DYNAMICAL EXPLORATION OF TWO-VARIABLE FUNCTIONS USING VIRTUAL REALITY[1]

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We present the rationale of an ongoing project, aimed at the development of a Virtual Reality assistant learning of limits, continuity, and other properties in multivariable Calculus. The Mathematics for which this development is intended is described briefly, together with the psychological and pedagogical elements of the project. What is Virtual Reality is explained and details are given about its application to the specific field. We emphasize the fact that this new technological device is suitable for self-teaching and individual practice, as well as for the better storing and retrieving of the acquired knowledge, and for identifying its traces whenever it is relevant for further advanced learning.

BACKGROUND

The institution and its pedagogical situation

The Jerusalem College of Technology (JCT) is a High-Tech Engineering School. During the Spring Term of first year, a course in Advanced Calculus is given, mostly devoted to functions of two, three or more real variables. A problem for many students is a low ability to "see" in three-dimensional space, with negative consequences on their conceptualization of notions such as limits, continuity, differentiability. Another bias appears with double and triple integrals, as a good perception of the integration domain is necessary to decide how to use the classical techniques of integration. Sik-Lányi et al. (2003) claim that space perception is not a congenital faculty of human being. They built a Virtual Reality environment for improving space perception among 15-16 years old students. With the same concern we address a particular problem of space perception with older students, using the same digital technology.

Berry and Nyman (2003) show students' problems when switching between symbolic representation and graphical representation of a 1-variable function and of its first derivative. They say that "with the availability of technology (graphical calculators, data logging equipment, computer algebra systems), there is the opportunity to free the student from the drudgery of algebraic manipulation and calculation by supporting the learning of fundamental ideas". Tall (1991) notes that the computer "is able to accept input in a variety of ways, and translate it's flexibly into other modes of representation, including verbal, symbolic, iconic, numerical, procedural. It therefore gives mathematical education the opportunity to adjust the balance between various

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modes of communication and thought that have previously been biased toward the symbolic and the sequential".

Until now, various technologies have been introduced as a tentative remedy to problems encountered with three-dimensional perception. Nevertheless, problems still remain. Numerous technologies have been introduced for the sake of visualization. Arcavi (2003) classifies the roles of visualization as a) support and illustration of essential symbolic results, b) provider of a possible way of resolving conflicts between (correct) symbolic solutions and (incorrect) intuitions, and c) a help to re-engage with and recover conceptual underpinnings which may easily be bypassed by formal solutions.

In the present paper, we focus on functions of two real variables, plotting and analyzing their graphs, considering especially the b) component in Arcavi's classification. A problem may appear inherent to all kinds of support: a graphical representation may be incorrect, either because of non appropriate choices of the user or because of the constraints of the technology (Dana-Picard et al. 2007). In order to overcome this problem we turn our attention towards another technology: Virtual Reality (VR). This technology is extensively used for training pilots or other professionals. Jang et al. (2007) discuss the usage of VR related to representation of anatomy, clearly a 3D situation too. But as far as the authors know, it has been implemented yet neither for Mathematics Education in general, nor for the Mathematics Education of Engineers. In this paper, we present the rationale for the authors to start the development of a VR assistant to learning Mathematics. We describe an environment where the learner is not passive and has some freedom to choose his/her actions. A VR environment offers cognitive assessment, spatial abilities, executive and dynamical functions which are not present in more traditional environments.

Representations of a mathematical object

Among the characters articulated in mathematics teaching cognitive aspects:

- Multiple representations of the same objects: textual (i.e. narrative) presentations, literal formulas, graphical representations, tables of numerical values, etc. These presentations may either be redundant or leave empty holes. Note that every presentation has to be accompanied by a narrative presentation for embodying the rule and for the sake of completing the given description of a rule. Mathematics educators generally agree that multiple representations are important for the understanding of the mathematical meaning of a given notion (Sierpinska 1992).
- When using together multiple representations in order to give a concrete appearance of composite consequences of the rule under consideration, it can be necessary to perform a transfer between an abstract concept and concrete representations. For example, Gagatsis et al. (2004) present a hierarchy among the possible representations of a function, calling tables as a prototype for

enabling students to handle symbolic forms, and graphical representations as a prototype for understanding the tabular and verbal forms of functions (for a study of prototypes, see Schwarz and Hershkowitz 1999).

