CERME 6 – SPECIAL PLENARY SESSION
Ways of working with different theoretical approaches in mathematics education research

The contributions on the following pages constitute a report about a CERME 6 special plenary session; they include an introduction by Tommy Dreyfus, two papers by Angelika Bikner-Ahsbahs and by John Monaghan, and a report on the discussion that followed the presentations by Angelika and John.

We thank Susanne Prediger for her help with recording the discussion.

TABLE OF CONTENTS

WAYS OF WORKING WITH DIFFERENT THEORETICAL APPROACHES IN MATHEMATICS EDUCATION RESEARCH AN INTRODUCTION

Tommy Dreyfus

Tel Aviv University

The development and elaboration of theoretical constructs that allow research in mathematics education to progress has long been a focus of mathematics education researchers in Europe. This focus has found its expression in many CERME working groups: some are focused around a specific theoretical approach and others allow researchers from different theoretical traditions and backgrounds to meet and discuss. For example, the working group on *Argumentation and Proof* at the present (CERME 6) conference has reported on passionate discussions about different theories and their relationships (Mariotti, 2009). More specifically, relationships between theories have been made the explicit focus of attention of the theory working group that started at CERME 4 in 2005. This group has been reconvened at CERME 5 as well as at CERME 6; this year, we discussed fifteen papers, twelve of which make use of at least two theories and deal with how or why they can be connected in some way (see the part on Working Group 9 on *Different Theoretical Perspectives in Research in Mathematics Education* in these proceedings). The plenary activity from which this report emanated inserts itself in this line of work of CERME; one of its aims was to broaden the discussion about relationships between theories to include members of all CERME working groups.

The undertaking of mathematics education is very complex; this complexity is well expressed, for example, in Paola Valero's diagram (Valero, 2009). It is not without reason that the field has developed from having a curricular focus via a cognitive focus in various directions including philosophical, socio-cognitive, anthropological, ethnographic, and other perspectives, all the while producing home-grown theories to deal with all these aspects – and I am not even trying to distinguish between paradigms, theories, theoretical frameworks etc. For example, Realistic Mathematics Education has variously been characterized (including by people from the Freudenthal Institute) as a theory for mathematics education, as an instructional design theory or simply a philosophy for mathematics education.

When one reads a journal like *Educational Studies in Mathematics*, it seems at times that every paper presents a new combination of existing theories, a new theory, or at least a development of an existing theory. This raises the question how to look at and deal with the diversity of existing theories in mathematics education. Does this diversity express richness or does it express lack of focus (Steen, 1999) or even arbitrariness?

The question is made all the more urgent and difficult since theories come in different 'shapes' and 'sizes' and have different functions. Some concern the micro-genetic analysis of a learning processes in a classroom on a time scale of seconds, others the development of an individual student over months (or even years) and still others the momentary functioning of entire education systems. The ‘mesh sizes’ of theories thus range from the individual student via groups, classes, and schools to entire educational systems; and time scales under consideration range from seconds to years. Nevertheless, in the end they all deal with the same fundamental issue: How can students learn mathematics (better)?

However, even for (roughly) the same type of issue and scale, several theories with possibly different outlooks may exist; take for example the role of the social aspects in learning processes at the scale of a lesson: Is the social unimportant since deep mathematics is learned mainly when individual students are thinking by themselves, is the social the very vehicle of learning, or is it something in between, part of the context of learning (see Kidron, Lenfant, Bikner-Ahsbahs, Artigue, & Dreyfus, 2008)? Such a fundamental difference is likely to express itself in terms of different theoretical notions and hence different means and ways to analyze data.

Quite a lot of work has been done and published over the past ten years by people aware of the issues raised by the existence and use of many different theoretical frameworks, and trying to ‘do something about them’. Approaches have been very diverse. A few group studies have been published, in which researchers have worked on a common set of data, each researcher illuminating these same data from a different perspective such as a recent special issue on Affect in Mathematics Education (Zan, Brown, Evans, & Hannula, 2006). While this constitutes an interesting learning experience for the researchers as well as for the readers, it does not help us make progress toward connecting between the theories. We should be more ambitious. Nobody is probably aiming at a grand unified theory (see, e.g., Grand Unified Theory, 2009) as are theoretical physicists - this may be impossible altogether in the social sciences, and even if it is possible, mathematics education certainly has not reached this stage. We cannot even expect our community to converge to a set of common basic notions because the very idea of common basic notions negates the option of a variety of analytic approaches, and such a variety is needed in order to understand the complex multi-scale phenomena we are dealing with.

But we do need to make efforts to realize to what extent we are doing similar things in different languages and to what extent we use the same language to do different things. And once we realize that, we may want to establish connections, eliminate redundancies and distinguish what can and needs to be distinguished. Even more importantly, we want to find points of contact between theories that are dealing with different but related areas and find a language to talk about such theories together, to link between them in ways that are robust in the sense that they can be used by other researchers. These issues are very complex because theoretical frameworks are culturally situated – we have long known this from the difficulties many of us have to connect to and deeply appropriate the Theory of Didactic Situation (Brousseau, 1997) that has emanated from the French cultural background and grown in the environment of mathematics education in France. A recent issue of *ZDM - The International Journal on Mathematics Education* (Prediger, Arzarello, Bosch, & Lenfant, 2008) emanating from the CERME meeting at Larnaca offers a number of concrete case studies for how different research teams dealt with the fact that several theories were relevant for their study. There exist also examples from outside CERME, for example an attempt to coordinate argumentation theory and Realistic Mathematics Education to provide a microanalysis of a whole-class discussion (Whitenack & Knipping, 2002).

In the following two papers, two researchers experienced in consciously using, combining, comparing and contrasting several theoretical frameworks in the same study, will present different and possibly complementary approaches to such an undertaking. Angelika Bikner-Ahsbahs has taken the initiative of creating and coordinating a group of researchers who continue the work taking place at the CERME conferences also in-between conferences. She has coined the term networking theories to describe her view of how theories can be linked. John Monaghan presents a point of view formed outside of the CERME theory working group, on the basis of his research; this research has led him, for example, to refine the theory of abstraction in context, which has enabled him to take a step of integrating work on instrumentation with a dialectical, situated view of processes of abstraction; he has also recently connected the purely cognitive ideas of concept image and concept definition with a social view of learning mathematics. In his paper, he stresses the role of the person of the researcher when selecting (parts of) theories to network with; these two papers will be followed by some excerpts of the discussion that followed the presentations.

