

COMPARING THEORETICAL FRAMEWORKS IN DIDACTICS OF MATHEMATICS: THE GOA-MODEL

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In this paper we propose a meta-model for comparing different theoretical frameworks in didactics, focusing on three components of the study object of didactics: a set of human beings with relations (e.g. students and teachers in a classroom), an organisation of human practice and knowledge, and a set of artefacts used to mediate and relate the previous two. We argue theoretically and through an example (related to the transition from secondary to tertiary education) that this meta-model helps identifying complementarities, similarities and differences among four leading theories or models of the didactical field, and thereby to facilitate rational justifications for selecting a theoretical framework with respect to a given purpose of research.

1. INTRODUCTION

The comparative study of theoretical frameworks in didactics of mathematics (for short, *didactics*) was the subject of a special issue of ZDM (no. 40, 2008), drawing on papers and discussions from working groups at CERME-4 and CERME-5 (cf. ermeweb.free.fr), as well as on other papers, many in previous issues of ZDM. Prediger et al. (2008, Fig. 1) subsumes the “landscape of strategies for connecting theoretical approaches” as ranging from “ignoring other theories” to “unifying globally”, between which we find intermediate positions for “finding connections as far as possible (but not further)” that the authors call “networking strategies”. Some consensus seems to have emerged to pursue the latter type of strategies, while considering the uses of a small number of theories (mostly 2-4) in concrete “cases” for research, such as studying or developing a classroom design based on a simple task. A general “metalanguage” to compare theoretical frameworks was proposed by Radford (2008, 320): a theory is considered as based on a triple consisting of a set of implicit and explicit principles of the theory, a methodology and a set of paradigmatic research questions. This idea seems to be applicable to theories in any field of research, and focuses essentially on aspects of the *epistemology* afforded by theories.

This paper proposes another, possibly complementary, approach to the issue: namely to compare the characteristic ways in which different theories build models of *the object of study in didactics*. The basic hypothesis is that significant differences among theories of didactics come from *focusing on different phenomena* within the complex reality of mathematics teaching and learning. In short, we propose a meta-model for the *ontology* of the theories, understood as the models they propose of their object.

2. EPISTEMIC SYSTEMS – THE GOA MODEL

Every science is about “something” – the *objects of study*. For an empirical science like didactics, which sets out to study a certain realm of mental, social or physical entities, the objects of study are delimited and to a certain extent constituted by the

development of theoretical *models*. Such models are more or less *systemic* in the sense that they imply relations among the objects; models are not simply lists of independently defined objects.

Without assuming (or saying) much, the “object” of didactics can be loosely described as the teaching and learning of a specific knowledge domain. Teaching and learning implies subjects who teach and learn – that is, teachers and students, or more generally a *structured group of people* (where structure implies that members of the group may have different roles and relations to each other, such as being teachers or students). The knowledge domain itself can be modelled and analysed as a coherent *organisation of knowledge and practice*. Finally, knowledge and “knowers” (be they teachers or learners) cannot be related without *artefacts* of different forms (texts, media, other tools and materials of various sorts). Given these basic observations we suggest that the systems of objects studied in didactics can be described as a triple

$$(G, O, A)$$

where: G is a group of people structured by a certain set of relationships, O is an organisation of knowledge and practice which G enacts, and A is a set of artefacts which G uses to access and communicate in and about O . Notice how relations on $G \cup O \cup A$ are crucial not just to study but also to define the triple. We call such a triple an *epistemic system* (ES) because the system involves use, circulation, development or even production of knowledge. Of course, not all ES are likely to be objects of didactical research, but surprisingly many types could need to be taken into account.

An ES may be considered in synchronic and diachronic ways, corresponding to a snapshot of its state at a given time (or a shorter period where it can be considered as relatively stable), and to its development over a period of time. It is also important to notice that (G, O, A) may be considered as general systems corresponding to an *institution* (e.g. a professional community or workplace) where the artefacts may include such diverse objects as buildings, tools, texts and so on, giving identity and delimitation to the institution. Finally, an ES may be naturally divided into “subsystems” (G_i, O_i, A_i) , such as different divisions within a workplace.

Here are four special cases which are of particular importance in didactics, in themselves and in interaction; they also show how varied phenomena ES include:

2.1. Didactic systems may be described as the case where G consists of one or (rarer) several *teachers* and a class of students, engaged in the teaching and learning of a knowledge organisation O while mobilising, possibly in different and changing ways, a set of artefacts A (including objects within the classroom). The knowledge organisation could be based on one or more problems or questions, mediated and tackled using A , and potentially mobilising or enabling the construction of the “intended knowledge or practice” (also part of O). In fact, these intentions – of the teacher(s) – are an important factor in didactic systems, but it could take many forms.

