the numbers is a well-defined task in the syllabus and is faithfully followed by the teachers according to the researcher's observations. We are going to develop our analyses concerning the following task: "recognizing which sets the given numbers belong to". This type of tasks is emblematic of the numeric domain worked on at the beginning of the year during the resumption of the school year. It is also equally symbolic of the Junior High/Senior High link by allowing a *reprise* of former knowledge and at the same time working on completely new knowledge (like the nomination of sets). This type of tasks will be written as T, this represents an essential problematic to the numeric domain. This restriction is found at the level of the discipline in Chevallard's terminology.

In Clotilde's and Mathieu's classes many specimens of T are worked on in the first chapter. In general the justifications are not asked for. In Clotilde's workbook the following affirmations without any justification are found: $\sqrt{18}$ irrational or 1/3 rational. The decision theory made in the relative class to this type of task T is incomplete. The technologico-theoretical block elements are absent, the expected response of the teacher rests on the numerous implicit elements which are certainly not shared by all the students.

The same observation concerning the incompleteness of the praxeologies relative to T was carried out on the 17th of September 2004 in Rosalie's class. We will take the same example which has been indicated and which concerns written numbers under the quotient form of two whole numbers. Rosalie does a particular study of two specimens $\frac{22}{7}$ and $\frac{103993}{33102}$ prompting this study with the fact that they are approximations of π. In other words, a cultural condition which is not based on a real mathematical problem.

For the first example, a possible technique known from Junior High, is to carry out the division of 22 by 7 in order to prove that the decimal writing of the number is unlimited and periodical. Rosalie expected this proof from the students as a relative technique to 22/ 7, which corresponds to an interesting *reprise* to continue to work on the concept of decimal numbers as is seen in this extract:

> A student wrote his answers on the board. Rosalie hears another student in the class:
> **Alexis:** It's a rational number
> **Teacher:** Why?
> **Alexis:** Because it's a fraction and the decimal part is infinite
> **Teacher:** How do we know that? …. It's best to write down the division because the calculator will always give a finite amount of numbers…of terms since it shows the numbers it has on its screen. Now this one here (she points out "$\frac{103993}{33102} \in R$" written on the board by a student) who doesn't agree?

The proof for the first quotient 22/7 is brought up orally, but it is not carried out effectively by the students, or the teacher. With the calculator experiment, Rosalie does not leave the students enough time to do it themselves. In doing this, she also avoids a debate which could have taken place on the nature of numbers displayed on the calculator screen. This certainly would have allowed her to consolidate the necessary learning of this tool and the numbers in play (moreover, registered learning in the syllabus as one of the numeric themes). The mathematical decision theory linked with T is just a draft, it is not completely developed yet. We can therefore ask ourselves what is going to remain of this for the students. We equally make a hypothesis that the personal relationship between the students and the mathematical activity in general runs a risk of not conforming to the institutional relationship. Rosalie may let her students believe that it is enough to bring up a possible proof during a demonstration.

For the second example, the possibility of articulation with the new parts of the Senior High syllabus is interesting. Indeed, the two rational numbers 22/7 and 103993/33102 are both *idecimal numbers*[4] (Bronner, 1997) but the choice of numerator and denominator for 103993/33102 makes it necessary to change the technique compared to the previous example. The technique expected by Rosalie for the first number, to know the division "by longhand" of 22 over 7 cannot lead to the underlining of idecimality for the second number. The quotient obtained for the first number is 3,142857 while the length of the period from the second quotient is too big for the quotient to be calculated by longhand. We see a change of the didactic variable between the two tasks. We wonder if this is really what the teacher anticipated. Indeed, in the observed session, the fact that the second number is idecimal is not shown and is not even questioned:

> **Teacher:** (…) Now this one (she points out $\frac{103993}{33102} \in$R written on the board by a student) who doesn't agree? Yohan, Kamel?
>
> **Kamel:** I agree but it's also a rational number
>
> **Teacher:** It is, that's true but the answer to the question lies in Q. It's the R of real and it's the Q from quotient (she corrects what is on the board at the same time). But we suppose that Xavier is using the notations that he knows. Now the last one... (she points out $\frac{167}{80}+\frac{\sqrt{10}}{3} \in$R).

The study of the nature of numbers, beyond knowing whether a number is rational or not, is not made. There is not even a technique brought up contrary to what is brought up for 22/7. Consequently there is no implementation of a new decision theory, it is avoided. A possible technique in grade 10 uses a theorem which is in the syllabus (optional). It is not available to the class at this moment of the year. The question of knowing if the number belongs to **D** is thus left aside. In the second case, the demonstration of the *idecimality* of the rational number is not even brought up, it is simply completely avoided.