Brousseau (1997). *Theory of Didactical Situations in Mathematics*. Norwell, MA: Kluwer Academic.

Grand Unified Theory (2006). In Encyclopaedia Britannica. Retrieved April 28, 2009, from http://www.britannica.com/EBchecked/topic/614522/unified-field-theory

Kidron, I., Lenfant, A., Bikner-Ahsbahs, A., Artigue, M., & Dreyfus, T. (2008). Toward networking three theoretical approaches: the case of social interactions. *Zentralblatt für Didaktik der Mathematik - The International Journal on Mathematics Education 40* (2), 247-264.

Mariotti, M.A. (2009). Report from the Working Group 2 on Argumentation and Proof. These proceedings.

Prediger, S., Arzarello, F., Bosch, M., & Lenfant, A. (Eds.) (2008). Special Issue on Comparing, combining, coordinating-networking strategies for connecting theoretical approaches. *Zentralblatt für Didaktik der Mathematik - The International Journal on Mathematics Education 40* (2).

Steen, L. (1999). Review of Mathematics Education as research domain. *Journal for Research* in *Mathematics Education*, 30(2), 235-241.

Valero, P. (2009). Plenary paper. These proceedings.

Whitenack, J. W., & Knipping, N. (2002). Argumentation, instructional design theory and students' mathematical learning": a case for coordinating interpretive lenses. *Journal of Mathematical Behavior, 21*, 441-457.

Zan, R., Brown, L., Evans, J., & Hannula, M. (Eds.) (2006). Affect in Mathematics Education. Special Issue of *Educational Studies in Mathematics, 63*(2).

NETWORKING OF THEORIES: WHY AND HOW?

Angelika Bikner-Ahsbahs

Universität Bremen

This contribution presents a short overview of the current discussion about a meta-theoretical standpoint of working with theories: the networking of theories as a practice of research. It explains some principles on which this kind of research practice is based. Based on a methodological frame, an example is worked out showing how the networking of theories can lead to deepening insight into a problem and to methodologically reflecting the process of connecting theories.

During the last four years a new kind of research practice has been investigated: the networking of theories (Bikner-Ahsbahs & Prediger, 2006; Prediger, Arzarello, Bosch & Lenfant, 2008; Prediger, Bikner-Ahsbahs & Arzarello, 2008). What does this mean? Networking of theories is regarded as a systematic way of linking theories (Bikner-Ahsbahs & Prediger, 2009). Linking theories is not a new idea. Within conceptual frameworks (Eisenhart, 1991) different theoretical approaches are used to build a consistent frame for research. In the case of design research, Cobb (2007) argues for connecting theories as a kind of "bricolage" in order to capitalize on different views. In addition, triangulation has developed as a kind of evaluation criterion for qualitative research (Schoenfeld, 2002; Denzin, 1989).

A lot of scholars in the community of mathematics educators have already triangulated different theoretical perspectives in their research projects to enhance insight. However, the networking of theories means more than that, it means going beyond triangulation and developing methodological tools for *systematically* connecting theories, theoretical approaches and theory use. To be a bit more precise, I will describe the networking of theories as a process of

- analyzing the same phenomenon in mathematics education from different theoretical perspectives or within different theories,
- reflecting the use of these different theories,
- respecting the identity of each theory,
- exhausting the possibilities for linking them, and
- linking them

Meanwhile some research has been executed which has led to the development of strategies, methods and techniques for the networking of theories and to some insights about the benefit that can be reached this way (Prediger et al., 2008). An interesting example is shown by Kidron (2008). Based on data she explains in detail why more than one theory is needed to understand limit concepts. She networks three theories analyzing the discrete continuous interplay of limits and shows how these three theories - the concept of procept, the instrumentation approach, and the theory of abstraction in context - provide complementary insights and, hence, deepens understanding of limit concepts like the definition of the derivative. This way, Kidron is also able to show strengths, weaknesses and the limitations of the three theories.

On a product level, the networking of theories might lead to types of networked theories. However, since only first steps have been made in this direction, e.g. at CERME 4, 5, and 6 and elsewhere (ZDM 40 (2) for an overview), it is not yet clear, how these products might look. As Radford (2008) stated, the kinds of products will depend on the aims of networking, for instance, developing the identity of theories, experiencing the limits of linking theories, developing new methodological tools and new kinds of questions etc. One current result of this effort is a landscape of networking strategies that was worked out on the base of the contributions to the theory working group of CERME 5 (Prediger, Bikner-Ahsbahs & Arzarello, 2008).

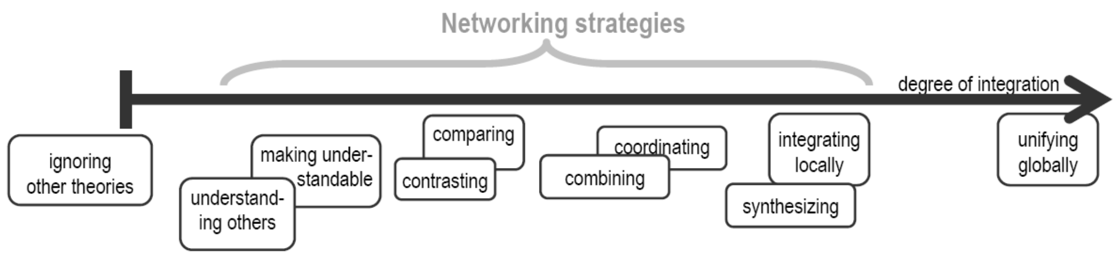

Figure 1: Networking strategies (Prediger et al., 2008)

This landscape represents a continuum of strategies for relating theories and theoretical approaches to each other including the extreme poles of non-relation between theories on the one hand and unifying them globally on the other. The term connecting theories means all kinds of building theory relations whereas networking strategies exclude the extreme poles. This landscape is ordered in complementary pairs of strategies according to their potential for integration. An example below will illuminate some of these strategies.

