TEACHERS' SUBJECT KNOWLEDGE: THE NUMBER LINE REPRESENTATION

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This paper considers the perceptions that trainee and experienced teachers have of the number line. Grounded within the theoretical perspective highlighted by Herbst (1997) the paper examines the interpretations that 'teachers' place on a core classroom representation advocated for teaching the number system in English schools (DfEE, 1999; 2006). The outcome suggests that primary school teachers have conceptions of the number line that do not portray conceptual understanding of its abstract nature as a representation of the number system. Descriptive characteristics of visual models, ambiguity and an emphasis on use overshadow the deeper understanding that would lead to the realisation of the potential as a valuable metaphor.

Key-words: Number Line, Teachers, Conception, Interpretation, Ambiguity.

INTRODUCTION

This paper brings to surface teachers' knowledge about the number line representation. A representation used extensively within English mathematics classrooms and that appears frequently within curriculum documentation – the National Numeracy Strategy (DfEE, 1999; 2006) and the National Curriculum for Mathematics (DES, 1991). Within these two documents, there is no explicit reference to the conceptual knowledge associated with the number line's form and use, despite the fact that this representation is identified as a "key classroom resource". The number line appears not only as an alternative version of the number track, but it also is frequently fragmented to emphasise particular features of the number system such as whole number and fraction. The difference between a number track and a number line lays both in the perceptual and the conceptual sense identified by Skemp:

> The number track is physical, though we may represent it by a diagram. The number line is conceptual – it is a mental object, though we often use diagrams to help us think about it. The number track is finite, whereas the number line is infinite. … On the number track, numbers are initially represented by the number of spaces filled, with one unit object to a space. … On the number line, numbers are represented by points, not spaces; … The concept of a unit interval thus replaces that of a unit object.
>
> (Skemp, 1989, pp. 139-141)

Using evidence drawn from the way in which practicing teachers and teacher trainees perceive and talk about the number line, this paper indicates that knowledge based on the perceptual characteristics of the number line together with an ambiguous use of the term number line with that of the number track, express an incomplete and

compartmentalised understanding of the conceptions associated with a representation which is used on a frequent basis in their primary school practice.

THEORETICAL FRAMEWORK

The association between number (real number) and line has been evident since Babylonian times (Wilder, 1968). The Greeks intuitively conceived real numbers as corresponding to linear magnitudes. The Greek idea of "magnitude", which is substituting magnitude for number, implied that one may think of "numbers as measured off on a line" (Bourbaki, 1984, p. 121). The number line is, therefore, an abstraction of a representation strongly associated with the notion of a measure instrument since continuity underscores it. Starting from the Euclidean line, a "sense of continuity" can be created for and by the individual and the result be used as a number line to represent natural numbers.

Herbst (1997) concurs that the number line is a metaphor of the number system and in order to form a number line:

> one marks a point 0 and chooses a segment u as a unit. The segment is translated consecutively from 0. To each point of division one matches sequentially a natural number. (Herbst, 1997, p. 36)

All kinds of numbers can be represented on it. If a series of different number lines each introducing different numbers is built, then the number line could be in one-to-one correspondence between numerical statements and number-line figures. Growing sophistication with its formation supports representation of a number line containing natural numbers, followed by number lines illustrating the positive rationals, the integers, the negative rationals and finally one containing all numbers — the real numbers number line, which would include all numbers. It is these features that would appear to suggest the use of the number line as a pedagogical tool whilst the "dense" quality of the number line enabled Herbst to write about what he calls the "number line metaphor" and the "intuitive completeness" (Herbst, 1997; p.40) of the number line, evolving from plane geometry.

Such features are relevant in the context of teachers' subject knowledge and awareness of conceptual issues associated with understanding the nature of the number line. Shulman (1986) defines subject matter (content) knowledge as "the amount and organisation of knowledge per se in the mind of the teacher" (p. 9) and distinguishes between the aspects of knowing "that" and knowing "why". Aubrey (1994) suggests that every teacher has different subject knowledge and personal beliefs about teaching and learning, which are factors affecting their work in classroom; and in order for teaching to be effective, conceptual understanding of knowledge is essential. It is suggested, therefore, that in the context of the number line, teachers would be effective if they conceptualized the representation as a "metaphor" of the number system.

Ball (1990) argues that subject matter knowledge for teaching not only entails '*substantive knowledge of mathematics*' – specific concepts and procedures – but also '*knowledge about mathematics*' – mathematics as a field. Examining what teacher trainees understood about division with fractions as they entered formal teacher education, she focusing on what they have learned as students and what they need to know as teachers. She concluded that the students' had narrow understanding of division that was compartmentalized and based on rules. This was a view supported by Ball, Hill & Bass (2005) who, as the result of their attempt to measure teachers' mathematical content knowledge (an amalgam of common content knowledge and specialized knowledge) for teaching, concluded that teachers in general lack strong mathematical understanding and skill.