2.2. *School systems* consist of a certain collection of didactic systems, e.g. with G comprising all students and teachers of a given school, or of all schools within a given region or country; the boundaries of a school system (as regards all three components) are sometimes institutional boundaries in the sense that they are defined quite explicitly, such as by law, or they could be considered pragmatically as (observable) systems of persons common aims, practices, and material surroundings.

2.3. *Teaching systems* are parts of school systems but with G being a group of teachers, who may work alone, or together, to construct or reflect upon one or more didactic systems. The knowledge organisations and artefacts involved in such systems may, of course, also be quite different from those involved in didactical or adidactical systems. For instance, teachers could be involved in developing or sharing teaching plans and other teaching material (artefacts) related to the teachers' knowledge and practice enacted within didactic systems.

2.4. *Noospheric systems* consisting of a group of people G involved in generating, delimiting or defining all or parts of the knowledge and practice organisations O to be worked on in didactic systems, using or producing artefacts to this end; for instance, G may be one or more authors of a textbook (part of A) aimed to support O , or a group responsible for producing standards for school systems (A in this case involves documents setting up requirements or recommendations regarding the practice, target knowledge and artefacts of these). The term *noosphere*, originally coined by Chevalard (1985), ironically refers to the "thinking about" school systems which takes place outside these systems from a peripheric yet superior position.

3. The GOA model as a "meta-model" for comparing theories in didactics

The above model of ES can be thought of as a *meta-model* since its use in practice requires finer models for each of its components and their interrelations. Supplying these details, we recover several "real" models or theoretical frameworks commonly used in didactical research. We now do this for some important ones, familiar to us.

3.1. *The theory of didactical situations* (TDS, cf. Brousseau, 1997) considers, as its primary objects, *didactical situations* evolving around *didactical milieus* and regulated by *didactical contracts*. The situations are themselves modelled as the interplay between students and teachers (forming G) and the milieu, which in turn is a compound of both material elements (forming A) and a particular organisation O_M of practice and knowledge. The system as a whole is analysed in terms of a wider organisation O of intended and prescribed forms of practice and knowledge, which includes also a dialectic between *personal knowledge* of the different members of G , and *shared knowledge* which develops over time. This means that the entire didactic system (G, O, A) is considered diachronically, albeit mostly over shorter periods (corresponding to a lesson or a sequence of lessons). The didactical contract consists of (mainly) implicit rules which govern the whole system, in particular the interactions within G and between G and the milieu. In sum, this theoretical framework models G

as consisting of a teacher and a group of students, with different relations to both O and A , a relation which varies over time and is interpreted as being governed of a rule system (contract) corresponding to expectations and obligations of the members of G . It can be said to be more “naturalistic” as regards G and A as such, and focuses particularly on the evolution of the relation between G and O . Moreover, diachronically, the theory focuses on subsystems existing at times where the teacher does not interact with the students, called *adidactical situations*; this refers to shorter time spans for a didactic system, which at other times involves interaction between teachers and students.

3.2 *The anthropological theory of didactics* (ATD) involves highly intricate models of O (mathematical and didactical organisations, cf. Chevallard, 2002), corresponding to forms of practice and knowledge related to mathematics and the teaching of mathematics, respectively. More precisely, it models both of these as organisations of *praxeologies*, each of which consist by definition in a quadruple (type of task, technique, technology, theory). Praxeologies are organised at various levels according to the techniques, technologies or theories they share. The researcher constructs a *reference model* to observe and analyse these organisations within different systems. Among artefacts explicitly considered in this theory are *ostensives* mediating and embodying the techniques and technologies of O , including also discursive media and tools. In this theory, G is mostly implicit, except for the strong emphasis on *institutions*, viewed as the human ecologies in which praxeological organisations live and between which they are transposed. The theory also contains a structured view of institutions successively determining each other at different levels (Chevallard, 2002), from a didactic system considered synchronically (e.g., Barbé et al., 2005), to the noospheric systems (including the level of societies) considered in diachronic development (e.g. Chevallard, 2002). Finally, a recent development in this theory, to describe the long term developments of didactic systems, is Chevallard’s notion of *research and study programme* (see eg. Barquero et al, 2006), focusing again on O but with a community of learners G being perhaps more explicit in recent empirical studies of how such a programme evolves (ibid.). However, even more than TDS, the ATD focuses primarily on the analysis of the O component.