The idea of the networking of theories is based on some principles, the principle of

1. regarding the diversity of theories as a form of scientific richness,
2. acknowledging the specificity of theories,
3. looking for the connectivity of theories and research results,
4. developing theory and theory use to inform practice.

The first two principles acknowledge the diversity of theories in the field of mathematics education and accept diversity as a resource for scientific progress (Bikner-Ahsbahs & Prediger 2009). The third principle assumes that research in mathematics education produces much more connectivity than is visible at first sight. Related to different viewpoints, the networking of theories provides the opportunity to make these implicit aspects more explicit. The different ways of connecting theories presented at the theory Working Group 9 at CERME 6 illustrate the value and variability of the third principle. The fourth principle does not necessarily need to be shared by all the researchers in our field; however, it helps to keep research about the networking of theories grounded in practical problems producing concepts with an empirical load that is not empty (Jungwirth, 2009).

We are all busy doing research within and about mathematics education. If research demands the use of different theories we should use them being aware that this has to be justified somehow. But why is it necessary to engage in a meta-theoretical discourse about theory use? Why do we need to reflect about linking theories?

1. WHY DO WE NEED THE NETWORKING OF THEORIES?

In order to inform practice, theories facing specific practical problems are needed. Therefore a variety of theories of middle range scope, so-called foreground theories (Mason & Waywood, 1996), have been developed, for instance different theories about abstraction (Mitchelmore & White, 2007). Furthermore, the objects of mathematics education research can be viewed from different theoretical perspectives, e.g. cognitive, semiotic, social, …. Thus, a variety of research perspectives and various theories have been used leading to theory development in different directions. Researchers normally know what their theory is about but often the theories' limitations remain implicit. Limitations of theories can be experienced through the failure to apply them. A systematic way to provoke these experiences is critique. It can lead to a change of view (Steinbring, 2008) but also to the development of theories in that concepts and their limitations become more precise, additional concepts are constructed or the theories' parts become interconnected more deeply. Therefore, the diversity of theories can be regarded as a resource for and a consequence of critique (see also Lerman, 2006) and is scientifically necessary.

However, the diversity of theories has also caused problems (Prediger, Bikner-Ahsbahs & Arzarello, 2008), for instance a language problem and a connectivity problem. The first problem arises whenever researchers from different theoretical traditions try to talk to each other, since different theories might use the same words in different ways (e.g. social interaction in different tradition, see for example Kidron et al., 2006) or different theories use different words for the same or very similar phenomena (for example *interest-dense situation* and *a-didactic situation*, see Kidron et al., 2006). The connectivity problem is related to the question of how research results from different theoretical traditions can be connected to understand and solve practical problems.

So we need scientific ways of dealing with the diversity of theories that encounter these problems. The idea of the networking of theories might be a promising concept for this task which has the potential to induce the development of a common language among different research traditions and to investigate the ways in which theories and research results can be linked.

I will now present an example that shows how these goals can partly be achieved.

2. HOW CAN THEORIES BE NETWORKED?

In order to connect theories, a framework is needed that allows building relations among them. Radford assumes a semiosphere that comprises the collection of the semiotic parts of the different theoretical cultures within mathematics education (Radford, 2008). He explains that a semiosphere is

> "an uneven multi-cultural space of meaning-making processes and understandings generated by individuals as they come to know and interact with each other." (Radford, 2008, p. 318)

Theories within this semiosphere can be described as triplets (P, M, Q) that establish languages and allow the building of relationships between them. In these triplets, P represents the system of principles, M is a sign for a system of methodologies that can be connected to these principles in an appropriate way, and Q represents a set of paradigmatic questions related to P and M. A connection between two theories establishes a specific relation that depends on the theories' structures and the goal of this connection.

Using this frame, I will present an example of the networking of two theories illuminating the benefit of critique for developing insight into a problem. Methodological reflections will uncover five steps through which the process of networking has passed. This example refers to a data set that was used by Arzarello and Sabena (Arzarello, Bikner-Ahsbahs & Sabena, 2009). I will use it to explicitly show benefits and limits of networking practices.

An episode about the growth of the exponential function

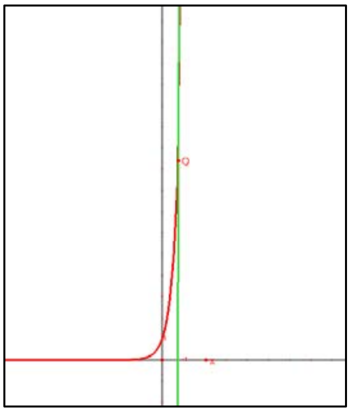

Figure 2

Two students of grade 10 are working in a pair on an exploratory activity on the exponential function and its growth. They use Cabri Geomètre to explore the graph's tangents. In this situation the teacher asks the students: What happens to the exponential function for very big x. The transcript shows the dialogue among the students G, C and the teacher.

Now I would like to invite the reader to participate in a short exercise using just a few pictures.

Figure 2 shows the computer screen the students observe.

Figure 3 presents two pairs of pictures. The left pair shows the student's gestures accompanying his utterances: his left hand goes up. The right pair illustrates the teacher's gestures accompanying his utterances: he crosses two fingers going to the right.

Figure 3: The student's gestures (left pair of pictures) and the teacher's gestures (right pair of pictures)

Please imagine for a moment what the teacher and the student are talking about. How does the student answer the question about the growth of the exponential function for very big x and how does the teacher react? – The student describes his perception of the screen meaning that the graph seems to approximate a vertical straight line. The teacher wants to show that this is wrong because every vertical straight line would be passed by the graph.

We now consider the beginning of the discussion.