This paper aims to present one aspect of primary school teachers understanding of the number line as identified by their conceptions of what the number line is. The insight may provide some indication of their potential effectiveness.

METHOD

The results presented in this paper form part of a broader study carried out during 2003 and 2004 (Doritou, 2006) that, given the explicit recommendations regarding the use of the number line within curriculum material, investigated the relationship between teacher's presentation and children's understanding of the number line. The study is a case study of an English primary school that follows guidance within the National Numeracy Strategy (DfEE, 1999). The issues addressed the primary school teachers' perception and understanding of the number line. This paper address one aspect of these issues but it draws its data from two samples that are considered to be related and complementary: (a) teacher trainees and (b) practicing teachers.

As part of an examination of their understanding and perception of the number line the full final year cohort of BA(Ed) students within the Education Department of a large Midland University were invited, through a questionnaire, to "Define a number line". The response to this question forms part of the focus of this paper. The 69 teacher trainees in the sample, had had the benefit of a four year course associated with the content and pedagogy of primary school (children aged 5-11) mathematics, were fully conversant with the contents of the National Numeracy Strategy, had experience teaching it within school and been provided core lectures associated with the number line. The respondents were followed a mixture of subjects, such as English, art, music and a third of them followed mathematics and science.

The full-time teachers' sample (also referred to as practicing teachers) contained teachers who taught mathematics within each of the year groups 1 to 6 (median ages 5.5 to 10.5). Through lesson observation and informal interviews on a one-to-one basis the teachers' perspective of the number line at a personal level and the way they presented it to the children as a pedagogical tool was investigated. Placing the trainees conceptions of the number line within a perspective associated with

practising teachers, it is hypothesised a valuable insight may be gained into what primary school teachers think a number line is.

RESULTS

Teacher Trainees' Conceptions of the Number Line

When the participating Teacher Trainees (TT) were asked as part of a questionnaire to define a number line, only one student provided a definition that implied that the number line was infinite and contained all numbers:

> A line that contains all rationals and irrational numbers. It is an infinite line. (TT4)

One other suggested it was:

> A continuous line of all of the numbers within our number system. (TT1)

Two others provided definitions that evoked either the notion of infinity but with no further explanation, one indicated that the number line was limited to rational numbers, whilst one other defined a number line with a response that may be interpreted as an association with magnitude:

> A sequence of numbers arranged on a line which has an infinite number of divisions. (TT23)
>
> A line of numbers on which any number can be placed. (TT48)
>
> A line where you may place all the rationals at some point on the line. (TT32)
>
> Representation of value according to how far the number is along the line. (TT43)

None of the above students gave any explicit reference to the notion of a repeated unit, which could be partitioned, although partitioning may be implied from the statement of TT4. However, almost one quarter of the students (16/69) did make reference to some form of equal spacing associated with the line, although there was some evidence of little formality about the way they articulated this underlying feature:

> A line which is separated equally into different portions. (TT2)
>
> A straight line with equal distances marked. (TT7)
>
> A piece of apparatus with equal divisions marked. (TT10)

13 of these sixteen students associated the notions of equal spacing with numbers although in two instances the students referred to digits:

> A line with digits equally spaced along it. (TT47)
>
> A line with numbers attached at equal intervals. (TT66)
>
> A line which numbers are spaced evenly across it in a specified pattern. (TT17)
>
> An equally segmented line, each segment numbered in ascending order. (TT20)

Although it is not certain, TT20's definition suggests that she is thinking about a number line that only has positive numbers. This type of definition was relatively common:

> Numbers placed at identical intervals marked on a line in ascending order. (TT15)

and indeed, no student made explicit reference to the notion that a number line could contain negative numbers.

TT17's reference to pattern was, together with notions of order and sequence, a feature of the number line identified by 42% of the respondents:

> A string of numbers in a pattern. (TT27)
>
> Numbers in a correct order. (TT9)
>
> A sequence of numbers in a row. (TT22)
>
> A sequence of numbers ordered from left to right. (TT24)
>
> A line in which there is a number sequence reaching from lowest to highest number. (TT11)
>
> An ordered set of numbers in sequence, horizontal. (TT6)

Here again we see no explicit reference to negative numbers. The implications in two of these quotes (TT24 and TT11) suggest that the number line only contains whole numbers, an issue confirmed by the comments of some trainees:

> A line with number patterns on it — or from zero to a number. (TT12)
>
> Numbers that have been arranged in some form of sequence mainly from 0 to 10. (TT35)
>
> A horizontal line with a series of digits on it that have a pattern: one to ten; ten to one hundred. (TT42)

The above comments also give the sense that the number line is finite and none of these particular trainees made any reference to the notion of partitioning the intervals. However, one student did provide an indication that partitioning was associated with the line by using the word "divided":

> A horizontal line divided into ten equal sections allowing it to be divided into fractions or quantities. (TT64)

Interestingly, in addition to these students who explicitly mentioned order, pattern or sequence, six others introduced the word "chronological" to define the number line:

> A chronological line of numbers. (TT37)
>
> A line with marked number intervals in chronological order. (TT56)
>
> A horizontal line where positive numbers ascend in some sort of chronological order. (TT61)

We can see from the definitions provided by the trainees identified through the above examples, that reference to the underlying qualities of Herbst's (1997) definition — the consecutive translation of a segment U as a unit from zero, the partitioning of U in an infinite number of ways — is extremely limited. We note that only three students referred to infinity, but only one of these implied that through partitioning all numbers could be represented. However, though there was no reference to the notion of "consecutive partition", almost 25% of the teacher trainees indicated that a number line possessed equal divisions but these definitions appear to be founded upon partitioning rather than the continued replication of a defined unit.

Herbst further indicated that a number line could be formed by choosing a unit, repeating it from zero and then attaching to the end of each repeated unit a natural number. Though just over 80% of the teacher trainees associated the notion of the number line with a number or numbers, the majority of the remainder focussed on defining the number line as a tool (see below) but, as TT6 (above) indicated, there was also some evidence that the reference to numbers was not linked to the notion of line.

The overall impression left from the trainees' definitions of the number line was that they did not define it, but instead indicated how it may be seen. The sense was that they were describing a specific number line but often this specificity was limited to the more obvious perceptual characteristics rather than conceptual aspects of the line. In doing this, essential features were often omitted. Only in the first six instances quoted above do we see the trainees' explanations rise above specificity to give more sophisticated responses.

An additional feature of the trainees' definition of the number line was its identification as a tool. Almost 10% of the trainees suggested that the defining feature of the number line was either its use in calculation or in solving mathematical problems:

> A continuous line in which numbers can be placed and used to aid calculations. (TT3)
>
> A piece of apparatus with equal divisions which children use to help them count. (TT10)
>
> A line with numbers on representing intervals, aid to solving mathematical problems. (TT34)

or associated it with the notions of counting:

> A device to aid learning, involving counting on and counting back. (TT39)
>
> A method used to count on or back horizontally. (TT62)
>
> To aid children when counting up or down. (TT65)

In one instance, the identified process was left open to interpretation:

> A way of roughly finding out any numbers between any two given extremes at each end.

(TT52)

Although the above responses emphasise the nature of the number line as a "helping tool" – used as a metaphor to support thinking – and although Herbst (1997) suggested that its dense nature meets such a requirement, there seems little indication from these particular responses that other qualities could be associated with the number line. Additionally, the responses suggest that those students who emphasise use are drawing upon experience, either as learner or as teacher and, it is hypothesised, were drawing upon episodes from within that experience.

Practicing Teachers' Perception of the Number Line

When the practicing teachers were each informally interviewed about their conceptions of the number line, one issue that was raised was whether or not they thought that the number line was a good representation of the number system.

3 of the 5 teachers identified the number line as a good representation of the number system because it carried the very ideas that 42% of the trainee teachers expressed with their definitions of the line. That is an emphasis on order and sequence:

> Yes! I suppose it is because it is natural order in a sequence, isn't it? (Y2 Teacher)

> It's a good representation for them to be actually able to see it! It has it (numbers) all in order and they can see it! (Y5 Teacher)

The fact that children could 'see' the number line was one of the reasons why a Year 4 teacher (teaching children with a median age of 8.5) thought the number line was a good representation of the number system

> Because it's visual and children like visual things, and they can come up and interact with it. (Y4 Teacher)

Having something to see enabled some of the teachers to be quite specific in talking about the number line although there was evidence that this could lead to the sort of confusion identified by Skemp (1989), particularly if we recognise the hundred square as a segmented number track:

> I have got the number line, which is really useful, but because it's so long, it is quite hard… It's at least two metres (a number line on laminated card under the board). I do refer to it quite a lot, but I do use the number square as well. I do try and encourage the children that it's the same. (Y2 Teacher)

This similarity between the hundred square and the number line was also volunteered by the Year 3 teacher. He indicated that the number line is a good representation of the number system when used to develop subtraction, but not so easy as the hundred square which is

