CERME 6 – WORKING GROUP 6
LANGUAGE AND MATHEMATICS

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INTRODUCTION
LANGUAGE AND MATHEMATICS

Candia Morgan, Institute of Education, University of London

The 21 papers presented to the Working Group were marked by a wide diversity of research focuses and theoretical perspectives. We therefore organised the discussion around five themes:

- Language and thought
- Classroom interaction
- Teacher development
- Theoretical perspectives to describe, analyse and interpret the semiotic aspects of students' mathematical activities
- 'Everyday' and mathematical language and learning

As will be seen from summaries of each of the sections below, there is some overlap between the issues considered in each theme. For example, the use of gesture has become of increasing interest and importance in the field and is found as a focus in papers in several of the themes. Similarly, while the relationship between everyday and mathematical language is a significant theme in its own right, it also emerges as an issue of relevance across other themes.

SECTION 1: 'LANGUAGE' AND THOUGHT

'Language' has a material, and therefore public, surface: either visible (writing and gesture - including sign language) or audible. On the other hand, thinking is invisible and inaudible. Therefore there is a challenge to render it observable, which must of necessity be by indirect observation. This sets up two fundamental tensions:

- Between the individual and the social
- Between implicit and explicit expression

The papers in this section propose different perspectives on how to make sense of the relation between language and thought.

- Focus on gestures, broad view on language (LaCroix)
- Reflection (Schülke/Steinbring)
- Inferential approach (Hußmann/Schacht)
- Argumentation: Toulmin model (Pimm/Sinclair)

- Critical thinking (Aizikovitsch/Amit)

SECTION 2: CLASSROOM INTERACTION

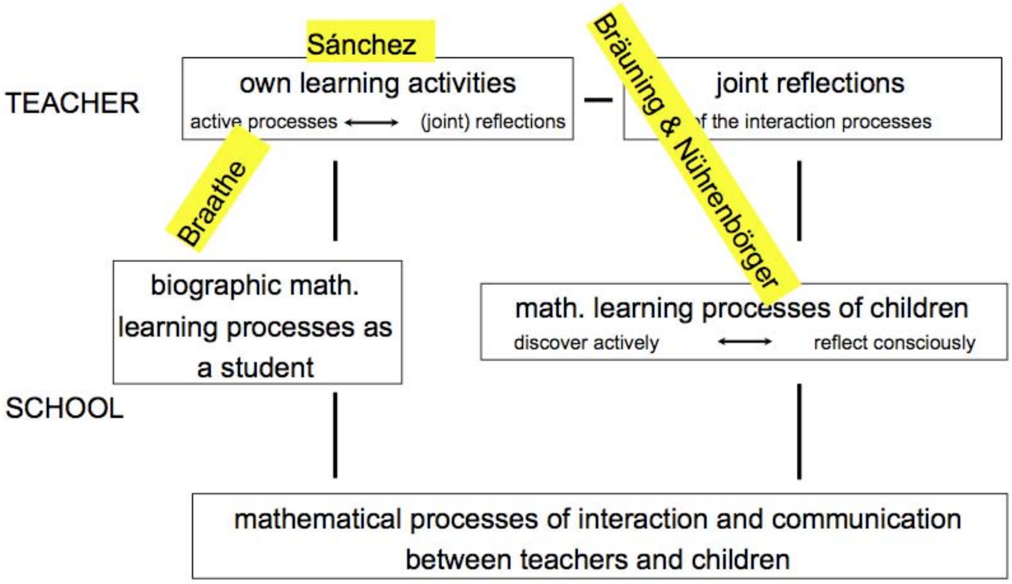
The theme “Classroom interaction” indicates that the papers in this section focus on the whole classroom, the relationships between teacher and students and among students and the role that language plays in establishing these relationships and in building mathematical discourse. The papers use a range of perspectives including the Wittgenstein’s language games, the notion of teacher as improviser, a focus on the use of gesture, shared thinking in group talk, and the interplay between everyday and mathematical discourse, aiming:

- to get deeper insight into processes of giving meaning to words in class (Meyer)
- to show how teacher and pupils co-construct new mathematical ideas using the improvisation metaphor (Dooley)
- to describe the communicative strategies of an experienced teacher when summing up pupil solutions (Bjuland et al.)
- to consider how discourse, as a theoretical and didactical concept, can contribute towards developing mathematics teaching (Riesbeck)

SECTION 3: TEACHERS’ PROFESSIONAL DEVELOPMENT

“Teachers’ professional development” is a major theme of the papers presented by HansJørgen Braathe, Kerstin Bräuning, Marcus Nührenbörger and Mario Sánchez. The understanding of different interaction forms of teachers` distanced view on communication and interaction processes is a necessary condition for their development, as Dewey (1916, 4) pointed out, “society not only continues to exist by transmission, by communication, but it may fairly be said to exist in communication.”

Each paper analysed ideas and thoughts expressed by teachers in written and oral form. But each paper deals with different aspects and schemas of professional development. The following diagram is separated in two levels: “teacher with distance to communication processes in school” and “the mathematical learning and teaching in school”. The level “Teacher” means that teachers are integrated in two different activities: On the one hand their own mathematical learning activities, and on the other hand their joint reflections. Each teacher has biographical mathematical learning processes. This aspect is located in-between the levels “Teacher” and “School”. The 2nd level “School” includes the mathematical learning processes of children and the interaction between teachers and children.



Each paper highlights not only different aspects and methodological approaches to teachers` professional development, but also refers to different theoretical frameworks – like positioning theory, inquiry cooperation model, epistemological and interactional theory. The variety of the theories deepens and broadens the insights in the special conditions of teachers` interactions and learning processes connected to language and mathematics.

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SECTION 4: THEORETICAL PERSPECTIVES TO DESCRIBE, ANALYZE AND INTERPRET THE SEMIOTIC ASPECTS OF STUDENTS' MATHEMATICAL ACTIVITIES

A common aspect of the four papers of this theme is the fact that their structure consists in the presentation of a *new* or *adapted* theoretical tool (or perspective), followed by some examples that are chosen to illustrate (and, possibly, discuss) the use and the potential of the proposed tool (or perspective). A *common*, problematic situation in mathematics education is particularly relevant in the *specific case* of these papers: the plurality of theoretical references (from different disciplines: linguistics, epistemology, psychology, sociology…) brings a proliferation of theoretical tools. Two legitimate questions are related to the previous remark: what educational need/problem should the theoretical tools (or perspectives) satisfy? And what effective educational implications do they have?

Boero and Morselli present a **comprehensive** tool derived from Habermas' construct of "rational behaviour" to describe and analyse student use of algebraic language. By integrating Blumer's "Symbolic interactionism" and Latour's "Actor -network -theory", Fetzer offers a perspective to analyse classroom interaction and discuss related interpretations. Font et al. present "Objectual metaphors", a particular kind of

(Lakoff & Nunez) “Grounding metaphor”, as a tool to analyze and discuss how the classroom discourse helps to develop students’ comprehension of the non ostensive mathematical objects. Morgan and Alshwaikh argue that a multi-semiotic environment not only affords rich potential for developing mathematical concepts, but may also affect more fundamentally the goals of student activity.

The discussion of the group of papers demonstrated openness to alternative theoretical perspectives. Not only may we consider what we can learn from others but attending to different perspectives serves to sharpen our understanding of our own theories. However, there are problems with the proliferation of theories that need to be managed, showing how various perspectives may be useful while being alert to the possibilities and constraints of combining or ‘merging’ theories. There is also felt to be a need to maintain links with the original sources of theoretical perspectives.