1 G: but always for a very big this straight line (pointing at the screen), when they meet each others, there it is again…that is it approximates the, the function very well, because…

2 T: what straight line, sorry?

3 G: this … (pointing at the screen) this, for x very, very big

With broken language the student tells something about the growth of the exponential function for big x. This broken language is an indicator for thinking aloud. Saying "sorry" the teacher interrupts the student's train of thought indicating that this question is important. However, the student does not answer the question. Instead, he defends the choice of the term "vertical straight line". The student reacts to the so-called illocutionary level (telling something through saying something) of the teacher's question. Illocutionarily, the teacher's disruption is an indicator that there is something wrong with the vertical straight line while on the locutionary level (what is said) the teacher wants to know what vertical straight line G refers to.

During the following dialogue the student and the teacher talk about the function's growth, but, illocutionarily they negotiate about whose train of thought will be followed. The student begins to become involved repeatedly but is disrupted every time. In the end the teacher wins.

We now have a look at the last utterances.

14 T: eh, this is what seems to you by looking at; but you have here x = 100 billion, is this barrier overcome sooner or later, or not?

15 G: yes

16 T: in the moment it (the vertical straight line) is overcome, this x 100 billion, how many x do you have at your disposal, after 100 billion?

17 G: infinite

18 T: infinite… and how much can you go ahead after 100 billion?

19 G: infinite (points)

We see: The teacher is involved in arguing and the student's involvement is reduced to one (or two) word sentences (for a more detailed analysis of this episode see Arzarello, Bikner-Ahsbahs & Sabena, 2009).

A case of networking

Two theories were used to understand the episode above (for a short introduction: Arzarello et al., 2009b); a theory about the emergence of interest-dense situations and a theoretical approach about how a semiotic game between the teacher and the students shape the transition of mathematical knowledge.

The perspective of interest-dense situations

The first analysis is done from the view of the theory of the emergence of interest-dense situations. This theory – regarded as a triplet – is based on the following principles, methodology and questions:

- P1: Mathematical knowledge is socially constructed through interpretations of the others' utterances (see as well: Kidron et al., 2008).
- P2: The object of research is "meaning-making" within the process of social interaction.
- P3: In an interest-dense situation successful learning takes place as learners are deeply involved in the activity of social interactions constructing mathematical meanings in a deepening way. In these situations learning with interest is supported.
- P4: If the teacher focuses on the students' train of thought the emergence of an interest-dense situation is supported, if the teacher pushes the student to follow the teacher's train of thought the emergence of an interest-dense situation is hindered.
- M: Main part of the methodology is speech analysis on three levels. On the locutionary level an interlocutor says something; on the illocutionary level he tells something by saying something; on the perlocutionary level the intentions and the impact are taken into account.

The analysis is executed according to three questions:

- Q1: Did an interest-dense situation emerge?
- Q2: What conditions fostered or hindered it?
- Q3: How was mathematical knowledge constructed?

From the perspective of the emergence of an interest-dense situation the dialogues do not lead to increasing student involvement. Locutionarily (what is said) the student and the teacher negotiated the growth of the exponential function for very big x.

Illocutionarily (telling something through what was said) the student and the teacher struggle whose train of thought is followed. In some instances the teacher starts to focus on the student's thinking process but changes his argumentation immediately according to his own train of thought, namely to work out a "proof of contradiction": Given a vertical straight line –seen as a asymptote- this line would be passed by the graph of the exponential function. The degree of the student's involvement decreases while the teacher follows his own ideas, although the teacher tries to connect them with the student's utterances. Several times, an interest-dense situation is about to emerge, but this process is interrupted by the teacher's behaviour forcing the student to follow the teacher's train of thought. The construction of mathematical knowledge is carried out by the teacher; the contribution of the student is very low.

The semiotic bundle approach (Arzarello, 2006; Arzarello et al., 2009a)

- P1: Mathematics is transferred through a semiotic game with the help of the teacher.
- P2: The object of research is the semiotic game and its semiotic bundle.
- P3: Successful learning is interiorisation of mathematics by the help of the semiotic game.
- M: Analysis of the semiotic game according to the use of the semiotic bundle meaning the interplay of speech, gesture, representations and the transition of sign use.
- Q1: How was the mathematical content transferred through the semiotic game?
- Q2: Did the teacher tune speech and gestures with the student's ones?

From the semiotic bundle approach the semiotic game seems to be successful: The teacher takes over the student's words, using more precise explanations or following the students' ideas for a while. He points to the computer screen showing what is wrong in the way of the student's perception. He underpins his explanation and the proof of contradiction using gestures and tunes his words with those from the student. As far as the teacher is concerned, the semiotic game seems to be fruitful. From the perspective of the teacher's options to engage in the semiotic game he has done a lot of things to successfully transfer the mathematical content to the student. The student seems to be convinced, since, in the end, he correctly answers the teacher's questions.

The networking of the theories

At first glance, these results seem to be contradictory. Each theory serves as a resource for criticizing the other. After the networking process we found that the results are complementary since we could add an aspect that provided the integration of the different results: The teacher tries to tune his words with those from the student; but the gestures show that the epistemological views of the teacher and the student are different and they do not converge. The student uses his perception and extrapolates the growth of the graph of the exponential function for very big x: the function seems to grow like a vertical straight line. The teacher's view is theoretical requiring potential infinity. Neither the teacher nor the student is able to bridge this gap.

Some methodological reflections

The contradictory results were a reason for us to meet and refresh our analysis. During this process five steps emerged:

1. *Re-analysis*: Analysing the data together again from both perspectives made our theories mutually more understandable.
2. *Comparing and contrasting*: As we contrasted and compared our theories we began to juxtapose some principles and methodologies. For example: our views on theory require different uses of the data.
3. *Establishing a common ground*: From the perspective of interest-dense-situations I could explain how the emergence of an interest-dense situation was hindered, but I could not explain why hindrance occurred. We agreed that the semiotic game was not successful as shown from the other theoretical perspective. The question was: why?
4. *Complementary analysis*: A hypothesis occurred as we looked at the semiotic game, the gestures and the speech complementarily: The student's epistemological resource was his perception of the computer screen: he extrapolated the growth of the exponential function for very big x. The teacher's epistemological resource was theoretical. This caused a gap that could not be bridged.
5. *Establishing an inclusive methodology*: We used the three levels of speech in a complementary way for the analysis of gestures and utterances and re-analysed the data carefully. Again we reconstructed the gap between the epistemological resources that could not be bridged through the semiotic game as it was executed.