3.3 *Socio-constructivist theory of mathematics learning* (SCT) exists in many forms and variants; we consider here the approach to didactic systems developed by Cobb and associates (e.g. Cobb et al., 2001). As the name suggests, the model has dual roots (ibid., 119-120): on the one hand, in *constructivist* learning theories going back to pioneers such as Steffe, Skemp, and ultimately Piaget; and in *socio-cultural* theories, with a lineage involving names such as Bauersfeld, Lave and Vygotsky. The idea is to study the learning – in particular mathematics learning – of participants in a classroom situation (students, teachers and even researchers) both as individuals and for the group collectively within a socio-cultural context. In fact,

there is an extremely strong relation between what we have described as the social and psychological perspectives that does not merely mean that the two perspectives are interdependent. Instead, it implies that neither perspective exists without the other in that each perspective constitutes the background against which mathematical activity is interpreted from the other perspective (ibid., 122).

The researchers' interpretation of classroom activity aims to clarify this dynamics of (mainly students') *individual beliefs* and *sociomathematical norms* developed and shared by G collectively. It is based on careful analysis of video recordings of classroom activity (as a primary form of data, among others such as field notes and interviews). This allows for observing not only *discursive* and *embodied* practices related to a mathematical task, and thereby the emerging organisation O of practice and knowledge found in the classroom, but also the role played by artefacts (discursive, semiotic, material...). It is important to note that while G and A are theorised e.g. as *communities of practice* and *semiotic ecologies* (ibid., 153), the individual and shared knowledge organisations (including beliefs and norms) are considered to *emerge* from the interaction within G and between G and A : *we take the local classroom community rather than the discipline as our point of reference* (ibid., 120). In other versions of SCT, such as Ernest (1997), a wider perspective is adopted.

3.4 *The cognitive-semiotic theory* (CST, cf. Duval, 1995) focuses on the relationships (mental schemes or processes) which exist for the members of G between a collection of signifiers (primary elements of A , organised in semiotic systems) and signifieds (mathematical objects within O). The fact that these relationships may be different for different members of G (and develop over time) is explicated in variants of this model through a *triadic* model of the sign relationship, including also the different interpretations or schemes for the relationship between semiotic artefacts and their "meaning". Particularly important for the objects of mathematics is *multimodal representations*, which occur in two distinctive forms (cf. Duval, 2000): different representations of an object within the same semiotic system (*register*), which are obtained by *treatments*, sometimes based on complicated algorithms; and representations in different *registers* (like a function being represented symbolically and graphically), obtained from each other by *conversion*. Coordination of different representations of a given mathematical object is a key requirement in many common mathematical tasks. Notice that this model may be applied to all kinds of ES, but with a special focus on "semiotic" artefacts and the corresponding schemes, and sometimes relatively implicit models of O (although for the case of mathematics, the mathematical objects and their properties are often considered as *constructed* or even *consisting* in those schemes, cf. Winsløw, 2004).

3.5 *Comparison*. The above four "snapshots" of theoretical frameworks enables a first comparison of them, as the modelling of certain *parts* of (G , O , A) occupy the foreground within each of them. In TDS, the interaction of teacher and students (G) around the didactical milieu (part of (O , A)), in ATD the praxeological organisations

(O) in their institutional context (G, A), in SCT the community of practice (G) with its evolving shared norms and individual beliefs which contribute to determine O , and in CST the semiosis and associated schemes (part of (G, A)) as a condition for accessing and enacting O . To compare these theoretical frameworks, it is crucial to realise that they model, to some extent, *different parts* of a common reality (such as a didactic system). To a much lesser extent do we find apparent oppositions in their basic constitution, such as the deliberate absence in SCT of reference models for O “outside the classroom”, versus the strong emphasis on such models within ATD.

4. CASE: THE TRANSITION FROM SECONDARY TO TERTIARY

For about a decade, I have been studying the transitions problems which arise for students at the beginning of university programmes in mathematics, along with development projects aiming at enhancing the outcome of students’ work. The difficulties students encounter – and the strategies one may envisage to help them overcome those difficulties – may be approached using any of the theoretical frameworks considered in the previous section (as well as others, of course). In this section, a simple example will be used to show how contributions associated to each framework are different because they model the relevant ES with different foci and notions. Notice that Gueudet (2008) presents an overview of literature explicitly addressing the transition from secondary to tertiary, including more theoretical perspectives than those considered here.