Theoretical ideas also have implications with respect to practice. They can provide language to help researchers see new aspects of practice. Moreover, through being introduced to theoretical ideas, teachers could develop awareness of complexities of the classroom

SECTION 5: ‘EVERYDAY’ AND MATHEMATICAL LANGUAGE AND LEARNING

All four papers of this theme group are in various ways occupied with links between everyday and mathematical concepts. Analysing classroom data the authors identify attempts to create such links. The discussion of the development of scientific concepts in children can be traced back to Vygotsky who describes this as a cooperative process between an adult and the child. Kyriakides discusses diagrams as a mediating tool in learning about fraction multiplication and points to an episode where the introduction of everyday language, instead of trying to remember an algorithm, proved to be an effective link to the scientific concept. On the other hand, Schütte describes an episode having to do with adding fractions, where the scientific concept *least common multiple* is lying behind. The teacher mainly uses everyday language, and the link to the scientific concept and her assisting function in the pupils’ development of mathematical language seem to be lost. In the paper by Vogel and Huth, the focus is on a combinatorial problem where two first graders, assisted by an adult, gradually start to use technical terms and the practical context become less and less important. Rønning studies a situation where the pupils are measuring milk, and where both teacher and pupils are moving back and forth between an everyday situation and a school situation. The two situations involve different semiotic representations and also different goals and actions, which can be seen to create a certain tension.

The following topics for discussion were identified.

- The function of everyday language in learning mathematics
- The function of diagrams in learning mathematics

– The teacher as a model for learning technical (scientific) language.

IMPARTING THE LANGUAGE OF CRITICAL THINKING WHILE TEACHING PROBABILITY

Einav Aizikovitsh, Miriam Amit
Ben Gurion University of the Negev, Israel

This paper reports a preliminary study of imparting to students a new kind of language, incorporating elements of ***critical thinking (CT),*** *in the course of a mathematics (probability) lesson. In the paper, we describe and analyse one probability lesson, which is part of an in-depth study that comprises fifteen math lessons of similar constitution. The purpose of this research is to determine whether the teaching methods we developed can improve students' critical thinking. Our approach favors immersion-teaching of CT, i.e. incorporating CT terminology and practice within the framework of a probability lesson, and is based on the specific taxonomy of CT skills proposed by Ennis. We focus specifically on critical thinking while distinguishing it from stochastic thinking, creative thinking and statistical thinking. This study involved 55 subjects. Analysis of interviews conducted with the students and an analysis of their submitted work indicated that students' critical and analytical capabilities greatly improved. These results show that if teachers consistently and methodically encourage CT in their classes, by applying mathematics to real-life problems, encouraging debates, and planning investigative lessons, the students are likely to develop the language of critical thinking as a result. This paper is a description of an initial study, a snapshot that focuses on one lesson and illustrates the orientation of the entire study.*

INTRODUCTION AND THEORETICAL FRAMEWORK

It has already been suggested that teachers should use a language of critical thinking as part of the attempt to change the method of teaching to enable meaningful learning of information (Perkins, 1992). This is an area in which a substantial research literature already exists.

Our focus in this paper is describing our approach and its initial results. In this paper, we are focusing on the language of critical thinking. When defining the term critical thinking (CT), it is important to realize that it is not a new concept; we can find it as early as ancient Greek times: Socrates, as reported by Plato, used to roam the streets of Athens asking people all kinds of philosophical questions about the purpose of life, morality, justice, etc., apparently for the purpose of stimulating a form of critical thinking. These questions and answers were collected and recorded in the Socratic dialogues. In the field of education, it is generally agreed that CT capabilities are crucial to one's success in the modern world, where making rational decisions is becoming an increasingly important part of everyday life. Students must learn to test reliability, raise doubts, and investigate situations and alternatives, both in school and in everyday life. Abundant definitions of critical thinking have been proposed, since this is a multidisciplinary subject that engaged teachers, educators, sociologists, psychologists and philosophers in all eras, but we would like to focus on Ennis' taxonomy, because for our purposes we needed to employ a hierarchical set of critical thinking skills isolated from other definitions. Ennis (1962) defines CT as “a correct evaluation of statements". Twenty-three years later, Ennis broadened his definition to include a mental element, defining CT as “reasonable reflective thinking focused on deciding what to believe or do” (Ennis, 1985). Our research is based on three key elements: a CT taxonomy that includes CT skills (Ennis, 1987); the learning unit "Probability in Daily Life" (Liberman & Tversky, 2002); and the infusion approach of integrating subject matter with thinking skills (Swartz, 1992).

Ennis’ Taxonomy (Ennis, 1987)

In light of his definition, Ennis developed a CT taxonomy of skills that include intellectual as well as behavioural aspects, e.g. judging the credibility of sources, searching for clarifying questions, defining the variables, searching for alternatives etc. In addition to skills, Ennis's taxonomy (1987) also includes dispositions and abilities. Ennis claims that CT is a reflective and practical activity aiming for a moderate action or belief. There are five key concepts and characteristics defining CT: practical, reflective, moderate, involving? belief and oriented towards? action.

Learning unit "Probability in Daily Life" (Liberman & Tversky 2002)

In this learning unit, which is a part of the formal syllabus of the Ministry of Education, the students are required to analyse problems, raise questions and think critically about data and information. The purpose of the learning unit is to teach the students not to be satisfied with a numerical answer but to examine the data and its validity in order to arrive at a more valid answer and develop their critical thinking. In cases where there is no single numerical answer, the students are required to know what questions to ask and how to analyse the problem qualitatively, not only quantitatively. Along with being provided with statistical instruments, students are redirected to their intuitive mechanisms to help them estimate probabilities in daily life. Simultaneously, students examine the logical premises behind their intuitions, along with possible misjudgments of their application.

The infusion approach (Swartz, 1992)

There are two main approaches to fostering CT: *the general skills approach* which is characterized by designing special courses for instructing CT skills, and the *infusion approach,* according to Swartz (1992), is characterized by providing these skills through teaching the set learning material. According to this approach, there is a need to reprocess the set material in order to combine it with thinking skills. In this report, we will show, on the example of one lesson, how we combined the mathematical content of "probability in daily life” with CT skills from Ennis' taxonomy, and evaluated the subjects' CT skills.

METHODOLOGY

The main paradigmatic aspects of methodology in mathematics education research have been broadly established (Scherer & Steinbring, 2006). Our methodological challenge was to investigate the development of the "language of critical thinking" through critical thinking skills incorporated into a structured mathematics lesson, such as a probability lesson. In this regard, the methodological approach is closest to the "Design Experiment" (as discussed by Cobb, Confrey, diSessa, Lehrer and Schauble, 2003). Through careful instructional design, a lesson sequence was constructed with the goal of consistently and methodologically encouraging and promoting crtitical thinking by applying mathematics to real-life problems, encouraging debates and using investigative lessons, in order to develop the "language of critical thinking". The research process examined student classroom products (primarily student submitted work) and post-lesson interviews with students to document changes in students' analytical capabilities. These changing capabilities could then be related to classroom activities, which were documented by video.

Setting, Population, and Data

Fifty-five children between the ages of fifteen and sixteen participated in an extra curriculum program aimed at enhancing the critical thinking skills of students from different cultural backgrounds and socio-economical levels. An instructional experiment was conducted in which probability lessons were combined with CT skills. The study consisted of fifteen 90 minute lessons, spread out over the course of an academic year, in which the teacher was also one of the researchers.

Data sources were: Students' products, Pre and post questionnaires, Personal interviews and Class transcriptions.

The students' products (papers, homework, exams etc.) were collected. Five randomly selected students were interviewed at the end of each lesson and one week after. The personal interviews were conducted in order to identify any change in the students' attitudes throughout the academic year. Not only was the general attitude examined, attention was paid to the development of critical thinking language (e.g., by asking the student to define critical thinking and to explain how they viewed critical thinking in the scope of the lesson; furthermore, they were also asked to assess whether they considered themselves to be critical thinkers, and it was the answer to this question that was used to establish the nature and frequency of critical thinking among them). All lessons were video-recorded and transcribed. In addition, the teacher kept a journal (log) on every lesson. Data was processed by means of qualitative methods intended to follow the students' patterns of thinking and interpretation with regards to the material taught in different contexts. Following Ennis' taxonomy (Ennis, 1987), data was analysed by employing three principles: (1) As the student is asked to articulate the question dealt with in a particular lesson, the level of critical thinking was deciphered (as will be discussed later on); (2) students' reactions to the teacher's attempt to induce critical thinking were examined through their responses as well as from the interviews; (3) proposition of alternatives was employed as an interview technique, in an attempt to identify critical thinking abilities.

