HOW CAN DIGITAL ARTEFACTS ENHANCE MATHEMATICAL ANALYSIS TEACHING AND LEARNING

Dionysis I. Diakoumopoulos
Department of Mathematics, University of Athens

Digital technologies seem to be still very promising to fruitfully support the construction of mathematical knowledge. Far more interesting is the way to incorporate them into the design of a learning environment framed by certain institutional constraints. Through this study we present some reflections and ideas arising from the dialectic interplay between the environment and the students in their effort to formulate a calculus theorem and construct its proof. Related teaching and learning phenomena providing information on instrumental genesis processes are primarily discussed.

INTRODUCTION

Elementary pre-calculus is at the heart of the syllabus at secondary level mathematics education and the entry-point to undergraduate mathematics as well. Many research studies witnessing students' problems to attain a satisfactory level of conceptualisation have been held on this field (for example, see Artigue, 1999). This fact is related to mathematically superficial strategies (Lithner, 2004) implemented by traditional procedure-oriented teaching practices. We claim that these practices are generated by both teachers' attitudes and institutional constraints implicitly or explicitly imposed by textbooks and curricular objectives (Ferrini-Mundy & Graham, 1991). Even at the university level, this situation results in detecting serious difficulties on behalf of the students when faced with non-algorithmic type demands which entail reasoning and conceptual understanding (Gonzales-Martin & Camacho, 2004).

On the other hand, the development of mathematics has always been dependent upon the material and symbolic tools available for mathematical computations (Artigue, 2002). Current research on mathematics education regarding the relationships between curriculum, classroom practices and software applications (Lagrange, 2005) offers the ground to address and develop questions concerning technology's fitting into learners' actual social and material environments, the problems users have that technology can remedy, and, furthermore, ways of conceptualizing the design of innovative learning tools as emergent from dialectics between designers and learners-users of those tools.

The learning environment is supported by a Dynamic Geometry software (DGS) enhanced by a function-graphing editor to help Mathematical Analysis teaching at the level of 12th grade.

The produced didactic sequence covers the introduction of global and local extrema definitions, Fermat's theorem (stationary points) with its proof, the mean value theorem, monotonicity definitions and the derivative sign/monotonicity theorem along with the proof and its applications. Selection of the exact targeted mathematical material on the field of differential calculus, as well as further elaboration of the activities, were attempted with the intention to form a rational succession of concepts to a coherent local unity of mathematical knowledge, mainly including introduction of definitions, formulation of theorems and construction of proofs. From this still on-going research, we present here some elements derived only from an activity concerning the teaching and learning of Fermat's theorem formulation and proof on the field of differential calculus.

THEORETICAL FRAMEWORK

Complexity and close interweaving of cognitive, institutional, operational and instrumental aspects obliged us to adopt a multidimensional approach (Lagrange et al, 2003) in order to design the learning environment and study the teaching/learning phenomena produced.

According to Duval (2002), construction of mathematical knowledge is strongly attached to the manipulation of different semiotic representations. This term refers to productions made up of the use of signs and formed within a semiotic register which has its own constraints of meaning and function. More specifically he defines a *"register of semiotic representation"* as a system of representations by signs that allows the three fundamental activities tied to the processes of using signs: the formation of a representation, its treatment within the same register, its conversion to another register. Interaction between different registers is considered to be of great importance and necessity to achieve understanding of a mathematical concept. Under this aspect, our tools were designed with the intention to mobilise and flexibly articulate semiotic representations within the numerical, the algebraic and the graphical register, so that to generate mathematical conjectures.

Very special and idiomorphic conditions existing within the local educational culture of Greek 12^{th} grade students obliged us to take into consideration the notion of didactical transposition (Chevallard, 1991). At this level, a huge amount of institutional pressure results in the development of an "exam-oriented mentality" on behalf of the students as well as their families, which promotes a procedure-oriented attitude towards the mathematical knowledge in context. Candidates' needs to be prepared for a final national university-entrance examination at the end of the year result, finally, in an implicit (or even sometimes explicit!) meta-didactical attitude leading them to ignore or reject conceptual approaches not strongly attached to exam demands. Through this perspective, we were obliged to take into account and reinforce both the epistemic and the pragmatic value (Artigue, 2002) of the mathematical knowledge to be taught without any decrease or discount of any of them, in the economy of the available didactical time. Relating this idea to the tools' design, we considered the possibility to teach basic mathematical concepts within a reasonable amount of learning time, and in ways compatible to both its institutional dimension and the transition to advanced mathematical thinking.