Conclusions

Did we move forward? Well – yes, we did. The starting point was the contradiction of our results that served as a resource for critique and a challenge for the networking of our theoretical backgrounds. We developed a common methodology including gesture analysis and the levels of speech into one analysis. We have gained a methodological overlap but we do not know yet whether our views will converge. If we do not dig too deep we can say we followed the same question: How is mathematical knowledge gained? However, this question is still understood a bit differently because our principles and paradigmatic questions remained the same. In the end, we deepened our insights and widened our theoretical perspectives. This was possible because the grain sizes of analysis were similar and the theories' principles were close enough to include the epistemological resource as a matter for explanation.

REFERENCES

Arzarello, F. (2006). Semiosis as a multimodal process. *Relime vol. Especial*, 267-299.

Arzarello, F., Bikner-Ahsbahs, A., & Sabena, C. (2009a). Complementary networking: enriching understanding. *Proceedings of CERME 6* (in press).

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009b). Gestures as semiotic resources in mathematics classroom. *Educational Studies in Mathematics, 70*, 97-109.

Bikner-Ahsbahs, A., & Prediger, S. (2009). Networking of theories – an approach for exploiting the diversity of theoretical approaches. In B. Sriraman & L. English (Eds.), *Theories of Mathematics Education: Seeking New Frontiers, first volume of the monograph series of the journal ZDM-The International journal on Mathematics Education*. (In press)

Cobb, P. (2007). Putting philosophy to work. Coping with multiple theoretical perspectives, In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning*. Reston: NCTM, 3-38.

Denzin, N. K. (1989). *The research act*. Prentice Hall: Englewood Cliffs.

Eisenhart, M. (1991). Conceptual frameworks for research. Ideas from a Cultural Anthropologist; implications for mathematics education researchers. In R. Underhill (ed.), *Proceedings of the Thirteenth Annual Meeting of Psychology of Mathematics Education – North America*. Blacksburg, VA: Psychology of Mathematics Education.

Jungwirth, H. (2009). On networking strategies and theories' compatibility: learning from an effective combination of theories in a research project. In B. Sriraman & L. English (Eds.), *Theories of Mathematics Education: Seeking New Frontiers, first volume of the monograph series of the journal ZDM-The International journal on Mathematics Education*. (In press)

Kidron, I. (2008). Abstraction and consolidation of the limit precept by means of instrumented schemes: the complementary role of three different frameworks. *Educational Studies in Mathematic*, *69*, 197-216.

Kidron, I, Lenfant, A., Artigue, M., Bikner-Ahsbahs, A., & Dreyfus, T. (2008). Toward networking three theoretical approaches: The case of social interaction. *ZDM - The International Journal on Mathematics Education, 40* (2), 247-267.

Lerman, S. (2006). Theories of mathematics education: Is plurality a problem? In *ZDM - The International Journal on Mathematics Education, 38* (1), 8-13.

Mason, J., & Waywood, A. (996). The role of theory in mathematics education and research. In A. Bishop, Clements, Ch. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *The International Handbook of Mathematics Education* (pp.1055-1089). Dordrecht: Kluwer A. Publ.

Mitchelmore, M., & White, P. (2007). Abstraction in Mathematics Learning. *Mathematics Education Research Journal*, *19*(2), 1–9

Prediger, S., Arzarello, F., Bosch, M., & Lenfant, A. (2008a) (Eds.). Comparing, Combining, Coordinating - Networking strategies for connecting theoretical approaches. *Thematic Issue of ZDM - The International Journal on Mathematics Education, 40* (2), 163-327.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008b). Networking strategies and methods for connecting theoretical approaches – First steps towards a conceptual framework. *ZDM - The International Journal on Mathematics Education, 40* (2), 165-178.

Steinbring, H. (2008). Changed views on mathematical knowledge in the course of didactical theory development. *ZDM –The International Journal on Mathematics Education*, *40* (2), 303-316.

Radford, L. (2008). Connecting theories in mathematics education: challenges and possibilities. *ZDM – The International Journal on Mathematics Education*, *40* (2), 317-327.

Schoenfeld, A. H. (2002). Research methods in (mathematics) education. In L. D. English (Ed.). *Handbook of international research in mathematics education*. (pp. 435-487). Mahwah: Lawrence Erlbaum Associates.

PEOPLE AND THEORIES

John Monaghan

University of Leeds

People are an essential consideration in networking theories. The dialectical relationship between people and theories is dynamic with regard to development. It is important to consider why people want to develop theories, their motive(s), and how they apply theories, their interpretation(s). Ascriptions of agency to theories, as ways of producing understandings or actions, need to be tempered by considerations of agency on the part of researchers applying theories. The appropriation of a theory by a person starts (and may end) with constructs from the theory.

I saw my role in this CERME plenary as that of reactor to Angelika's contribution. My first reaction is *I think what Angelika is doing is very interesting*. Indeed, "interesting" may be too weak a word as networking local theories is something that I expect to rise to prominence in mathematics education research in the near future. Angelika, Tommy, Ferdinando and I agreed at the outset that the CERME plenary should generate debate. With "debate" in mind I wanted a theme to my reaction that was honest (I did not want to generate debate by simply saying the converse of what Angelika was saying) but addressed issues that Angelika did not address and I focused on people because people network theories.

This paper follows my talk very closely and is in three parts. In the first part I argue that theories cannot be separated from the people theorising. In the second part I look at researchers' motives for adopting/creating theories and their interpretations of data. In the third part I argue that in practice researchers often appropriate parts of theories. I preface these three parts with some preliminary remarks.