The Intervention- Unit Description

As already mentioned, the probability unit combines CT skills with the mathematical content of "probability in daily life". This new probability unit included questions taken from daily life situations, newspapers and surveys, and combined CT skills. Each of the fifteen lessons that comprised the probability unit had a fixed structure: a generic (general) question written on the blackboard; the student's reference to the question and a discussion of the question using probability and statistical instruments; and, an open discussion of the question that included practicing the CT skills. The mathematical topics taught during the fifteen lessons were: Introduction to set theory, probability rules, building a 3D table, conditional probability and Bayes theorem, statistical connection and causal connection, Simpson's paradox, and judgment by representativeness. The following CT skills were incorporated in all fifteen lessons: A clear search for an hypothesis or question, the evaluation of reliable sources, identifying variables, “thinking out of the box,” and a search for alternatives (Aizikovitsh & Amit, 2008). Each lesson followed the same four part structure.

1. Given Text

At the beginning of the lesson the teacher presented a short article or text.

2. Open Class Discussion in Small Groups

Discussion in small groups about the article and the question.

•Initial suggestions for the resolution of the question

•No intervention by the teacher

3. Further Discussion Directed by the Teacher

Open class discussion. During the discussion the teacher asked the students different questions to foster the students’ thinking skills and curiosity and to encourage them to ask their own questions.

• Various suggestions from students in class.

• Interaction between groups of students.

• Reaching a consensus across the whole class (or just across the group).

4. Critical Thinking Skills and Mathematical Knowledge (Teaching)

The teacher referred to the questions raised by the students and encouraged CT, while instilling new mathematical knowledge: the identification of and finding a causal connection by a third factor and finding a statistical connection between C, and A and B, Simpson's paradox and Bayes Theorem.

Case study- The Aspirin Case

Below, I have provided a detailed description of one lesson called the Aspirin Case. Following the description, I outline the analysis of the lesson using the following techniques: referring to information sources, raising questions, identifying variables, and suggesting alternatives and inferences. The lesson topic was conditional probability. The CT skills practiced in the lesson were evaluating source reliability, identifying variables, and suggesting alternatives and inference.

1. A Given Text

Your brother woke up in the middle of the night, crying and complaining he has a stomachache. Your parents are not at home and you don't know what to do. You gave your brother aspirin, but an hour later he woke up again, suffering from bad nausea and vomiting. The doctor that takes care of your brother regularly is out of town and you consider whether to take your brother to the hospital, which is far from your home. You read from a book about children's diseases and find out that there are children that suffer from a deficiency in a certain type of enzyme and as a result, 25% of them develop a bad reaction to aspirin, which could lead to paralysis or even death. Thus, giving aspirin to these children is forbidden. On the other hand, the general percentage of cases in which bad reactions such as these occur after taking aspirin is 75%. 3% of children lack this enzyme.

(Taken from "probability thinking" p. 30+slight changes made by researcher)

2. Open Class Discussion in Small Groups

Discussion in small groups about the generic question:

Should you take your brother to the emergency room? What should you do?

Can aspirin consumption be lethal?

3. Further Discussion Directed by the Teacher

The generic question on the blackboard was:

Should you take your brother to the emergency room? What should you do?

21 Teacher: What do you think?

22 Student 1: Where is the information taken from? Can we see the article for ourselves?

23 S2: Is the source reliable? How can we check it?

24 S3: Where is the article taken from? What is its source?

25 S1: Should I answer the identification of the sources question?

26 T: Not yet. We are focusing on searching for questions. Please think of other questions.

27 S3: What connection does the article discuss?

28 S2: first we need to identify the variables!!!

29 T: Right. First, we ask what the variables are.

30 S4: You can infer it from the title that suggests that a connection exists between aspirin and death.

31 T: According to the data from the article, Can we find a statistical connection? (the student already know this subject)

32 S2: I know! We can ask: suggest at least 2 other factors that might

cause the described effect.
33 S5: The question is what causes what?
34 S6: Can aspirin consumption be lethal?
35 T: What do you think?
36 T: How can you be sure?
37 S6: Umm…
38 S3: Are there other factors, such as genetics!?
39 T: Very good. What did student 3 just do?
40 S1: He suggested an alternative!!
41 T: How can we check it? Do you have any suggestions? Can you make a connection between this problem and the material we have learned in the past few lessons? Can you offer an experiment that would solve the problem?
42 S3: Of course. An observational experiment.

In paragraph 21 we encounter skills such as "searching for the question"- a fundamental skill. First there is a need to clarify the starting point for the interaction with the student. We also need to clarify to ourselves what is the thesis and what is the main question before we approach decision making. The paragraph also demonstrates relevance to daily life. In paragraph 26 the students are taking a step back, we refer to "identifying information source and evaluating the source's reliability" skill. This step is crucial, as it helps us to assess the quality and the validity of the article discussed. This skill was practiced in past lessons. See paragraph that summarizes the article. In paragraph 26 we encounter "searching for the question" skill again. We will continue searching for the main question through practicing the "variables identification" skill. Raising the search for alternatives. Posing questions enables the practice of this skill. Paragraph 30 deals with identifying the variables and understanding them by a 2D table and a conditional probability formula. In paragraph 36 the teacher builds the students' self esteem by encouraging them to express their ideas and opinions (even if they are not always correct or relevant). She prevents any intolerance of other students. The method of instruction that aims at fostering the confidence and the trust of the students in their CT abilities and skills is, according to Ennis "referring to other peoples points of view" and "being sensitive towards other peoples' feelings". In paragraph 23 the student is referring to other sets and finding the connection between them. Paragraph 31 depicts the skill of "Searching for alternatives". Paragraph 42 refers to a controlled experiment or an observational experiment. An additional grouping and finding the connection between the variables by Bayes theorem or a 2 dimensional table.

4. Critical Thinking Skills and Mathematical Knowledge (Teaching)

This phase of the lesson focused on encouraging critical thinking and instilling new mathematical knowledge (Bayes formula) statistical connections by referring to students’ questions and further discussion.

A teacher-led discussion focused on methods of analysis using such Critical Thinking skills as: Source identification: Medicine book; Source reliability: High; Variable identification: A – enzyme deficiency, D – adverse reaction to aspirin; Mathematical Knowledge: Data: $P(D/A)=0.25$ $P(D)=0.75$ $P(A)=0.03$, To prove: $P(A/D)=?$
Using Bayes formula (or a two dimensional matrix) the result is:
Lesson Conclusion is that only 1% of the children without the enzyme develop an adverse reaction to aspirin, thus there is no need to go to the hospital.
Even so, is it worth taking the risk? What do you think? (question to the class).

DISCUSSION

Research analysis according to critical thinking skills in this case study
Through the infusion approach, students practice their CT while acquiring technical probability skills. In this lesson, the following five skills are exercised: raising questions – asking question about the article and probing on the main question about the connection between aspirin and death; referring to information sources and evaluating the source's reliability - the text took from Medicine book; the students skepticism and identification of variables – students identified the enzyme deficiency and adverse reaction to aspirin. Following these skills, another skill, searching for alternatives (paragraph 38), was presented. In class the teacher and the students spoke about suggesting alternatives, not taking things for granted, but examining what had been said and suggesting other explanations. Hence, the skills that were practiced in the described lesson were: raising questions, evaluating the source's reliability, identifying variables, and suggesting alternatives and inference. In order to understand and monitor the students' attitudes toward CT as manifested by the skills specified above, interviews were conducted with five students after the aforementioned lesson. In these interviews, the students acknowledged the importance of CT. Moreover, students were aware of the infusion of instructional strategies that advance CT skills. Examples from two of the interviews follow.
Student 4 was interviewed and was asked to define CT. His answer was:
"I think CT is important when you study Mathematics, when you study other topics and when you read the paper, but it is most important when you deal with real life situations, and you need the right instruments in order to do so (deal with these situations)."
When Student 2 was asked about important components during the last few classes and the present class, she answered: *"first we should check the information source's reliability and despite all the numerical data, I don't accept the researcher's conclusion."*
Additional data, consistent with these two examples suggest that infusion of CT into the formal curriculum in mathematics can equip students with CT skills that are applicable to wider disciplines.