The theory of didactic situations (Brousseau, 1998) helped us conceive the whole learning environment (milieu) as a source of contradictions, difficulties, and disequilibria stimulating the student (on his own responsibility to control it) to learn by means of adaptations to this environment. At this point, we took also into account activity theory (originated in socio-cultural approaches and mediation theories rooted in Vygotski, 1934) to assign to the environment a character sometimes antagonistic to the subject (as pointed by TDS) but also sometimes cooperative and oriented to an educational aim, guided by distinctive didactical intentions.

In order to best incorporate digital artefacts in our didactical engineering, we considered the potential technology offers for linking semiotic registers within the frame introduced by the instrumental approach (Rabardel, 1995, Artigue, 2002, Trouche, 2004). A cultural tool or artefact, designed to mediate mathematical activity and communication within a socio-cultural context, differs from the corresponding instrument into which this artefact can be transformed. The artefact, as the final result, encompasses a psychological component; a construction by the subject, in a community of practice, on the basis of the given artifact by means of social schemes. This transformation is developed through an instrumentation process directed towards and shaping the subject's conceptual work within the constraints of the artifact and an instrumentalisation process directed towards and shaping the artifact itself. Both constitute a bidirectional dialectic and sometimes unexpectedly complex process called instrumental genesis (Artigue, 2002). Concerning tool design, we tried to keep simplicity and friendliness to the user, in the sense that their implementation demands, as far as possible, a short process of appropriation by the user and an easy way to be transformed into mathematical instruments to be utilised in the context of the activities. The necessity of any technical support by the teacher was also minimised as far as possible.

The crucial question to answer through our research is whether a design philosophy under the norms mentioned above has the potential to determine a set of effective digital learning tools, pre-constructed on the dynamic software, which can be easily transformed to learning instruments successfully integrated into the teaching of important calculus concepts at the level of theorem formulating and proof. By the term *successfully integrated* we mean that, firstly, they can make visible phenomena previously invisible, secondly, they can potentially generate innovative approaches to important mathematical concepts, and, thirdly, they shape and better our understanding of some productive or problematic dimensions of the computer transposition regarding the mathematical knowledge accessed through the instrument's mediation.

METHODOLOGY

The activity (of total duration 90 min) was developed in two different schools in groups of 12th grade students (10 in one group and 5 in the other) during the month of February, 2008. The main differences between the students of the different schools were identified on the socio-cultural and financial background of the corresponding families (we did not address any comparison issue in our research goals) and as well to the fact that comparatively more students belonging to a certain school had a facility for using mathematics software, being exposed several times in the past at different kinds of technology enhanced approaches. For the latter we did not find enough evidence to support the idea that different software cultures of the students have great impact on their attitude and capabilities of manipulating the pre-constructed software tools induced by our activities.

The informatics laboratory of every school was used and the pupils were at couples situated in a PC-environment. This time the researcher played the role of the teacher as an orchestrator of in-class situations. A Teacher-Analysis sheet has also been developed to provide necessary details so that other teachers can handle the in-class orchestration.

At the beginning, a worksheet was given to the students to work with and at the end of the session they received a corresponding post-assessment sheet including several T/F type questions of mathematical nature, which they returned back completed next day. The whole didactic sequence (consisting of four Sessions) was recorded by a voice-recorder and, the whole sequence being completed, a post-questionnaire was passed to the students in order to collect and save some of the instrumental marks being left on them through the entire approach. Finally, four students (two for each group) were interviewed to explicitly clarify their answers at this questionnaire concerning the instrumented actions performed and the students' attitude towards mathematics teaching before and after the whole experience.