PRELIMINARY REMARKS

I am not sure, in general terms, what a theory is. I am aware of discussions in mathematics education and in the social sciences of discussions of this issue. Prediger, Bikner-Ahsbahs & Arzarello (2008) consider the variety of theories in mathematics education research and conclude that 'We can distinguish theories according to the structure of their concepts and relationships' (ibid., p.168). A recent consideration of this issue in the wider social sciences is Ostrom (2005) who considers the difference between frameworks, theories and models with regard to her research interest, institutional analysis. Acknowledging that these terms 'are all used almost interchangeably by diverse social scientists' (ibid., p.27) she goes on to differentiate them according to their function in analysis: frameworks help to identify elements; theories help to specify relevant components for specific questions; models clarify assumptions regarding variables. These authors provide cogent considerations but I am still not sure what a theory is; but I know one when it is presented to me, e.g.

the Theory of Didactical Situations in Mathematics (TDS; Brousseau, 1997). So I will speak of theories with regard to the theories I am aware of that are referred to in mathematics education research.

Mathematics education researchers employ what might be called “out-of-mathematics-education theories” as well as those created within mathematics education. There are, I feel, problems for mathematics education researchers in both of these kinds of theories. The majority of us in mathematics education research are not experts in out-of-mathematics-education theories; most of us do not have a critical insight into all of their ramifications due to a lack of immersion in the academic literatures of philosophy, psychology, sociology etc. Out-of-mathematics-education theories can also miss fine mathematics detail (people interacting with mathematical relationships) that we are so very interested in. Mathematics education theories, on the other hand, can miss the big picture; the sites (classroom, workplace) of most mathematics education research are but a part of the lives of the participants.

THEORIES CANNOT BE SEPARATED FROM THE PEOPLE THEORISING

I present seven statements under this theme.

1 Theories do not exist without people

A theory without someone to interpret the theory is only words (and maybe symbols). A theory accordingly can be considered as a pair, (theory, person). For any given theory and *n* people there will be *n* such pairs. Some pairs will be almost identical, some will differ greatly; any given pair will depend on the interpretation of the theory by the person in the pair.

2 Theories develop and people develop them

(theory, person) pairs are dynamic, they change/develop. It is a bit sad if this does not happen! People develop in their understanding of a theory and through scholarships and research they develop theories. It can also be the case that a person appropriates particular development in the history of a theory, e.g. I am influenced by Davydov’s (1990/72) mid 20th century use of activity theory but activity theory has developed in numerous ways since his time.

3 People hold implicit and explicit theories

I have heard it said that people can only see via a theory and that people adopt theories. I think both claims, without further explanation, are rubbish. We “see” via the artefacts (including implicit and explicit theories) available to us in our phylogenic and ontogenic development (Wartofsky, 1973). The word “adopt” is too passive. I think there is, to draw a close analogy with Guin & Trouche’s (1999) ‘instrumental genesis’, a theoretical genesis in which people with initial ideas (I_I) interact with a theory (T), the person with I_I and T reviews experiences and, if T is convincing for that person, then (T′, P) develops. NB This account is certainly too simple but suffices, for my purposes, as an initial hypothesis.

4 Many people subscribe to more than one theory

Theory_1 informs us on … and Theory_2 informs us on … With regard to person-theory pairs we do not just have (T1, P) and (T2, P) but (some combination of T1 and T2, P). Maybe this is where networking theories becomes really important.

5 A continuum with regard to theory expertise

At one extreme there are leading theorists; at the other extreme there are those who do not appear to understand a theory; and there are many intermediate positions. In France there is a maximal element in the pair (TDS, Brousseau) but I believe that it is intellectually dangerous to grant absolute authority to leading theorists.

6 Mathematics education researchers network and partially absorb others' ideas

We (mathematics education researchers) read but we also talk – to people. I was introduced to the anthropological theory of didactics (ATD; Chevallard, 1999) by talking to J-b Lagrange. I did eventually read the paper but my understanding of the theory was through my conversations with J-b Lagrange and his research.

1-7 Theories arise in communities and cultures

As academic we may aspire to objectivity but we cannot escape cultural and community influences in our work. The plenary debate took place in France and I have alluded to ATD and TDS above in homage to mathematics education theories from France. I referred to J-b Lagrange introducing me to ATD above but our relationships with this theory will be distinct simply because J-b Lagrange is a French mathematics educator and his, and not my, identity is partially shaped in relation to this French theory.

A different example is provided by Nkhoma (2002), a black South African mathematics educator. This paper comments on attempts to import learner-centred instruction from the USA into Black SA classrooms:

> It is not beneficial to stereotype classrooms practices into, simply, teacher-centred therefore bad, and learner-centred therefore good ... rich experiences can be provided in practices that appear teacher-centred. (p.112)

In reading Nkhoma's paper it is difficult not to feel his anger at the importation of a "foreign" theory.

MOTIVES AND INTERPRETATIONS

I now look deeper into people and theories and examine researchers' motives for adopting/creating theories and their interpretations of data within theoretical frameworks.

To examine motives I consider a paper by Kieran & Drijvers (2006) and a response to this paper by Monaghan & Ozmantar (2007). Kieran & Drijvers worked in a form of ATD with they call "task-technique-theory" (TTT). It is a long and interesting paper on the interplay between computer algebra systems (CAS) techniques and by-hand techniques. The students were working on factorisations of x^n-1, and the CAS required specific values for *n* and did not give the classic factorisation every time. They state:

> According to the TTT … a student's mathematical theorizing is deemed to be intertwined with the techniques … tasks … we distinguish the following three theoretical elements.
>
> 1. Patterns in the factors of $x^n - 1$: Seeing a general form and expressing it symbolically
>
> 2. Complete factorization: Developing awareness of the role played by the exponent in $x^n - 1$ …
>
> 3. Proving: Theorizing more deeply on the factorization of $x^n - 1$ (pp.242-243)

We viewed this with regard to Hershkowitz, Schwarz & Dreyfus's (2001) abstraction in context (AiC) recognising and building-with actions. Students' prior work had involved factorising binomial expressions with regard to the difference of squares and sums and differences of cubes. They *recognise* that expressions of the form x^5-1 can be factored and *build-with* this knowledge artefact to produce factorisations $x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1)$.