RESEARCH LIMITATIONS

This case study presents one lesson which was designed in a fixed pattern – a generic question, a discussion of the question, the practice of statistical connection, introduction to causal connection and experiencing the use of CT skills such as: raising questions, evaluating the source's reliability, identifying variables, and suggesting alternatives and inferences. On the basis of the interviews conducted and questionnaires that were qualitatively analyzed, it is not established, at this stage, the extent to which these skills have been acquired. Skill acquisition will be evaluated in much greater detail at a later phase in this study, using quantitative measures – the Cornell Critical Thinking Scale and the CCTDI (Facion, 1992) scale. At this stage we have provided only an introductory picture of our approach and an indication of the form of our analysis and results. However, this case study provides encouraging evidence of the effectiveness of this approach and further investigation in this direction is needed.

CLOSING REMARKS

The small scale research described here constitutes a small step in the direction of developing additional learning units within the traditional curriculum. Current research is exploring additional means of CT evaluation, including: the Cornell CT scale (Ennis, 1987), questionnaires employing various approaches, and a comprehensive test composed for future research.

The general educational implications of this research suggest that we can and should lever the intellectual development of the student beyond the technical content of the course, by creating learning environments that foster CT, and which will, in turn, encourage the student to investigate the issue at hand, evaluate the information and react to it as a critical thinker. It is important to note that, in addition to the skills mentioned above, in the course of this lesson it appears that the students also gained intellectual skills such as conceptual thinking and developed a class culture (climate) that fostered CT. Students practiced critical thinking by studying probability. In this lesson, the following skills were demonstrably practiced: referring to information sources (paragraph 22), encouraging open-mindedness and mental flexibility (all questions), a change in attitude and searching for alternatives. A very important intellectual skill is the fostering of cognitive determination – to be able to express one's attitude and present an opinion that is supported by facts. In this lesson, students could be seen to be searching for the truth, they were open-minded and self-confident. In other words, they practiced critical thinking skills. A new language was being created: the language of critical thinking.

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TOWARD AN INFERENTIAL APPROACH ANALYZING CONCEPT FORMATION AND LANGUAGE PROCESSES

Stephan Hußmann, Florian Schacht

Institute for Development and Research in Mathematics Education, Dortmund, Germany

This paper introduces a theoretical approach to study individual conceptual development in mathematics classroom. It uses the theory of a normative pragmatics as an epistemological framework, which Robert BRANDOM made explicit in 1994. There are different levels of research in mathematics education on which BRANDOM's framework offers a consistent theoretical approach for describing such developments: a linguistic perspective, the theory of conceptual change and the theory of conceptual fields. Using that framework, we will outline an empirical example to describe technical language developments as well as developments of conceptual fields and of the students' conceptualizations.

INTRODUCTION

Many results of large-scale studies monitoring the education system (PRENZEL et al. 2007, ARTELT et al. 2000, BAUMERT et al. 1997, BAUMERT et al 1998) show for mathematics education that German students have difficulties with tasks that challenge their conceptual understanding. These difficulties seem to be caused by the German classroom practices, which do not challenge enough the students' individual cognitive skills, which lack teachers' diagnosis abilities, and which do not offer enough room for creative and individual work (e.g. PRENZEL et al. 2004).

Research is required in both mathematical learning environments and in formation of concepts and conceptualizations in order to find out in how far (i) the use of the specific potential which certain tasks offer and (ii) the dealing with students' conceptualizations have an effect on the formation of conceptual thinking. In Germany, there are only some studies which focus on the analysis of individual concept-formation (HUßMANN 2006, BARZEL 2006, HAHN / PREDIGER 2008, Prediger 2008a/b). There is also a demand for research with regard to dealing with certain individual students' conceptualizations.

Because mathematical thinking is genuinely conceptual thinking, the formation of mathematical concepts has gained big interest in the mathematics education research community. The multiple approaches and theories for describing and explaining conceptual processes and developments differ a lot in terms of their theoretical framework, e.g. developmental psychology or cognitive psychology. In this study, we choose a social-constructivist approach (COBB, YACKEL 1996).

With his theory of inferentialism, the philosopher Robert BRANDOM (1994) has introduced a convincing, comprehensive and coherent theoretical framework to analyze such language processes.

EPISTEMOLOGICAL FRAMEWORK: INFERENTIALISM

In his influential book on reasoning, representing and discursive commitment "Making it explicit" (1994) BRANDOM chooses an inferential approach to describe semantic content of concepts in terms of their use in practice: it is the idea that propositional semantic content can be understood in terms of the inferential relations they play in discourse, which means for example to know what follows from a proposition or what is incompatible with it. BRANDOM gives an analysis of discursive linguistic practice, describing a model of social practice - and especially a model of linguistic discursive practice - as a game of giving and asking for reasons, which means a normative pragmatics in terms of deontic scorekeeping. Using his theory to describe linguistic practice and based on the theory of a normative pragmatics introduced by BRANDOM (1994), we will develop an analytic tool to describe the formation of concepts. For BRANDOM, understanding

> can be understood, not as the turning on of a Cartesian light, but as practical mastery of a certain kind of inferentially articulated *doing*: responding differentially according to the circumstances of proper application of a concept, and distinguishing the proper inferential consequences of such application. (BRANDOM 1994, p. 120)

In this sense, discourse can be described as a game of giving and asking for reasons, a term that can be traced back to WITTGENSTEIN'S 'Sprachspiel' (language game). Therefore, every 'player' in the game of giving and asking for reasons keeps score on the other players. This deontic score keeps track on the claims that every player (including oneself) is committed to and it keeps track on the commitments each one is entitled to. With every assertion – so with every move in the game of giving and asking for reasons - which one player is making, the score may change.

The inferential relations are commitment - and entitlement- preservations and incompatibilities. BRANDOM'S normative pragmatics gives an understanding of conceptual content on the basis of using the concepts in practice. "The aim is to be able to explain in deontic scorekeeping terms what is expressed by the use of representational vocabulary - what we are doing and saying when we talk about what we are talking *about*." (BRANDOM 1994, p. 496)

BRANDOM claims that the fact that propositions have a certain (propositional) content should be understood in terms of inferential relations. Accordingly, propositions are propositions because they have the characteristic feature to function as premises and conclusions in inferences (that means they function as reasons).

> Thus grasping the semantic content expressed by the assertional utterance of a sentence requires being able to determine both what follows from the claim, given the further commitments the scorekeeper attributes to the assertor, and what follows from the claim, given the further commitments the scorekeeper undertakes. (…) In such a context, particular linguistic phenomena can no longer reliably be distinguished as 'pragmatic' or 'semantic'. (BRANDOM 1994, pp. 591/592)

It is important to note that it is not necessary for an individual to know all the inferential roles of a certain concept to be regarded as someone that has conceptualized a certain concept. “To be in the game at all, one must make *enough* of the right moves - but how much is enough is quite flexible" (BRANDOM 1994, p. 636).