- The more numerous the rule's implications (in Physics, Biology, Engineering, Finance, etc.), the more important is the requirement of creative skills (e.g. interpolations, extrapolations, which the learner will have to apply). Here the teacher will generally try and guide the learner with examples, graphical representations, and animations.
- The more fundamental the rule, the more important for the learner to store it, to internalize it and its consequences for a long duration. This will enable him/her to build more advanced rules. More than that, the learner needs ways to extract the knowledge and to find its traces whenever it is relevant for further learning (Barnett et al, 2005).
- Regarding a mathematical rule with geometrical implications and representations, its complete mastering requires from the learner, according to the Gestalt conception, a permanent transfer from one kind of representation to another kind (see Hartmann and Poffenberger, 2007). On the one hand, it is necessary to understand how a change in the parameters of the rule influences the representation. On the other hand, abstraction skills enable to conjecture the rule from the graphical representation and to modify the parameters in the formula according to the changes in the graphical representation. This is the rationale for the usage of software for dynamical geometry.

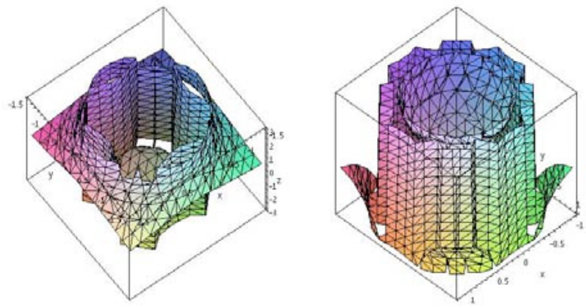
The graphical representation has been made using either Maple 9.5 or the free downloadable software DPgraph (www.dpgraph.com). Because of the dynamic character of a VR device, we do not include screenshots. Suitable presentations can be found at URL: http://ndp.jct.ac.il/companion_files/VR/home.html.

LIMITATIONS AND CONSTRAINTS ON THE CONVENTIONAL REPRESENTATION TOOLS

Real functions of two real variables may have various representations: symbolic (with an explicit analytic expression $f(x,y)=...$), graphical (the graph of the function, i.e. a surface in 3D-space), numerical (a table of values), not necessary all of them at the same time. This last kind of representation is generally not easy to use in classroom; the plot command of a CAS uses an algorithm which provides numerical data, and the command translates this numerical data into a graphical representation. Generally the higher level command is used, and the user does not ask for a display of the numerical output. The VR device that we develop uses this numerical output to create a *terrain* (a landscape) over which the student will "fly" to discover the specific properties of the function, either isolated or non-isolated singularities, asymptotic behaviour, etc.

It happens that a symbolic expression is unaffordable. This creates a need, central for teaching, for suitable tools to illustrate the function and make it more concrete. An example is given by Maple's **deplot** command for plotting the solution of a Differential Equation without having computed an analytic solution; of course this command uses numerical methods. Within this frame, educators meet frequently obstacles for their students to achieve a profound and complete understanding of the behaviour of such functions. Examples of the limitations have been studied by Kidron and Dana-Picard (2006), Dana-Picard et al. (2007) and others. The student's understanding of the behaviour of a given function depends on the representations which have been employed.

Dana-Picard et al (2008) show that the choice of coordinates has a great influence on the quality of the plot produced by a Computer Algebra System (CAS). Compare the plots of $f(x,y)=1/(x^2+y^2-1)$, displayed in Figures 1 and 2. Cartesian coordinates have been used for Figure 1 and polar coordinates for Figure 2. The discontinuity at every point of the unit circle is either not apparent or exaggerated. Moreover Figure 1b shows a kind of waves which should not be there.