The way of obtaining results-serving the a posteriori analysis-from the raw input data has to be explained here. The whole content referring to the 2nd Activity (Fermat's Theorem: Stationary Points) has been divided (according to the conceptual meaning development) into 12 *Episodes* and each one of them potentially to one up to four *Phases*. Next, for every one of the 24 *Phases* produced, we used the transcribed outcomes of the recorded class discourse, along with the written notes and answers of the students on the worksheet, to produce some discrete entities of information we called *Events*. An *Event* in this terminology is characterised and differentiated by components of mathematical or didactical or instrumental nature which can probably coexist. The study and analysis of these *Events* provided our a posteriori analysis with the material to compare the results composed up to this point to the analysis of the students' answers to the corresponding post-assessment sheet-being sorted out and analysed separately. Finally, we took into consideration the students' answers on the final post-questionnaire as well as the transcribed explicitation interviews in order to enhance our vision and come up to some final conclusions.

LEARNING ENVIRONMENT

Concerning the tools' design (and being sensitive to the complexity of instrumental genesis processes), we tried to reduce, at least, the complexity of the interface. We tried also to keep tools' implementation strongly attached to the mathematical needs emerging within the predefined context. The learning environment regarding the whole activity was, thus, perceived with the intentions to:

a) Mobilise students' interest to estimate local extrema departing from a real problem, b) Make up a link with the students' previous knowledge on the subject of local extrema and the limit concept, c) Stimulate the students to construct the targeted mathematical knowledge by mobilising different registers of representation (graphic, numerical, symbolic, and verbal) for the same concept and favouring representational interconnections between them, d) Use the in-class discourse to generate an activity space favouring students' effective instrumental processes, e) Support conjecturing, conceptualisation, and institutionalization, f) Insert certain examples or counter-examples when necessary (Gonzales-Martin & Camacho, 2004).

We focus especially on the activity designed to introduce the concept of Fermat's theorem. As far as the students were concerned, our specific didactical aims were: to conjecture the theorem, construct its formal statement and proof realising the absolute necessity of its presuppositions and its application range, to perceive that the reverse form of the theorem is not valid, and, finally, to apply it in calculating the local extrema of the function-given by a formula-induced by the problem.

In the following we describe and analyse some selected *Events* drawn out of two different *Episodes*. The material that will be presented is coming from a blend of actual events produced by both groups of students, whose comments and actions have been complementing each other over the flow of the activity.

Remark: The term S-Tools refers to the specific on-Screen pre-constructed tools on the software.

Episode A: Introduction to the *Line y*=k, *IntersectionPoints*, *Magnification* S-Tools and applications in approximating local extrema positions on the function graph.

Tool Description: The students were prompted to open *Line y=k* and *Intersection Points* S-Tools. The first one draws a horizontal parametric line, whose position can be controlled by the active parameter *k* (a number in yellow background on the screen that can be modified by the user, see Image 1). If this line has some common points with the function graph then the second S-Tool *IntersectionPoints* draws these intersection points and provides their x-coordinates. Furthermore, a technique permitting the students to change the decimal length and the digits of any active parameter was explained to them by the teacher.
The following question given by the corresponding worksheet came to stimulate students to S-Tools utilisation:

Q1: Could you find or estimate points of local extrema for function *P*?

Aim Description: The main intention of the constructed situation was to encourage students to explore and use the S-Tools in order to estimate several intervals of x-axe that could enclose positions of internal local extrema and to get approximate values for these positions by shortening the length of the corresponding intervals. Moreover, they had to identify the kind of local extremum (maximum or minimum) and perceive which of them are internal to the interval.

Events: The teacher asked the students to change the active parameter k and see what happens. Some of them could not understand the changes on the counters of intersection points coordinates and that was clarified by the related discussion in the class community. Then, the students were asked to use these tools to numerically estimate the local extrema positions (Question Q1) on the graph the better they could (Image 1). Some students could not cope with changing the decimal length and the values of several digits so they were given additional technical instruction for that. The teacher asked them to find an interval including the abscissa of a local maximum (this was done very easily) and then to try to shorten this interval by means of the tool. This was not so easily done by every pupil but remarks made by several students and on-screen indications gave good results.