> easier than sometimes using the number line. Really, they're sort of similar things, but this goes zero to one hundred, this goes from one to one hundred, so it's the same really… (Y3 Teacher)

Other evidence associated with seeing and with accessibility came from the Year 1 teacher, who when asked if there was a difference between a number line and a ruler, replied:

> I just use the ruler, because it's a good individual tool and easily accessible. So if they want to use the number line it's immediately accessible. (Y1 Teacher)

Within her teaching of the classroom lessons, this teacher and the Year 2 teacher both drew an analogy between the number line and the ruler:

> A ruler is a bit like a number line. (Y2 Teacher)

> The number line here is like a ruler. Use a ruler[1] as a number line to help you. (Y1 Teacher)

However, the Year 2 teacher preferred to use the hundred square

> I do use the hundred square as well in the classroom, coz that's easier to display to be honest. (Y2 Teacher)

One of the teachers explicitly thought the number line was a good representation of the number system, because of the arithmetic that could be done with it:

> Yes! Very good! Use it to bridge through multiples of ten. Partition the numbers and then the tens and then the units, if they're doing addition. And if you're working out subtraction. (Y3 Teacher)

The teacher teaching Year 6 among other classes was the only teacher who gave a response that made any reference to the fact that the number line (although finite in her terms) was a representation of the number system:

> … I think Year 6 children are quite good to see that the number line represents quarters, halves, numbers up to a thousand or even negative numbers.

This teacher's response to the question "Is the number line a good representation of the number system?" bore remarkable similarities to the trainee teachers' conceptions of the number line. Two of the five teachers referred to pattern, order and sequence. There was reference to the number line as tool, but only one reference to the variety of numbers that could be represented on it. However, whilst all of the teachers could talk about what it may look like or what it may be associated with, none provided a sense of its continuity and density. Those teachers who referred to the hundred square or to the ruler did not make a distinction between the abstract nature of the number

[1] The ruler the teacher referred to and given out to the children was one that represented a number track. It was a wooden 30cm stick divided in squares, with the first coloured yellow, the next green, the next yellow and so on and so forth. Within each box a natural number was written, starting from 1.

line representing continuity and the more concrete nature of the alternatives that represented the discrete nature of number.

DISCUSSION

In their consideration of effective teachers Askew, Brown, Rhodes, Wiliam & Johnson (1997) suggest that effective teachers can be distinguished from less effective teachers in terms of increased fluency in discussing conceptual connections in the context of classroom practice whilst less effective teachers may express a more procedural rather than conceptual personal subject knowledge. The former, generally identified as "connectionist", appeared to value both pupils' methods and teaching strategies, in an attempt establish links with mathematical ideas. The latter, those associated with the notion of "transmission", appeared to prioritise teaching over learning and considered mathematics to be a collection of routines and procedures.

The data presented in this paper would suggest that connectionist values associated with the number line seldom featured in the responses of either trainees or practicising teachers. Indeed, most of the English curriculum material presents the number line as a concrete model supporting actions with little if any reference to its strength as an abstract representation of the number system. Such a focus may be more strongly associated with, and possibly even instrumental in, promoting beliefs that are associated with transmission. Though the classroom teachers in this survey applauded the pedagogical benefit of the number line as a tool, neither they nor the trainee teachers provided little explicit or implied indication that this benefit had a formality based upon the repetition of a unit interval and the partition of this interval. Instead we see that perceptual features, frequently implicitly associated with episodes and with a particular "line", dominated the definitions and additional comments obtained, though, particularly in the case of the teachers, these were frequently tempered by representational ambiguity and supported by counting episodes associated with moving backwards and forwards.

Gray and Doritou (2008) suggest that such conceptions lead to similar conceptions amongst children and though these do not appear to mitigate against the success of younger children in elementary arithmetic they eventually led to confusion amongst the older children. Specific interpretations of the features and use of a number line fail to provide children with a platform from which they may recognise its potential to contribute to the development of a global perspective on the number system. They also fail to contribute towards procedural efficiency as number size increased.

Bright, Behr, Post & Wachsmuth (1988) suggest that the number line is currently an extensively used model in the teaching of mathematics in elementary school, and whilst generally effective is also the source of difficulty both in instruction and its use by children. This paper provides one explanation for this difficulty – a very limited conceptual understanding of the representation by the teachers who use it. It was more general for the number line to be conceptualised as a series of discrete

representations of particular elements of the number system. The notion that it evolved from a unit that could be repeated and partitioned was less important than the notion that actions could be carrying out with it. This emphasis essentially associated with transmission caused ambiguity in the way teachers referred to a number line and, it is hypothesised, a consequent limited understanding of a sophisticated representation by the children who are faced with it.

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