DERRY (2008) outlines the characteristics of an inferential view for education. Referring to BRANDOM and VYGOTSKY she notes that the

> priorisation of inference over reference entails, in terms of pedagogy, that the grasping of a concept (knowing) requires committing to the inferences implicit in its use in a social practice (…). Effective teaching involves providing the opportunity for learners to operate with a concept in the space of reasons within which it falls and by which its meaning is constituted. (DERRY 2008, p. 58)

CONCEPTUAL DEVELOPMENT RESEARCH IN MATHEMATICS EDUCATION

Using Robert BRANDOM’s ideas of a normative pragmatics, it is the aim of the project to develop a coherent theoretical framework within which the formation of concepts in mathematics education can be described. This theoretical framework uses inferential (instead of representational) vocabulary. There are different levels of research in mathematics education on which BRANDOM’s framework offers a consistent theoretical approach for describing such developments.

Theory of conceptual fields

Using Robert BRANDOM’s theory of a normative pragmatics as an epistemological background to describe formations of concepts, VERGNAUD’S theory of conceptual fields offers a consistent framework within which long- and short-term conceptual developments can be analyzed. Within his framework, he gives respect to both mathematical concepts and individual conceptualizations.

WITTENBERG says that mathematics is “thinking in concepts” (1963). What distinguishes us as human beings is the fact that we are concept users (Brandom 1994). Accordingly, not only mathematics is thinking in concepts: everything obtains a conceptual meaning for us and concepts are the smallest unit of thinking and acting. This decisive linguistic perspective of conceptual understanding was pointed out by SELLARS: “grasping a concept is mastering the use of a word” (see BRANDOM 2002, p. 87). Accordingly, it is necessary to research concept formation, which means it is necessary to study the classroom discourse. For that, VERGNAUD (1996, 1997) offers a solid theoretical framework. With his theory of conceptual fields, VERGNAUD developed a theoretical framework which picks up BROUSSEAU’S theory of didactical situations (1997) and which offers a tool to describe, to analyze and to understand both short- and long-term formations of concepts. For him, a conceptual field refers to a set of (problem) situations, conventional and individual concepts.

> [A] *conceptual field* is a set of situations, the mastering of which requires several interconnected concepts. It is at the same time a set of concepts, with different properties, the meaning of which is drawn from this variety of situations. (VERGNAUD 1996, p. 225)
>
> A concept is a three-tuple of three sets: C = (S,I,S) where S is the set of situations that make it meaningful, I is the set of operational invariants contained in the schemes developed to deal with these situations, and S is the set of symbolic representations (natural language, diagrams (…)) that can be used to represent the relationships involved, communicate about them, and help us master the situations. (VERGNAUD 1996, p. 238)

In the latter definition, VERGANUD points out that language is essential for focusing on conceptual fields. Language is the surface on which we analyze formations of concepts. Conceptual fields are equally related to situations, to mathematical concepts, to individual conceptualizations and to operational invariants such as *theorems-in-action* or *concepts-in-action*. On the one side, those operational invariants are *theorems-in-action* which are "held to be true by the individual subject for a certain range of the situation variables" (VERGNAUD 1996, p. 225). On the other side, they are categories- or *concepts-in-action*,

> that enable the subject to cut the real world into distinct elements and aspects, and pick up the most adequate selection of information according to the situation and scheme involved. Concepts-in-action are, of course, indispensable for theorems-in-action to exist, but they are not theorems by themselves. They cannot be true or false (VERGNAUD 1996, p. 225).

In every new situation, the individual schemes develop. Because of the strong connection between situation and scheme, the short-term perspective on concept formation is important to study. At the same time, because of the individual development within the learning process and the different situations the individual deals with, the long term perspective is equally important to study.

Linguistic approach

Besides the theory of conceptual fields, there is a specific linguistic approach that can be drawn from BRANDOM'S epistemological framework. Therefore, SIEBEL (2005) refers to developments from colloquial to technical language by making implicit concepts explicit.

Thought and language is not the same, otherwise we would not be able to form sentences like "I don't know how to say it" or "that is not what I meant". Still, we can only get a precise picture of conceptual developments by observing the use of language, the discourse, that what's made explicit. To get an idea of what is implicit in use, we have to ask for reasons and commitments.

In her linguistic approach categorizing and analyzing technical language used in elementary algebra books, Siebel (2005) picks up that distinction. She distinguishes between explicit and implicit technical terms. Explicit ones are explicitly defined, e.g. by "x is called variable". Explicit technical terms are characteristic for explicit knowledge ('know-that') which can be made explicit in either words or formulas. In contrast, the meaning of implicit terms is characterized by their use (SIEBEL 2005, p. 120). Implicit technical terms are characteristic for implicit knowledge ('know-how') which can only be learnt by practical exercising. SIEBEL points out that most of our concepts are implicit and that we can only make some of them explicit (see SIEBEL 2005, p. 122). Referring to BREGER (1990), SIEBEL describes how knowledge and concepts develop from "know-how" to "know that" knowledge, from implicit to explicit knowledge – by making them explicit (2005, p. 122). That linguistic approach offers a description of developments from colloquial to technical language, lining out how implicit concepts and knowledge ("know-how") become explicit ("know-that").

Judgments as basic units

Following BRANDOM, the linguistic perspective cannot be separated from the propositional content. With every commitment and every judgment, we have taken on a certain kind of responsibility and committed ourselves to some explanation of the given phenomenon. Those explanations and judgments correspond to the theoretical schemes (see VERGNAUD 1996) which are intimately interwoven with the specific situation.

Theory of conceptual change

Following BRANDOM and VERGNAUD, learning and formation of concepts is closely linked to a specific situation. The developments that proceed in these situations are closely connected to the conceptualizations we have. These conceptualizations maybe have to be revised, expanded or modified in every new situation which we have to commit ourselves to, for example to a certain scheme or an explanation. The theory of conceptual change (e.g. DUIT 1996) picks up that distinction between individual conceptualizations and scientific conceptions.

The conceptual change theory is a constructivist approach to describe learning processes in terms of reorganization of knowledge (Duit 1996, p. 158, Prediger 2008b for an example in mathematics education). That means for the students to learn that their preinstructional concepts do not give sufficient orientation in certain scientific situations and for them to activate scientific conceptions in those situations (see DUIT 1996, p. 146). Learning scientific concepts often leads to conflicts with prior knowledge and familiar everyday concepts because certain features of both – familiar and new scientific concepts - seem to be incompatible. FISCHER and AUFSCHNAITER (1993) for example studied developments of meaning during physics instruction, focusing on the terms charge, voltage and field. Against the background of different levels of perception, they describe how the use of certain words changes during the learning process: "For this reason, at the beginning of the development of a subjective domain of experience it might be possible that words, as properties of objects, are not yet generated." (p. 165)

Summary

In all the perspectives above, there is a similar line of thought concerning the analysis and description of conceptual developments: intuitive concepts-in-action to consolidated mathematical concepts, implicit meaning of use to explicit technical language, pre-instructional conceptualizations to scientific concepts. The aim of our project is to follow those lines among linguistic descriptions of expressions in mathematics classrooms and to develop learning environments which considering the formation of concepts in mathematics classrooms.

For this purpose, we study the development of individual long- and short-term conceptualizations and of formations of mathematical concepts within learning processes: what is the connection between (problem) situations and operational invariants (such as *theorems-in-action* or *concepts-in-action*)? What is the connection between the formation of concepts and symbolic expressions? In how far is it possible to classify the (problem) situations against the background of individual operational invariants?

Three aspects can be inferred from those questions: How does technical language develop? How do individual conceptualizations develop? How do conceptual fields develop? To examine these questions, we develop an empirical study to describe the individual learning processes.

ONE EXAMPLE ON (TECHNICAL) LANGUAGE DEVELOPMENT

To give an example of how the research questions outlined above can be approached, we offer some results of a small-scale study on technical language development (SCHACHT 2007). This example shows how technical language in chance-situations can develop, how individual conceptualizations develop and how the conceptual field of chance-situations has developed.

Short introduction to the study

For this purpose a fifth grade mathematics classroom of 30 students was observed and videotaped over a period of about six weeks. The central goal of the unit for the students was to develop a concept of ‘chance’. That means that in chance situations, the individual case will not be predictable, but focusing the long term, chance has a certain kind of mathematical structure (HEFENDEHL-HEBEKER 2003). Accordingly, one special focus of this unit was for the students to discover and experience the law of large numbers.