We also viewed it via Davydov's ascent to the concrete (an inspiration for AiC) whereby an abstraction progresses from an initial entity to a consistent final form. This progression depends on the

> disclosure of *contradictions* between the aspects of a relationship that is established in an initial abstraction ... It is of theoretical importance to find and designate these contradictions. (p.291)

One student says, with regard to $x^{135} – 1$, 'how are we supposed to know if it's valid or not?' – the initial abstraction is fragile and limited to specific whole number exponents. The teacher introduced $x^n – 1$ and this required a *vertical* reorganisation of their knowledge. Student work shortly after includes *recognising* and *building-with* but on a higher vertical level than when the exponents were specific whole numbers.

Later in the Kieran & Drijvers paper we see students grappling with contradictions created from attempts to reconcile paper-and-pencil and CAS techniques and how this attempt at reconciliation led to synthesis and further insights.

> The difference-of-squares 'proof', for example, with its accompanying treatment of the case $x^{n/2}$ +1, for odd values of *n*/2, helped to extend even further the thinking of students in the class. The $x^n – 1$ conjecture, which had issued from the earlier work of some students with the factoring of $x^{10} - 1$, helped others to integrate their ideas about odd, even, and prime exponents - theoretical ideas that had been generated in interaction with various CAS and paper-and-pencil techniques ... (p.253)

We also viewed Noss & Hoyles' (1996) situated abstraction and webbing as closely related to Davydov's ascent to the concrete

> learning as the construction of a *web* of connections – between classes of problems, mathematical objects and relationships, 'real' entities and personal-specific experiences. (p. 105)

We are certainly networking theories (though in a different sense to how Angelika networks theories) but we also attend to differences arising from our ascribed personal motives of the theorist to theorise. Minimal ascriptions of motive are:

- Kieran & Drijvers – to understand the interplay of machine and by-hand techniques;
- Davydov – to develop theory to aid instructional design;
- Noss & Hoyles – to account for mathematical meaning making and the structuring of mathematical activities;
- Hershkowitz et al. - dissatisfaction with empirical theories of abstraction re students' actual development.

I now briefly consider interpretation. Angelika talks about a case where two theories are successfully networked. In CERME 6 Working Group 9 "Different theoretical perspectives and approaches in research: Strategies and difficulties when connecting theories" some papers focused on difficulties in networking theories. This is not new, six years ago Even & Schwarz (2003, p.283) commented 'We exemplify how analyses of a lesson by using two different theoretical perspectives lead to different interpretations …' I have no problem with this but question whether different interpretations are only the result of different theoretical perspectives. Research is often a team effort. Have you ever disagreed with a colleague during data analysis? I have and the outcome is usually compromise or an impasse. I think this tends not to get reported in papers. This is a further refinement to my point 1 'any given pair will depend on the interpretation of the theory by the person in the pair' but with regard to the interpretation of data via the theoretical perspective. Is at least one interpretation wrong?

ISSUES, CONSTRUCTS AND CONSISTENCY

In this final section I consider the extent to which theories lead research, theories and constructs and consistency issues. This section expands on my "motive" considerations above in that people often turn to/develop theories in order to address issues that they regard as important. But often it is parts of theories that they appropriate and this can lead to potential consistency problems.

Radford (2008, p.320) states that 'a theory can be seen as a way of producing understandings and ways of actions based on' a system of basic principles, a methodology and a set of paradigmatic research questions. I take partial issue with this, as a generality. I consider issues and specific research projects.

In my experience there are fundamental issues that mathematics education researchers return many times over their working lives. In my case one of these is the link between school mathematics and out-of-school mathematics. I have grappled with this over many decades. Research questions, methodologies and principles have come and gone but the issue remains. The construct "transfer" often arises in discussions of this issue for something akin to transfer is central in linking school to out-of-school maths. Personally I hate the term and largely agree with the old Lave (1988) critique but the issue haunts me and I am prepared to consider any theory that will further my understanding of this issue.

I now consider research projects with regard to theories. These generally have a shorter time scale than "issues". I, like most CERME delegates, write formal proposals with a theoretical framework, research questions and methodology. Almost every time, however, I develop in the process. I encounter unexpected phenomena (and revise the research questions) or experience problems in data analysis (and revise the methodology) or develop the theoretical framework. My point is that sometimes theories lead research, sometimes they do not; and, whatever the case, researcher development is in the dialectic mix. I now consider constructs.

A construct may be regarded as a proper part of a theory, e.g. *didactical contract* in TDS. I think people often appropriate a construct of a theory without appropriating the whole theory. I further think that if a person appropriates a theory, then they appropriate constructs of that theory prior to appropriating the theory. I included 'I think' in the previous two sentences because I base these remarks on my reflections of my own development; with regard to my point on "theoretical genesis" in (3) above I am not aware of research that traces the genesis of theory acquisition amongst academics but such research would be relevant to my reflections.

As an instance of construct appropriation in my own development I return to my comments above that J-b Lagrange introduced me to ATD. This is true, he did introduce me to ATD, but what I initially appropriated was the 'task-technique' part of ATD (and this focus as the only part of ATD I made sense of lasted several years). This focus was, I am sure, due to my prior experience. I had long experience of working with students and with teachers on using ICT-mathematics tools and the term "technique" in my country's everyday mathematics-education-speech refers to value-free manipulation. To view, as Lagrange's exposition of ATD does, techniques as not only being not value-free but techniques having both epistemic and pragmatic values and being viewed with respect to tasks was, quite frankly, a huge revelation and very relevant to my ICT work. Monaghan (2000) provides published evidence of this narrow focus. Perhaps it was due to the big impact this construct had on my thinking that appropriating other aspects of ATD took me a longer period of time.

I do not think the above (ATD, me) is an isolated example. I think the theory-person development is similar to that which I outline in (3) above: a person with a theoretical approach (T, P) interact with a construct C, the person with theoretical approach and construct reviews experiences and, if C is convincing for that person, then (T+C, P) develops. As with my comments in (3) this is almost certainly simplistic.

As I prepared for the CERME plenary I kept returning to T and C, in (T+C, P), with the thought that T and C must, in some sense, be consistent. I tried to formulate consistency criteria but failed, my attempts to frame consistency criteria ended with grand but empty phrases. This failure may be a personal failure but it may be that there is not a suitable meta-language in which to couch consistency criteria for non-specific theories and if this is the case, then perhaps we just need to resolve consistency tensions in our own research in case and theory specific ways.