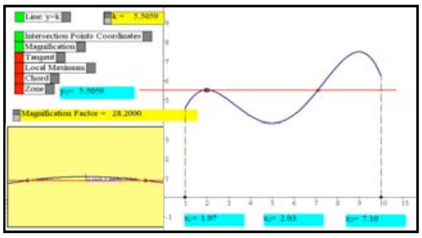

Image 1

Interesting events identified on behalf of the students were:
-During exploring with decimal digits many students observed two intersection points approaching each other and, finally, coinciding to only one but the indications on the corresponding counters were different.
-Six of them noticed that they could see intersection points on the screen but the indications on the coordinate counters did not attest such an existence.
Concerning these two events, the teacher's proposition was to use the *Magnification* S-Tool.

Tool Description: This S-Tool could be used to magnify a selected region around a point which can be displaced anywhere on the graph and is controlled by the *Point-Abscissa* and the *Magnification Factor*.

Subsequently, the students were asked to use the same process to estimate the values of every local extremum they could perceive on the graph.

Remarks: Students' written answers on the worksheet revealed that the whole class succeeded at the qualitative level (number of local extrema, approximate position and characterisation). However, at the numerical level, only a small part (26% of them) tried to test in the extreme the instrument's potentialities and even less (6,6%) achieved at exhausting them-providing the values asked at 3rd or 4th decimal digit accuracy as we had anticipated. A technical weakness versus time disposal has been estimated as a possible reason for that.

Results: This first contact with the notion of approximation opened up the ground for a further in-class discussion. The discourse came up to the point that the tool is able to provide visual images of a certain validity only as an indication generator (which in certain cases can be of great importance for the mathematical knowledge) but not always to produce an arithmetic value in absolute accuracy. The teacher reinforced this situation by asking what would happen if the extremum in search had the real value $2/3$ or $\sqrt{2}$. This fact conducted the discussion to bring into light the inherent inadequacy of every computing system to represent infinite decimal numbers in a complete way. So the students realised that, through this attempt, and also in general, they could obtain only relative accuracy for the local extrema values. The necessity of devising new mathematical tools that could probably provide absolute accuracy for these values came in the discourse.

Episode B: Introduction to the tangent: Relating line y=k when passing through an internal local extremum to the function graph – Derivability

Next Question Q2 had the intention to sensitise students' attention and make them focus to what is going on locally at the area near an internal local extremum point.

Q2: When line *y=k* is passing through an internal local extremum point on the graph, how is this line related to the curve at an area near this point?

Description: Within this *Episode* the students were asked to express their thoughts regarding the visual relation between the line y=k when passing through an internal local extremum point on the graph and the curve itself near the extremum point. The first attempt was made on normal view and the second by means of the *Magnification* S-Tool (Image 2). Subsequently, at the third phase of the *Episode* a new subroutine program file was invoked, where the students could alternatively observe under magnification the behaviour of functions $y = x^2$ and $y = abs(x)$ in the neighbourhood of $x = 0$ (Image 3). This was done by changing only the function formula through a menu of the file. The necessary technique was shortly explained by the teacher.

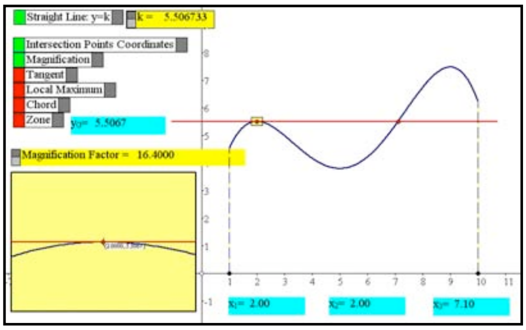

Image 2

Events: The class discourse developed at this *Phase* helped many of the students to communicate their thoughts and formulate them in an intelligible way. They came up with the visualisation of the inequality relations $f(x) \leq k$ or $f(x) \geq k$ near the local extremum. Relating this fact to the image produced by the function graph and the horizontal line, they could easily conjecture that this line when passing through a local extremum point on the graph *"leaves the whole curve on one side"* or *"does not cut it"* at the area near this point.

Remarks: Analysis of students' written answers on the worksheet showed that the big majority of them (80%) succeeded in perceiving the visual relation between the curve

and the line and, moreover, 26,6% of them were able to connect it with the corresponding symbol relation. 26,6% of the students proceeded to conjecture that, at this case, this horizontal line should be a tangent of the graph, whereas even fewer (13,3%) mentioned that there was only one common point of the line and the curve at the area near the local extremum.
To the question of the teacher if these two conditions (namely existing of a single common point and "not cutting" in the area near a local extremum) are able to assure the existence of a local extremum, confusion arose and the community could not provide a clear answer. This event, along with the term tangent mentioned earlier, was used as a bridge to the discussion of next question:

Q3: At the area near the extremum point, can you observe any additional relation between the curve and the line *y=k* when the latter is passing through this point?