Main features of the unit concerning the research interests of the study were the focus on *discursive elements* in mathematics classroom, the focus on *reflection tasks* during the mathematical learning process and the focus on *student*- rather than teacher-*activity* (cf HUßMANN/PREDIGER 2009).

Based on a functional pragmatic approach, language was analyzed in terms of its use (e.g. EHLICH/REHBEIN 1986, KÜGELGEN 1994). The features of the unit mentioned above formed a solid base to analyze language developments of some students especially because they were often challenged to make their concept of chance explicit (either in written form or verbally).

Some results of the study

The results of this small scale study show some interesting phenomena which could be observed. We will outline one prototypical example of the study and describe its main features concerning technical language development as well as individual conceptualizations and conceptual fields development.

In this example, the task for the student Ralf was to describe and compare results of dice throws in different situations (10, 100, 500 and 1000 throws). Because of the qualitative differences of the situations which he is working in (description of absolute values □ description of relative values), the technical language he uses leads to a paradox situation (distance ("Abstand") is 'small' and 'large' at the same time). A couple of days after this situation, he uses a different and new term which seems more sufficient and viable.

More precisely, the student Ralf first uses the term 'distance' to compare some results of dice-throws. In the first scene, he uses the term 'distance' to compare absolute results.

Number	1	2	3	4	5	6
10 throws	3	1	0	4	0	2
1000 throws	165	174	169	161	171	160

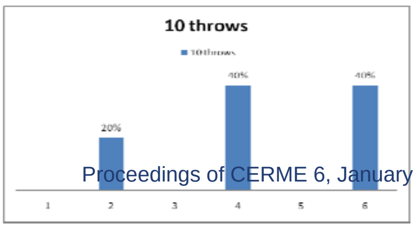
Table 1: Similar example of dice results in absolute values (10 and 1000 throws)

Comparing results similar to Table 1, Ralf observes:

102 And in the situation with small numbers of throws
103 the distances ("Abstände") get smaller.

There are two aspects to point out concerning the use of the term 'distance'. First, he compares the dice results by noticing that the "distances get smaller" (line 103) the smaller the number of throws is. In the example above, that means that there is a little distance between the one time '2' and the two times '6' but there is a greater distance between the 160 times '6' and the 171 times '5'. Second, he uses the term distance to distinguish between situations with a high number of throws (e.g. 1000 throws) and a small number of throws (100 throws).

Later in the same lesson, he uses the same term ('distance') again to compare dice results, except now they are given in relative values (in percent). The teacher asks the students to compare a couple of histograms which show the results of 10-100-500-1000 throws. The histograms which show results of 10 throws of course look quite different to those with 1000 throws. The latter ones show the stabilization of the relative distribution (law of large numbers) while the others show that the results with for example 10 throws differ quite a lot.



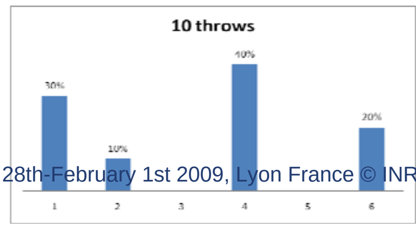
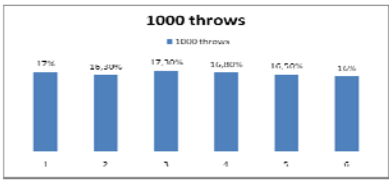


Table 2: Similar example of histograms showing dice results in relative values (10 throws)



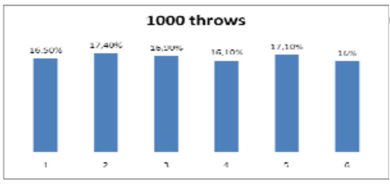


Table 3: Similar example of histograms showing dice results in relative values (1000 throws)

The teacher asks Ralf, what he noticed. Ralf answers:

8 Ralf: I observed that,
9 given a small number of throws,
10 the distances ("Abstände") become larger
11 and given a large number of throws,
12 the distances ("Abstände") become smaller.

In this situation Ralf describes that the distances become larger given a small number of throws. It seems plausible that he has a horizontal perspective and compares all histograms showing the results of 10 throws whereas the "distances become smaller" comparing the others with for example 1000 throws.

At the same time, like in the situation above, Ralf is using the term 'distance' again to distinguish the small and the large 'number-of-throws situations'. Except that he uses the term conversely: in the first situation he described the distances to become smaller when the number of throws becomes *smaller* (lines 102/103), in the latter situation he observes the distances to become larger when the number of throws becomes *smaller* (lines 8/9).

Comparing both examples, the difficulty is that the quality of the situation changes: in the first situation, Ralf compares the absolute values of the dice results of 100 and of 1000 throws. He recognizes that the distances of the results with 10 throws are smaller than the ones with 1000 throws (lines 102/103).

Accordingly, although the term 'distance' is a quite helpful and viable term *in each situation* to distinguish between small and high number of throws, it is overall not sufficient because it seems to lead to paradox and incompatible results.

Some days later the students are asked to give a written comment on the following sentence: "You cannot predict the result of throwing a single dice, but in the long run you don't have a random result." Ralf writes:

130 Given a small number of throws
131 you cannot predict
132 chance, but
133 given a higher number of throws, that works better
134 because it is more distributed ("verteilter") there.

The next day, he adds on a working sheet in a similar situation:

5* in the situation of thousand throws, the distribution ("Verteilung") is: (…)

In both quotes, Ralf uses the term 'distribution' / 'distributed' to distinguish between small and large numbers of throws. For him, this term works without inconsistencies to distinguish both situations. He is also able to predict a distribution in the large number-of-throws situation (line 5*).

Summary

Focusing on technical language development from a linguistic perspective, this example describes a development of the intuitive and implicit use of the term ‘distance’ to an explicit use of the technical term ‘distribution’ that is viable to distinguish between small and large number of throws.

There are two different concepts-in-action Ralf uses: in the first situation, he has a binary concept for comparing the results. In the other situation, Ralf observes a certain structure given a high number of throws. Here, his concept-in-action is that given a high number of throws and a certain mathematical structure, chance is predictable. That effects his theorem-in-action: given a high number of throws, the (relative) distribution can be predicted quite precisely.

This development shows his conceptual change regarding chance situations: whereas his intuitive conceptualization focuses on the term ‘distance’, he then is able to activate a mathematical conception on chance situation using the technical term ‘distribution’ which focuses on the long-term perspective on chance situations. The conceptual change is in line with the dynamic development of Ralf’s theorem-in-action: the new problem situation leads him to come up with a new theorem-in-action.

This example shows in how far all three levels are connected in terms of the inferential epistemological approach that BRANDOM introduces: both conceptual change and conceptual fields help to observe the formation of concepts. But these processes can only be studied because we are concept users (BRANDOM): language is the surface on which the linguistic analysis of the formation of concepts operates.

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ICONICITY, OBJECTIFICATION, AND THE MATH BEHIND THE MEASURING TAPE: AN EXAMPLE FROM PIPE-TRADES TRAINING [1]

Lionel LaCroix

Brock University, Canada

This paper examines an adult student's efforts as he works intensely, with the help of the researcher, to make sense of the fraction patterns on a measuring tape marked in inches. The multi-semiotic analysis of this encounter is framed using Radford's Theory of Knowledge Objectification. From this socio-cultural perspective, mathematics learning involves the social and semiotically mediated process of objectification, i.e. a process in which one becomes progressively aware and conversant, through one's own actions and interpretations, of the cultural logic of mathematical objects. This paper contributes to Radford's notion of iconicity by showing, through fine-grained analysis, relevant aspects of its dynamics as well as by calling attention to a form of iconicity that, to my knowledge, has not been reported elsewhere.