Nevertheless a decision theory corresponding to the syllabus could have been built into this class for task T. Here is the description: a possible technique in grade 10 is

[4] Idecimal: in Bronner's terminology, following the model of rational/irrational, decimal/idecimal

to determine the irreducible fraction which is equal to the given quotient. In this case, Euclide's algorithm allows us to demonstrate that the numerator and the denominator are coprime, and that the given fraction is irreducible. The denominator has a decomposition in product of prime numbers $2\times3^3\times613$, it is not a product of powers of 2 and 5, the number is *idecimal*. This technique is possible only from grade 10 onwards, but it also uses tools which are taught in Junior High, like the idea of irreducible fractions. This also permits another way of conceiving the decimal number in the register of fractional writings (Duval, 1995). Therefore, it gives us the opportunity to really strengthen our knowledge of numbers. So, T is indeed in a moment of *reprise* in the numeric space, which allows us to connect past knowledge, and new knowledge.

The comparison between what could have been done with T and what was effectively done clearly shows what is avoided in the targeted mathematical organization. We wondered why Rosalie made these choices:
- Is it about a lack of reflection in the analysis of the session?
- Is it the decision about the mathematical theory regarding the syllabus which is seen as not being a suitable teaching form in this class?
- Does Rosalie anticipate that the technique is too difficult to set up and might discourage students at the beginning of school year? This technological element of the professional gesture was confirmed in an interview with her. She said that she does not want to put students off learning mathematics.

This observation brings to light one of the difficulties that teachers have in building numeric space. The work in this numeric domain assumes a very precise study of the mathematical decision theory in accordance with the knowledge of the students. Another symptom of the problem of the profession is probably the misunderstanding of teachers on these difficulties. It asks the following question: what is the knowledge necessary for teachers in order to achieve the process of didactic transposition between the reference mathematical knowledge and the knowledge to be taught (Bosch et al., 2005)?

But what are the *raisons d'être* of this emblematic task? What essential mathematical problem for the discipline motivates the mastery of decision theory linked to T? By asking these types of questions, we refer to Yves Chevallard who denounces the teaching of mathematics as being like a museum visit, or the traditional way of teaching answers, even when the original questions have been lost (Chevallard, 2000). He questions what motivates the calculation of numbers in order to express them under these particular forms. He makes us become aware of the problem which legitimizes this work in the numeric domain:

> "We come to […] a big problem in mathematics: how to recognize if two mathematical objects of a certain type are or are not the same object? How to know for example if $7\times5-8 = 23$? Or if $\frac{60}{84} = \frac{380}{532}$? Or again if $\frac{n(n+1)(2n+1)}{6} - \frac{(n-1)n(2n-1)}{6} = n^2$? There is one solution to this one generic, universal problem: to respond to the question asked. We

need to use a considered type of written system for the objects, where each of these objects has a writing expression and a written expression of its own. The calculation of the «canonic» writing of the objects to be compared therefore allows us to answer: so we have $7\times5 - 8 = 35 - 8 = 27$, which shows that $7\times5–8 \neq 23$. Similarly it comes from a part $\frac{60}{84} = \frac{4\times15}{4\times21} = \frac{3\times5}{3\times7} = \frac{5}{7}$, from another part $\frac{380}{532} = \frac{190}{266} = \frac{5\times19}{19\times7} = \frac{5}{7}$, meaning that we can positively conclude this time that we have equality $\frac{60}{84} = \frac{380}{532}$."

In this citation Chevallard wishes to show that the only question about numbers which is important is to know how to write a number in relation with its nature. Different kinds of writing are possible, and we have to know the canonic one, useful to compare and calculate with several numbers. So it is not the knowledge of the nature of the number that is important, but the knowledge of the canonical writing given for a type of number. This necessity is backed up by another necessity of mathematical work, which is the rule of the *institutional contract of calculation* (Bronner, 2007). For demonstration work in mathematics, we are obliged to use exact values. The following reasons explain then why it is important to know the exact values of trigonometric lines of particular angles such as: $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and why we keep this way of writing with a radical. We are going to further develop this example, various types of numbers appearing within the framework of trigonometry, a *reprise* of work on the numeric is then possible.

THE EMBLEMATIC TASK AND TRIGONOMETRY

In the part concerning irrational numbers we are going to come across "products" (Bronner, 2007) within the framework of trigonometry, but neither their appearance nor their nature is questioned. In Mathieu's and Clotilde's classes, the chapter on trigonometry was approached late in the year, for Mathieu from May 23rd, 2007 and for Clotilde from April 30th, 2008. By using our methodology, a work of comparative analysis was able to be carried out.