Globally, transition concerns *students* who move from one type of ES, (G, O, A), to another one, (G', O', A'), in which there may be some overlap in all three components, including (by definition) the students themselves within $G \cap G'$. An obvious place to locate the obstacles for students within (G', O', A') is in the set of practices and knowledge components O' which they have to acquire, as opposed to those they have previously known (O). As difficulties appear most strikingly in the setting of concrete tasks which the students experience as difficult or impossible, many studies focus on such tasks and how they relate to the global transition. Here, we shall consider the following task, and expand our analysis of it as presented in (Winsløw, 2007):

- a) Show that $f(t) = t/(1+t)$ defines an increasing function on $[0, \infty)$.
- b) Show that with f as above, $f(s+t) \leq f(s) + f(t)$ for all $s, t \geq 0$.
- c) Show that the formula

$$d(a,b) = \frac{|a-b|}{1+|a-b|}$$

defines a metric on \mathbb{R} .

In fact, c) is the enunciation of a text book task (Carothers, 2000, p. 37) while a) and b) are provided as hints in the book. This is a typical task given to students as they

begin to study the concept of a *metric*, defined axiomatically by three properties: on a space M , a metric is a function on $M \times M$ which satisfies, for all x, y, z in M : $d(x, x) = 0$, $d(x, y) = d(y, x)$ and $d(x, z) \leq d(x, y) + d(y, z)$. In the case of c), the first two properties are immediate and the last one is verified using a) and b), bearing in mind that $\delta(a, b) = |a - b|$ defines a metric on \mathbb{R} (corresponding to the usual concept of distance on \mathbb{R}).

In actual practice, our observations of numerous exercise sessions show that students take the “hint” to rephrase the exercise as a three step procedure, as formulated above; that almost all solve part a) by computing the derivative and showing it’s positive; that few students solve b), often by round-about methods involving functions of two variables (and sometimes even a computer algebra system); and that very few students were able to make use of a) and b) to solve c), in fact in 6 out of 8 groups of 25-30 students observed, no student had succeeded to do so. Another point is that students having failed with b) did not even try to tackle c).

4.1. TDS approach. The exercise can be considered as a didactic milieu devolved by teachers to students and presenting certain obstacles, the overcoming of which are the price of the experience which the teacher aims for the students to have, of applying the definition of metrics. The first parts seem more familiar to the students and its form activate existing contracts, in the sense that surface parts of the enunciation (“increasing” and “ \leq ”, corresponding to artefacts in the milieu) trigger certain techniques of calculation. In particular, to show that a concrete function is “increasing” one computes the derivative and “see” that it is positive. To “see” an inequality may take some rewriting; the presence of two variables in b) (again, artefacts of the milieu) is responsible for many complicated attempts to use partial differentiation and the like. Finally, no contract has been established for a “right way” of showing that something is a metric; the tripartite nature of the definition is no doubt part of the problem (one has to verify three properties instead of one). In the classroom situation, the teacher can get no further than to make the students recite (or look up) the text book definition. The teacher’s (and the text book author’s) expectation that this will be a simple experience with applying the definition thus fails because the milieu leads the students to identify the problem with contracts at the “micro”-level of the individual steps. A more global contract is visible in the fact that failure with b) kept many from even considering c), amounting to an understanding of exercises as built of from increasingly difficult parts (“if you can’t do b) then you certainly can’t do c)”). The outcome of this analysis, in the setting of TDS, is that the milieu will have to be *re-designed* to better fit the teachers’ intentions (the target knowledge, surely to be further analysed!) as well as the contractual phenomena evidenced by the students’ response to the original tasks. In particular, the properties of metrics – the key element of O' for the above task – may have to be (re)constructed by students first, as a response to a situation with a milieu that relates to their existing experience with (O, A) .