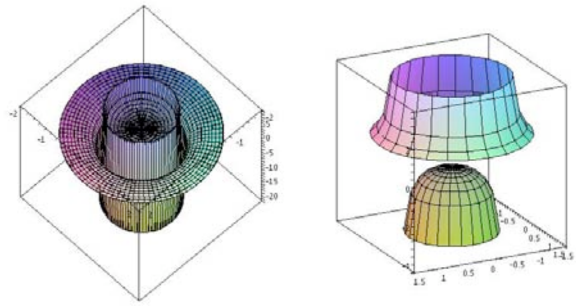


(a) **(b)**

Figure 1: Plots of a 2-variable function, with Cartesian coordinates

The choice of suitable coordinates is not the sole problem for getting a correct plot. Figure 2a shows that our discussion on "correct coordinates" is not the ultimate issue, and even with these coordinates, other choices influence the accuracy of the graph, whence the student's understanding of the situation. In Figure 2a the discontinuities are totally hidden, as a result of the interpolation grid chosen by the software. This issue is discussed by Zeitoun et al. (2008).

A "wrong" choice of coordinates may *hide* important properties of the function, but may *show* irrelevant problems, whence numerous problems with the figure and its adequacy to the study. A central issue is to decide what "correct coordinates" are and what a "wrong choice" is. It has also an influence on the possible symbolic proof of the properties of the function. A couple of students have been asked why they have hard time with such problems; they answered that the reason is a lack of basic understanding of the behaviour of the represented mathematical object (no matter whether the representation is symbolic, numerical, or graphical). A problem can arise when checking that data of two different kinds actually represent the same function. Experience must be accumulated by the learners.



(a) **(b)**

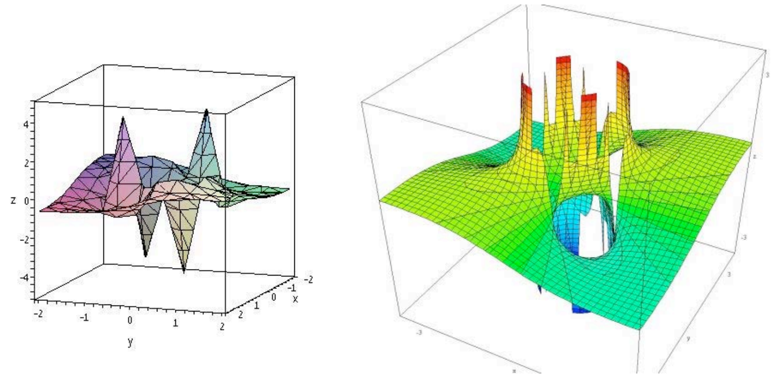
Figure 2: Plots of a 2-variable function, with polar coordinates

Moreover, the students may receive a proof of a certain property using an abstract-symbolic representation of the mathematical object under study. Despite the proof's precision, it happens that the student needs a more concrete presentation. In a practice group of 25 students, the teacher chose the function defined by $f(x,y)=1/(x^2+y^2-1)$ and showed plots like those displayed in Figure 1. Two thirds of the students saw immediately that the function has a lot of discontinuities (intuitively, without giving a proof), but could not explain immediately what is wrong with Figure 1.

The graph of a 2-real variable function is a surface in 3-dimensional space. A function of three real variables can be represented by level surfaces. Excepted at certain points, this is the same mathematical situation as before, because of the Implicit Function Theorem. At the beginning of the course, about 70% of our students have problems with surface drawing. A lack of intuition follows, for example concerning the existence of discontinuities. This may incite the student to make successive trials, i.e. to multiply *technical* tasks not always relying on real mathematical *thinking*. Afterwards a symbolic proof is required, and maybe a graphical representation will be needed to give the "final accord".