The comments of the syllabus of grade 10 state: *"The definition of sin x and cos x for a real x will be made «rolling up **R** » on the trigonometric circle. We will make the link with sine and cosine of 30°, 45° and 60°".*

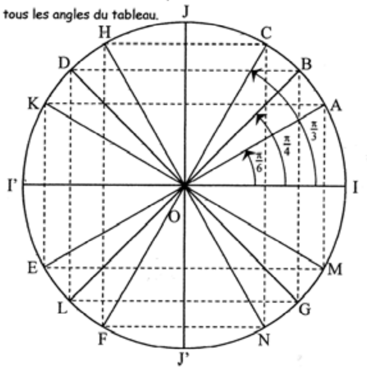
During Clotilde's lesson on May 16th, 2008, at the end of the sequence on trigonometry, she gives out a table which the students have to complete.

Exercice : On a donné les valeurs exactes du sinus et cosinus de quelques angles remarquables entre 0 et $\frac{\pi}{2}$.

Point								I	A	B	C	J				
x	$-\frac{5\pi}{6}$	$-\frac{3\pi}{4}$	$-\frac{2\pi}{3}$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos x$								1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0				
$\sin x$								0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1				

a. Retrouver le point qui correspond à chaque angle.

b. En déduire les valeurs exactes des cosinus et sinus de tous les angles du tableau.



This document presents an extraordinary showcase of numbers which appear in the numeric space of grade 10 with whole relatives, decimals, irrationals formed with the typical examples often used like $\pi, \sqrt{2}$ and $\sqrt{3}$. We observed that it does not become the student's responsibility to know that it is necessary to keep complex writings of these numbers, for example $\frac{\sqrt{2}}{2}$. If the teacher had given the responsibility of this question to the students, then he would have been able to carry out a *reprise* of the emblematic task T to justify the canonical writing of these numbers. But the awareness of the nature of the numbers is completely absent in this entire sequence even though it is very rich in respect to possible work on the numeric. The only justifications are under the form of conventional rules not referred to as necessities of the discipline. So, Clotilde does not accept the answer $\frac{1}{\sqrt{2}}$ and transforms it into $\frac{\sqrt{2}}{2}$ by arguing that: *"as we already said we did not like the roots of 2 under the line of fraction, we write it like that"*.

Thus, teachers accustom the students to practices of exact calculation, which are governed by conventional rules only decided on by the teacher, while epistemological reasons support them. The institutional contract of calculation remains in this context of trigonometry entirely the responsibility of the teacher. Nevertheless, the underlying questions could be seen by the student as being an aspect of the mathematical work.

The numeric space elaborated in grade 10 is so enriched by new elements which are *operators* (Bronner 2007), namely the operators cosine and sine, generators of tables of real numbers containing many irrational numbers. These operators allow a production of numbers in a procedural way. The interest is centered on the way of obtaining the numerical values, and not on their nature. In the same way, there is no interest in the change of status of the number which must be seen as a variable of the function cosine.

The *dynamic* implemented by both teachers is a *numerico-geometrical dynamic* (Bronner 2007). Numbers of various natures are generated by the operator cosine from the trigonometric circle and from the right-angled triangle. However, another dynamic remains implicit, it is an *inter-numeric dynamic.* This one could exist thanks to the numeric resumption of work at the beginning of the year linking with the symbolic task and the canonical writing of the numbers according to their nature. However, it would seem that this symbolic task is not exportable except the sector "Numbers" of the domain "Calculations and functions". This place of trigonometry in grade 10 would allow the numeric to work, because irrationals come "naturally". But, the awareness of the nature and the writing of these numbers is not the responsibility of the student. Nevertheless, it would be interesting to ask the question about the exact value of a number like for example cos17 and to make the students aware that the writing of the exact value is cos17, in the same way that the exact value of $\sqrt{34}$ cannot be written without using a radical. These examples could enrich the usual prototypes used as irrationals. Nevertheless, from the synthesis of numbers encountered in the vast *mixed-bag* of school, this type of number has been popular and can be reused as an example.

IDENTIFICATION OF A PROBLEM IN THE PROFESSION

We asked the question of the *reprise gestures* concerning the study of the nature of numbers by focusing our gaze on an essential problem in mathematics: writing numbers according to their nature. Obviously, this question takes its meaning only in the context of a problem. The most relevant register of writing is conditioned by the work to be done with these numbers. But what we also observed with the teachers was the absence *reprise* whenever the problem arose. The notions are only worked on as objects, the "raisons d'être" posed about the writing of the numbers becomes nothing more than a question of habit.

In the reality of our observations, the teachers introduce T to the students at the beginning of the year in a certain number of cases in accordance with the syllabus. They do this without taking into account the specific problems of the discipline, nor is it used later to pursue the study of synthesis relative to numbers. Nevertheless we have seen that a *reprise* of T is possible during the grade 10 syllabus (we have only quoted the case of trigonometry). Teachers do not see these new niches for reactivating this type of tasks no matter how essential it is to work on the numeric. Our study opens new ways for identifying specific teachers' knowledge in the matter of numeric domain. It is especially useful for the formation of teachers and the necessary practice of particular *gestures of interweaving*.

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