4.2. *ATD approach.* Winsløw (2008) presents a two-step transition in terms of the praxeological organisations present in secondary schools and in universities: first practice blocks are completed to entire praxeologies (with theory blocks), then new practice blocks are built with tasks that take objects from “old” theory blocks. In the task above, it is mainly the second step which is in play: the function f and its (theoretically proved) properties are used to build a new object for a practice block related to metrics (task type: show that a given two variable function is a metric; technique: verify the axioms). Also, the “standard distance” δ (fundamental to the theory of calculus on \mathbb{R}) becomes one among an infinity of objects that this task type takes as an object. The institutional point of view provides a framework for explaining the apparent necessity of this two-step transition $O \rightarrow O'$. In secondary school, epistemic systems are constrained by noospheric systems pursuing aims that go much beyond the school institution itself (a range of continuing study programmes, a central examination, etc.). At universities, two types of ES coexist, those of research and those of teaching (cf. Madsen and Winsløw, to appear); the overlapping group of users consists of “professors” (research mathematicians who also teach). The complete praxeologies O' pursued in undergraduate mathematics programmes in research intensive universities aim to converge towards those pursued in research (O''). In particular, the practice of checking that a given object is a metric, as well as theories based on metric spaces, are indispensable in several branches of research mathematics. Tasks of the kind considered above are thus, at least to some extent, consequences of the *choice* that O' should approach O'' .

4.3. *SCT approach.* Sharing some concerns with the TDS analysis presented above, a SCT analysis focuses more sharply on the beliefs and norms evidenced by the discourses found in the classroom where the exercise is discussed. While we have no space to provide even excerpt of relevant data, these might well turn out to present a cleavage in the group G' , between the teachers and the few students on the one hand who have formed a conception of metrics – and a technical level in algebraic manipulations – that allow for understanding and completing the task; and those students who try, quite desperately, to relate the task to norms and beliefs which they have acquired in secondary school practices. This may or may not proceed towards a progressive inclusion of the latter subgroup into a community of practice with shared norms and beliefs; but in this isolated episode, this seems to be out of reach. The fact that the majority of students did not get to consider the “real” task – because of their inability to follow the “hints” (or complete the preliminary tasks) – leads to a form of (at least) local alienation from the intended meaning-making. Such phenomena are frequently observed in studies of undergraduate mathematics within a socio-constructivist approach. Dreyfus (1999, 106) subsumes the transition which many students fail to make as moving from questions of type ‘What is the result?’ to questions of type ‘Is it true that ...?’, after arguing that even within the community of mathematicians, there are no universal criteria for whether an answer to the latter

type of question is complete and correct (cf. also Ernest, 1998). A SCT approach thus focuses on the process of building at least local consensus in G' about this matter, given that G has mainly been engaged in practices where it does not occur.

4.4. CST approach. Here, the first question could well to be: when addressing the three parts of the task, what are the forms of representation (including the variety of semiotic artefacts) available – actually and potentially – to students (relation of G to A)? For part a), the students can easily graph the function and thus become intuitively convinced of the claim (part of O). For a more formal argument, they can use the algebraic register and either differentiate f to get $(1+t)^{-2} > 0$, or use a treatment like $t/(1+t) = 1-1/(1+t)$. The latter (and simpler) *ad hoc* argument did not occur among students or even teachers, because it is not produced by an algorithm, and the standard procedure is reasonably easy. For b) there is no easy general procedure and the treatment required is equally non-standard; so only few students succeed. In fact, the complexity of required treatments *not given by a standard algorithm* also explains why the almost all students fail with c): here one has to *combine* previous results with the validity of the axioms for δ . Moreover, unlike a) and to some extent b), representations of the involved objects in other registers, such as the graphs or tables of the functions d and δ , are of no help to understand or intuitively support the sought conclusion. By contrast, in secondary level mathematics, the predominant mode of thinking involves *coordination* of several registers (e.g. graphical, symbolic, numeric) of the objects considered. While this is both a challenge and a support at the secondary level, it tends to disappear for more abstract mathematical objects at tertiary level. In the concrete case, a task on metrics on \mathbb{R}^2 (with ample possibilities for illustrating the different metrics) might help to enable a multimodal first encounter with the notion, on the condition that students succeed in coordinating the involved forms of representation.

5. CONCLUDING REMARKS

In this paper we compared four theories in general (see 3.5). While it would be too simplistic to maintain that the considered theoretical frameworks model only one or two of the three components of ES, they do exhibit very different foregrounds in the sense that each provides highly developed notions and principles for analysing certain components or relations between them, while leaving other in the background. One might also talk of *ontological foregrounds* in the sense that different parts of didactical reality are identified or constructed through these models. This is also illustrated by the case study (section 4). Serious integration of theoretical frameworks may eventually become possible and even useful to some extent; but I personally feel that it is more urgent to develop the rationality with which we *choose* frameworks according to *a given purpose of research*. Analysing theoretical frameworks within the GOA model may contribute to this end because research purposes and ontological foregrounds are strongly interdependent.

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