Graphical features of a Computer Algebra System are used to enhance visual skills of our students, hopefully their manual drawing skills. With higher CAS skills, an animation of level surfaces can help to visualize graphically a 3-variable function. We meet two obstacles:

- The dynamical features of a CAS are somehow limited. In many occurrences, it is possible to program animations, and/or to rotate the plot, but not more.
- A CAS cannot plot the graph of a function in a neighbourhood of a singular point. In this paper we focus on limits and discontinuities. The CAS either does not plot anything near the problematic point (Figure3b) or plots something not so close to the real mathematical situation (Figure 3b: where do these needles come from?). Note that this occurs already with 1-variable functions, but with 2-variable functions the problem is more striking.



(a) false incomplete plot **(b) incomplete plot**

Figure 3: Two Problematic plots for $f(x,y)=\dfrac{xy}{x^2+y^2-1}$ **.**

VIRTUAL REALITY

What's that?

The technology called **Virtual Reality** (VR) is a computer-based physical synthetic environment. It provides the user with an illusion of being inside an environment different from the one he/she is actually. This technology enables the building of a model of a "computerized real world" together with interactive motion inside this world. The VR technology gives the user a feeling that he/she an integral "part of the picture", yielding him/her *Presence*, *Orientation*, and even *Immersion* into the scenario he/she is exposed. After a short time he behaves like it’s the real world.

The goals: VR-concretization and its added value

A CAS is not a cure-all for the lack of mathematical understanding when dealing with discontinuities of multi-variable functions. A more advanced, more dynamical concretization is given by a VR environment. It is an additional support to Mathematics teaching completing the classical computerized environments, beyond the traditional representations (symbolic, tabular-numerical, and graphical). Actually VR provides an integration of computer modes previously separate (Tall 1991):

- Input is not limited to sequential entry of data using a keyboard. Devices such as a joystick are also used.
- A working session and its output mix together the iconic, the graphical and the procedural modes.

When reacting to the student's commands, the VR device computes anew all the parameters of a new view of the situation. The student takes a walk in a landscape which is actually part of the graph of the function he/she studies. At any time, VR simulates only part of the graph, the discontinuity is never reached, but it is possible to get arbitrarily close to it. The VR may provide the student what is missing in his/her 3-dimensional puzzle, by eliminating the white areas appearing in CAS plots, such as Figure 3a. It is intended to provide him/her a real picture of how the function he/she studies behaves.

A VR environment provides compensation to the limitations and the constraints of the imaging devices already in use (CAS and plotters). It presents an image of a real world and gives a direct 3-dimensional perception of this world, as if the user was really located in it. The higher the quality of the VR environment, the more powerful the impression received from this imaginary world's imitation of the real world.

In our starting project, the simulation provided by VR is intended to improve the students' understanding of continuity and discontinuity, and afterwards give also a better understanding of differentiability of a multi-variable function. Among other affordances, the VR simulation cancels problems of discontinuity related to graphs because of its local and dynamical features.

COGNITIVE CHARACTERISTICS AND SIMULATION FEATURES OF A VR ENVIRONMENT

The final rules may be represented in a concrete fashion by interaction with the environment and by showing to the learner the limitations of the rules, as they appear in a (almost) static environment generally yielded by a CAS. Non graphical representations of functions, such as numerical representations, cannot show continuity and discontinuity. This comes from the discrete nature of these representations, a feature still present in the computerized plots.

The new knowledge afforded by the learner is a consequence of his/her own efforts to explore the situation. His/her ability to change location, to have a walk on the graph, will lead him/her to internalize in a better way the mathematical meaning of continuity and discontinuity. An added value is to help him/her to understand the meaning of changing parameters in the geometric representation. This added value is made possible by the *live experience* of the behaviour of the function, no matter if the transitions are discrete or continuous (according to changes in the variables or in the parameters). The mental ability to feel changes, their sharpness, their acuteness, comes from the immersion into the topography in which the learner moves.