INTRODUCTION AND THEORETICAL FRAMEWORK

This paper is based upon a small part of an impromptu tutorial session involving a pre-apprentice in the pipe-trades with the researcher serving as mathematics tutor. It is part of a larger case study that focuses on the manner in which the pre-apprentice attempts to make sense of, and become fluent with, the mathematics embedded in a measuring tape marked in feet and inches–an essential skill for the pre-apprentice's chosen vocation. While Canada has officially adopted the metric system and most students study measurement exclusively using metric units in their mathematics courses in elementary and secondary school, the use of imperial units of linear measure (e.g. feet and inches) remains common practice in the construction trades. Consequently, is it not unusual to find students at the start of workplace training in the construction trades who struggle with the cultural practice of measuring lengths in fractions of an inch using a measuring tape.

The study draws upon Radford's (2002, 2008a, 2008b) socio-cultural theory of knowledge objectification (TO) to examine the manner in which the pre-apprentice begins to notice the mathematics embedded within the inscriptions on a measuring tape. In this theory, learning is conceptualized as the active and creative acquisition of historically constituted forms of thinking. Such an acquisition is thematized as a problem of *objectification*, that is, as a problem of becoming conscious of, and

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critically conversant with, the cultural-historical logic with which mathematical and other objects have been endowed. One of the aspects that makes the idea of objectification distinctive is the close relationship that it bears with the Vygotskian concept of consciousness and the mediated nature of it (Vygotsky, 1979, also Leont'ev, 1978). Consciousness is formed through encounters with other voices and the historical intelligence embodied in artifacts and signs with which we mediate our own actions and reflections. Within this context the efforts that the pipe-trade pre-apprentice undertakes to make sense of the mathematics of a measuring tape are seen as a process of objectification. One of the questions is to investigate how the cultural meaning of the mathematics behind the measuring tape becomes "recognized" by the pre-apprentice. The question is not only the manner in which personal and cultural meanings become tuned, for personal meanings can only arise and evolve against the backdrop of forms of activity. Here the TO departs from other approaches. The problem is precisely the very social formation and evolution of personal meanings as they evolve within goal directed activity and are framed by the cultural meanings conveyed by socio-cultural contexts.

Several contemporary approaches emphasize, for various theoretical reasons, the embodied dimension of thinking (see, e.g. Arzarello, 2006; Lakoff & Núñez, 2000; Nemirovsky & Ferrara, 2008) and the role of artifacts (Bartolini Bussi & Mariotti, 2008). In the TO, the sensuous and artifact mediated nature of thinking leads, methodologically, to paying attention to the semiotic means through which objectification is accomplished. These means are called *semiotic means of objectification*. Much more than being simple aids to thinking, semiotic means of objectification are constitutive and consubstantial parts of thinking and include kinesthetic actions, gestures, artifacts (e.g. rulers, tools), and/or signs, e.g. mathematical symbols, inscriptions, written and spoken language (see Radford, 2008c); they allow one to draw one's own attention and/or the attention of another to particular aspects of cultural objects (Radford, 2003; Radford, L., Miranda, I. & Guzmán, 2008).

In his recent work, Radford has identified two main (and interrelated) processes of objectification, namely *iconicity* and *contraction* (2008a). While contraction refers to the process of making semiotic actions compact, simplified and routine as a result of acquaintance with conceptual traits of the objects under objectification and their stabilization in consciousness, iconicity is a link between past and present action: it refers to the process of noticing and re-enacting significant parts of previous semiotic activity for the purpose of orienting one's actions and deepening one's own objectification (Radford, personal communication, September 29, 2008). One of the goals of this paper is to contribute to this idea of iconicity by showing, through fine-grained analysis, some relevant aspects of its dynamics as well as to call attention to a new form of iconicity that, to my knowledge, has not been reported elsewhere.

METHODOLOGY

Data collection

The data for this study was collected in a pipe-trades pre-apprenticeship training class being conducted at a trade-union run school in British Columbia, Canada. This program involved pencil and paper work in the classroom as well as practical work in the workshop. It was designed to give the pre-apprentices a head start with important skills that would be addressed subsequently in the early years of their formal apprenticeship training in a number of different pipe-trades.

Throughout this pre-apprenticeship course the researcher served as a math tutor for any pre-apprentices who sought out his help. At other times, the researcher observed pre-apprentices and engaged them in discussion about their mathematics related coursework as they were working on it. The activity of individual and groups of pre-apprentices, working either with the researcher or working on their own, was documented using a video camera. Copies of the course print materials and copies of pre-apprentices' written work were also retained for analysis. The data for this paper was selected from this collection of data.

The individual who is the focus of this analysis, was a secondary school graduate. He had been in the workforce and completed a small number of courses in an electronics-technician training program at a community college during the three and a half years between the time that he finished secondary school and the time he began the pre-apprenticeship program in the pipe-trades. Throughout the pre-apprenticeship course he actively sought out the researcher for help with his mathematics related work.

Data analysis

A *multi-semiotic analysis* was conducted of the pre-apprentice's and the researcher-as-tutor's joint activity during their one-on-one tutoring session to investigate process of knowledge objectification. This involved the construction of a transcript of the dialogue from the video-recording of the session, along with a detailed account of significant actions, semiotic systems, and artifacts used. This process required, at times, a slow-motion and frame-by-frame analysis of video tape to assess the role and coordination of spoken language with the use of artifacts and gestures during the encounter.

The analysis to be discussed here focuses on an excerpt from the beginning of the tutoring session with the pre-apprentice, who will henceforth be referred to as "C". The researcher will henceforth be referred to as "L". This session took place at a table in the classroom immediately after L discovered that C was having difficulty reading fractions of-an-inch from his measuring tape while working on a pipe-fitting project with his colleagues in the workshop. The focus here is on C's objectification of the difference in the fraction marking patterns on the measuring tape below and above 12 inches, or one foot, where they are marked to thirty-seconds of an inch and sixteenths of an inch respectively. These two marking patterns can be seen in figure one. This is one of a number of mathematical patterns inscribed on the measuring tape that C comes to notice and coordinate as he becomes proficient with reading the measuring tape over the course of the entire thirty-two minute tutorial.

Figure 1. *The marking lines to the left of one foot indicate fractions to thirty-seconds of an inch. On the right side of one foot the markings indicate fractions to sixteenths of an inch. (C has inscribed a line across the measuring tape with his pencil at 11 1/8", partly obscuring the measuring tape inscriptions, and another short line over the marking at 11 5/32".)*

RESULTS AND DISCUSSION

The shared goal of C and L's work together in the tutoring session is for C to learn how to read fractions on the measuring tape to sixteenths of an inch or, using the language of the TO, to objectify the system of fractions-of-an-inch crystallized within this cultural artifact (the measuring tape). C needs to learn this to be able to complete a pipe-fitting project that he is working on, as well as for his ongoing training, and for his future work as a trades person. L's immediate goal in this particular episode is for C to begin noticing differences and similarities in the marking patterns on the measuring tape.

Semiotic means of objectification using gestures and signs

The measuring tape from C's tool box is extended on the table top in front of both C and L and the session begins with L asking C what difference he notices between the pattern of spaces on his measuring tape below 12 inches and above 12 inches.

75. L: … What do you notice here between the spaces here, up to twelve [Gesture-uses the index finger of his left hand to sweep up from the zero end of the measuring tape and pauses at 12" just before saying "up to twelve"]
76. C: Yeah its,
77. L: and the spaces after twelve? [G-now pointing with the fourth finger of his left hand to sweep through the exposed interval of the tape measure above 12"]

Here L asks C to explain what he notices while using two distinct sweeping gestures separated by a static pointing gesture at the twelve inch point. This in an attempt to draw C's attention to, and initiate his objectification of, these two intervals as distinct regions of the measuring tape. L emphasizes this distinction by using different pointing fingers to sweep through each of the intervals and a contrasting static pointing gesture at the end of his sweep up to 12 inches to highlight the boundary point between them. As every educator knows, posing a question like this one is an effective means of drawing a student's attention to, and having him or her engage in a more critical way with, an object at hand. In this short excerpt L's question is framed through the coordinated use of spoken language to describe the two regions of the measuring tape, and the use of a static pointing gesture and two different forms of sweeping gestures. Together, spoken language and gesture serve as semiotic means of objectification for C.