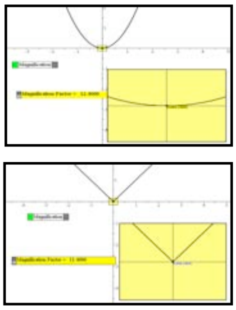

Image 3

Remarks: Class discourse concerning this question resulted in the assertion on behalf of the students that under magnification the curve tends to become a horizontal line or to coincide with it. Moreover, there were some more students stating in a clear way the conjecture that the horizontal line when passing through a local extremum point on the graph *keeps the position of a tangent of the graph at this point*. This conjecture provided the bridge through which the teacher introduced the issue of the existence of the tangent at such a point. Additionally, as a natural consequence of the previous discussion, the subroutine file was used to support students' exploring and help them visualise the difference between the function graphs of $y=x^2$ and $y=abs(x)$ on point $x=0$ under magnification (Image 3) and relate it to the derivability of the function at this point. Most of the students' expressions were for example *"Oh, there's an angle there!"* or *"… in this case we have a peak point …"* etc.

DISCUSSION AND PRELIMINARY RESULTS

In this paper, we tried to describe a few situations concerning only the instrumental dimension of our research. The *Episodes* presented above contribute, as a first step, to Fermat's theorem construction departing from an intuitive approach. This goal is achieved by exploring and visualizing the local extrema positions related to the premises of the theorem.

As it has been pointed by Guin and Trouche (1999), students' answers were strongly dependent on the environment:

At a first attempt, many students tried to configure the artefact regarding the needs of the specific work: screen view adaptation by transposition of toolboxes and active parameters configuration (i.e. changing the decimal length and the values of certain digits of parameter *k*). These facts confirm, on their behalf, an effort to adapt the artefact to the demands of the specific task induced by the first question Q1 (Estimation of local extrema values) and we consider that as a step to the direction of instrumentalisation in the evolution of instrumental genesis processes (Rabardel, 1995, Trouche, 2004). As instrumentation processes had intently been designed and anticipated not to be very complex, soon after, we could observe automaticity towards certain instrumented action schemes to the execution of necessary tasks (i.e. utilising active parameters).

We point to an internal constraint (Trouche, 2004) of the instrument, which is related to computer's inherent deficiency in providing absolute preciseness through computations, regarding infinite decimal numbers. This issue was discussed with the students during several activities and, finally, was used as an entry to the discussion concerning the notion of approximation. Additionally, a common feeling was developed pointing out that computers will not solve all the mathematical questions inserted. This fact was also used to encourage students to develop their knowledge so as to overcome these limitations.

Students' answering to questions of the post-assessment sheet regarding the statement of Fermat's theorem or its applications within only the graphic register showed that the great majority of them (86,6%) could cope very good at this level. However more complex questions relating this register to the algebraic one have been far too difficult for the students, proving that more work is necessary to be done at this level.

Analysis of students' answers to the final post-questionnaire testified a generally positive attitude towards *"this way of teaching"*. For example, to the question: "*Could you identify any positive or negative points through this series of activities you have been attending?*", some of their answers were: *"We could discover and see by ourselves most of the things on the screen…"* or *"By the aid of the computer we could really see and work on the staff we treat usually in the class"*, or *"It was easy-going because, we first, … we didn't realize that we had made the proof of the theorem and only at the end we got the typical statement"* or *"It was very helpful to recollect the images on the screen, but the problem was that we didn't solve many exercises!"* etc. Of course, more work and analysis need to be done on this issue in order to obtain some reliable results.

Due to the lack of space, we did not address issues concerning the ways through which the rest of our theoretical perspectives shape our research. However, some results seem to deepen our reflection. They show the potential of such a learning environment design to produce didactical phenomena giving an illumination to both problematic and productive aspects of the mathematical knowledge developed through the educational use of digital technologies.

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