This added value is still more important when the function under study encodes a concrete situation, in Physics, Engineering, Finance, etc. The interactive experience enables the learner to translate the rules to which the function obeys, to find analogues of these rules for other concrete situations. The concrete sensations provided by VR improve the learner's understanding of interpolation and extrapolation, and to translate this understanding into the graphical situation (see also Dana-Picard et al., 2007). The more immersive features of the mathematical knowledge that are incorporated into VR representation for the learner, the faster he/she will find the traces of it whenever it is relevant for further learning. Besides, the more immersive features are incorporated into VR knowledge representation the greater the longevity of preserving the acquired knowledge. This means a slower extinction of it in the memory system (Chen, et.al. 2002).

Interactivity improves the learning experience. Numerous studies show that the more deeply lively experienced the learning process the more internalized its results (Ausburn and Ausburn, 2004; Barnett et al, 2005). The internalization is assessed by an improved conservation of the knowledge, i.e. a slower decrease of the knowledge as a function of elapsed time. Therefore, a Virtual Reality assisted learning process yields a better assimilation of the mathematical notions than with more conventional simulations devices, as it provides this live sensorial experience. This is a more than a realization of the request expressed by a student involved in a research made by Habre (2001); this student wished to be able to rotate surfaces in different directions. A Computer Algebra Systems does this already. VR meets a further requirement of this student, namely to have "a physical model that you can feel in your hands".

According to the brain mapping, the numerical representation of functions is acquired by the left hemisphere of the brain, and the space-live experienced acquisition in a learning process is devoted to the right hemisphere. The transfer from the symbolic rule to a 3D representation and vice-versa requires transfer between two brain lobes with different functionalities. Concerning conceptualization, especially when it must be applied to a concrete domain, there exists a mental difficulty to "move" from one lobe to the other (in terms of longer reacting time, or of completeness of the process). An interactive environment where functional parameter changes are allowed, and where the environment changes can be sensitively experienced, enables a faster building of bridges between the different registers of representation, symbolic, numerical, and graphical.

Finally, the usage of a VR assistant to learning is purely individual. The teacher can show a movie, but it is only an approximation of the requested simulation. The student's senses are involved in the process, the hand on the joystick, the eyes and the ears in the helmet, etc. Therefore the VR device should take in the learning computerized environment a place different from the place of other instruments.

OUR VR DEVICE AND FUTURE RESEARCH

The digital device described above is now in its final steps of initial development. The user can fly over (or walk along) the terrain, i.e. over the graph of the given function. The details of the graphs, the possible discontinuities, are made more and more visible. This effect is not obtained by regular zooming, as this operation only inflates the size of the cells of the interpolation grid. For new details to appear the data has to be computed anew and only part of the surroundings is displayed.

Furthermore, a VR environment seems to contribute an added value by representing more holistic characteristics of the mathematical knowledge. Among the main contributions are the dynamics or flow traits. A more integrated one is the ability to understand its place in the whole mathematical or physical context it is playing with.

In cognitive terms it means that by VR environment, the teacher should provide to the student a more accurate mental model of the mathematical knowledge, including the applicable images of it (Croasdell et al, 2003).

In particular, the dynamical properties of a VR device and their appeal to various sensitive perceptions (vision, audition, etc.) induce also the need of the integration of the hand into the educative schemes. As Eisenberg (2002) says, the hand is not a peripheral device, but is as important as the brain. He discusses the issue of the importance of physical approximations to purely abstract concepts, rejected by Plato's point of view. Here we use the hand totally coordinated with vision and sensorial perception.

As noted by Artigue (2007), "The increasing interest for the affordances of digital technologies in terms of representations have gone along with the increasing sensitivity paid to the semiotic dimension of mathematical knowledge in mathematics education and to the correlative importance given to the analysis of semiotic mediations". In this perspective, a preliminary double blind research is on its way, with two groups of JCT students. We intend to report on the results in a subsequent paper.

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