REFERENCES

Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics*. Norwell, MA: Kluwer Academic.

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques,* 19, 221-266

Davydov, V. V. (1990). *Soviet Studies in Mathematics Education: Vol. 2. Types of Generalization in Instruction: Logical and Psychological Problems in the Structuring of School Curricula* (J. Kilpatrick, ed. and J. Teller, trans.). National Council of Teachers of Mathematics Reston, VA (Original work published in 1972).

Even, R. & Schwarz, B.B. (2003). Implications of competing interpretations of practice for research and theory in mathematics education. *Educational Studies in Mathematics*, 54, 283-313.

Guin, D. & Trouche, L. (1999). The complex process of converting tools into mathematical instruments: The case of calculators. *International Journal of Computers for Mathematical Learning,* 3(3), 195-227.

Hershkowitz, R., Schwarz, B.B. & Dreyfus, T. (2001). Abstraction in context: epistemic actions. *Journal for Research in Mathematics Education,* 32(2), 195-222.

Kieran, C. & Drijvers, P. (2007). The Co-Emergence of Machine Techniques, Paper-and-Pencil Techniques, and Theoretical Reflection: A Study of CAS use in Secondary School Algebra. *International Journal of Computers for Mathematical Learning*, 11 (2), 205-263.

Lave, J. (1988). *Cognition in Practice*. CUP: Cambridge.

Monaghan, J. (2000). Reaction to indispensable manual calculation skills in a CAS environment. *Micromath*, 17(1), 9-11 (2001).

Monaghan, J. & Ozmantar, M.F. (2006). Comments on the co-emergence of machine techniques, paper-and-pencil techniques, and theoretical reflection. *International Journal of Computers for Mathematical Learning*, 11 (3), 351-360.

Nkhoma, P. (2002). What successful black South African students consider as factors of their success. *Educational Studies in Mathematics*, 50(1), 103-113.

Noss, R. & Hoyles, C. (1996). *Windows on Mathematical Meanings: Learning Cultures and Computers*. Dordrecht: Kluwer.

Ostrom, E. (2005). *Understanding Institutional Diversity*. Princeton: Princeton University Press.

Prediger, S., Bikner-Ahsbahs, A. & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: first steps towards a conceptual framework. *ZDM – The International Journal on Mathematics Education*, 40(2), 165-178.

Radford, L. (2008). Connecting theories in mathematics education: challenges and possibilities. *ZDM – The International Journal on Mathematics Education*, 40(2), 317-327.

Wartofsky, M. (1973). *Models*. Dordrecht : D. Reidel.

DISCUSSION

The discussion was opened and guided by the following questions:

- How can we link our theories to the mathematical background? Is this necessary?
- Why should we care about theories as an object of research? Working with theories and constructing theories within mathematics education is our job. It is not our job to investigate the epistemological processes within ME themselves. We have enough problems to work on if we restrict ourselves to the teaching and learning of mathematics.
- What does consistency mean? Taking bits and pieces from different theories includes the danger to merge inconsistent parts. Does consistency depend on the grain size? How can we link different grain sizes?

The following comments, relating to the above questions, were made by members of the audience and the presenters:

Concerning the link to mathematics, on the one hand, the experience of participants in the *Advanced Mathematical Thinking* group of CERME is that linking the theories to mathematical content domains is crucial, especially crucial when trying to network with mathematicians. On the other hand, most mathematicians hesitate to go into didactical theories, and those who do sometimes point out difficulties of communication (Quinn, 2008).

Concerning the importance of reflecting at the meta-level how we work with more than one theory, other working groups than those mentioned above also reported that they were coming up against this issue; specifically, the working groups on *Affect*, on *Mathematics and Language*, on *Early Years Mathematics*, and on *Comparative Studies* were mentioned. In addition, two comments were made, namely that networking will never end since theories are dynamic entities, and that the important aspect of John Monaghan's presentation is the human one, never mind whether individual or social. However, the characteristic trait of this part of the discussion was that contributors tended to ask questions rather than make comments; these questions included:

- Isn't our research necessarily linked to what happens outside of the discipline since the research needs are defined by politicians, funding agencies, and teachers?
- How do we deal with theories that we adopt from other disciplines such as psychology, epistemology, or even medical science? In particular, how do we integrate mathematics (or at least a mathematical view) into these theories? How do we integrate theories from other disciplines into the area of mathematics education?

- To what extent do home-grown theories integrate, adopt or adapt elements from general (outside) theories?
- How would one distinguish local from global combining of theories?
- Shouldn't networking efforts also include cases where the researchers attempting to connect do not start from a specific phenomenon?
- Why not using the useful (but relative) distinction between background and foreground theories?
- Would it promote networking to start by comparing metaphors?

Concerning the issue of consistency, participants commented that criteria for consistency might better be found outside our community, that it might be preferable to use the term 'compatibility' rather than 'consistency', that the distinction between 'theories of' and 'theories for' could be useful, that looking at complementary phenomena could be a starting point for networking theories – a point already made by Steiner (1985), and that the main reason for connecting theories might be the complementarity of their aims, which is important by itself and might make their convergence rather less important.

While it is far from clear whether our community has already made substantial progress in its attempts to find ways of working with different theories in mathematics education research, this plenary session has made it amply clear that the issue of how to work with different theories is deep, that it occupies a central position for a large number of researchers and plays a important role in the discussions of a majority of the CERME working groups. It is therefore recommended that CERME continue its support of efforts to make progress on this issue and to discuss scientific ways of dealing with the diversity of theories in a manner that is comprehensive and includes researchers from different areas and backgrounds within mathematics education.

Quinn, F. (2008). Communication between the mathematics and mathematics education communities. Retrieved April 27, 2009 from http://www.math.vt.edu/people/quinn/education/communication.pdf

Steiner, H.-G. (1985). Theory of Mathematics Education (TME): an Introduction. For the Learning of Mathematics 5 (2), 11-17.