Gestures dominate C's response to L's question. This is clear by considering his spoken words alone, which provide only a vague and partial response. It is only through C's use of spoken language, interspersed with an elaborate and coordinated sequence of ten gestures, each positioned in a precise way relative to the measuring tape that it becomes clear that he is, indeed, becoming consciously aware of the way in which the marking patterns on the measuring tape are different from one another.

(Transcript note: The spoken words in the transcript below are printed in bold to assist the reader to differentiate these from the descriptions of the accompanying actions.)

78 C: **There's,** [G(Video frame 1, 26:52)–sweeps up through the first few inches of the tape measure with the fourth finger of his left hand in a manner similar to the gesture just enacted by L]

there's more. [G(Video frame 2, 26:53)–makes two chopping motions aligned with the markings on the tape measure with his left hand, the first significantly larger than the second just before he says "there's more" in reference to the markings inscribed on the measuring tape.

G(Video frame 3, 26:54)–points to the 12" mark with the fourth finger of this left hand before withdrawing it from the measuring tape].

		
Video frame 1 (26:52). C sweeps up through the first few inches of the measuring tape.	**Video frame 2** (26:53). C makes two chopping motions aligned with the markings on the measuring tape.	**Video frame 3** (26:54). C points to the 12" mark.

In line 78, C begins his description of the difference between the two marking patterns on the measuring tape. He starts by sweeping the fourth finger of his left hand upwards through the first few inches of the measuring tape (Video frame 1). This is the same type of one finger indexical sweeping gesture that L had just used (albeit using a different finger) to draw attention to this region of the measuring tape. C embellishes L's original gesture sequence by including a chopping gesture midway up this interval. This chopping gesture is aligned with the series of parallel markings inscribed on the measuring tape and reflects the familiar action of physically dividing or chopping up the interval on the measuring tape in the same way as is indicated by the inscribed measuring tape markings (Video frame 2, 26:53). Immediately following this gesture C says "there's more" (line 78), a confirmation that he is, indeed, referring to the closely packed markings inscribed on this region of the tape measure. C resumes and finishes his sweep through this region of the tape measure by pointing with the same finger of his left hand to the 12 inch point, the endpoint of this interval (Video frame 3, 26:54), before taking this hand away from the measuring tape. This use of a static single-finger pointing gesture at the 12 inch point separating the two regions of the measuring tape is the same type of gesture that L used a few seconds earlier to separate his sweeping gestures at the 12 inch point as well.

(line 78 continues) **It's like it's more spread out** (in reference to the markings on the tape measure after the 12 inch point.) [G(Video frame 4, 26:55a)–points briefly to the 12" mark on the tape measure now with the first finger of his right hand, replacing the previous pointing gesture expressed by the fourth finger of his left hand.

G(Video frame 5, 26:55b and Video frame 6, 26:56a)–starting with his thumb positioned at the 12 inch point, sweeps his right hand up the measuring tape a short distance while holding an approximately 2.5" wide interval between the thumb and first finger.]

		
Video frame 4 (26:55a). C points again to the 12" mark on the measuring tape.	**Video frame 5** (26:55b). C begins to sweep an approximately 2.5" wide interval up the measuring tape starting with his right thumb at 12".	**Video frame 6** (26:56a). C continues his wide-interval sweep up the measuring tape.

(line 78 continues) **when** [G(Video frame 7, 25:56b)–grasps the tape measure with his right thumb and first finger on opposite edges at the 12" point and G(Video frame 8, 26:57)–sweeps his hand in this configuration upwards a short distance from 12"] **you pass one**,

79 L: **Yeah,**

80 C: **one foot** [G(not shown)–while maintaining the same grasping position, repeats this sweep upwards for a second time]

Video frame 7 (26:56b) C grasps the measuring tape at the 12” point.

Video frame 8 (26:57) C sweeps his hand in this configuration upwards a short distance from 12” and then repeats this motion a second time.

When line 78 continues, C replaces, briefly, his left hand pointing gesture at the 12 inch point with the first finger of his right hand (Video frame 4). This reflects, in part, L’s earlier set of indexical gestures, i.e. using different pointing fingers to distinguish between the two different regions of the measuring tape. C then forms a wide-interval gesture using his right thumb and first finger and without hesitation sweeps this up the measuring tape with his right thumb starting from the 12 inch point (Video frame 5 to Video frame 6). As he does this he says “it’s more spread out” (line 78). This reflects the wider interval spacings between adjacent fraction markings inscribed here. C then grasps the measuring tape at 12 inches with his right thumb and first finger in a position that looks like he is grasping or pinching it (Video frame 7), and then sweeps his hand up the measuring tape from 12 inches and Video frame 8) and then repeats this a second time (not shown). This series of three sweeps up the measuring tape from the 12 inch point (one wide-interval sweep and two grasping sweeps) serves to sustain both his own and L’s attention on this region of the measuring tape.

(line 80 continues) **and when you’re before one foot its more um,** [G(Video frame 9, 27:01)–makes a very brief and narrow-interval gesture with the thumb and first finger of his right hand with this hand now positioned above the region of the tape measure between 0” and 12”.]

Video frame 9 (27:01) C makes a very brief narrow-interval gesture with the thumb and first finger of his right hand with this hand now positioned above the region on the tape measure between 0” and 12”.

81 L: **Okay.**

82 C: [silence]

C’s explanation comes to an end as he says “below one foot its more um” (line 80) while making a very brief but distinct narrow-interval gesture with the thumb and first finger of his right hand (Video frame 9). This gesture is positioned above the region of the measuring tape between 0 and 12 inches and reflects the narrower intervals between adjacent markings on this region of the measuring tape in comparison to the intervals above 12 inches that C had described using a wide-interval gesture seconds earlier.

By responding to L's question in lines 78 and 80, C enacts a coordinated series of semiotic actions that serve to draw his own awareness to the marking patterns on the tape measure and thus mediate his thinking and deepen his consciousness of these patterns. This was, after all, the outcome L was aiming for by posing his initial question in lines 75 and 77. C's use of gestures and spoken language in this excerpt are examples of semiotic means of objectification for oneself.

Forms of iconicity and mathematics as reflexive praxis

Radford describes iconicity as the process of noticing and re-enacting significant parts of previous semiotic activity for the purpose of orienting one's actions and deepening one's own objectification. We can find three forms of iconicity within this brief and intense exchange between L and C.

The first form of iconicity involves C noticing and re-enacting all of the hand gestures and corresponding hand positions that L had used while posing the question to him at the start of their exchange. These included his use of different fingers for pointing at the different regions of the measuring tape in line 79–Video frames 1 and 4, the sweeping gesture for identifying the region of the measuring tape below 12 inches in line 78–Video frame 1, and the static one-finger pointing gesture directed at the 12 inch point in line 78–Video frame 3.

The second form of iconicity involves C noticing the different inscription patterns on his measuring tape below and above 12 inches and re-enacting these using different forms of semiotic actions, in this case using hand gestures. The examples here include C's chopping gesture to describe the closely packed pattern of marking lines below 12 inches in line 78–Video frame 2, his wide-interval gesture to describe the relatively wide intervals between markings above 12 inches also in line 78–Video frame 6, and his narrow-interval gesture to describe the relatively narrow intervals between the markings below 12 inches in line 80–Video frame 9.

The third form of iconicity to be found coincides with the second form of iconicity just described in this set of data. It involves C noticing a form of gesture that he has enacted himself and then re-enacting this within a different context. I refer here to C's use of a narrow-interval gesture using this thumb and first finger to describe the marking pattern below 12 inches on the measuring tape in line 80–Video clip 9. This occurs after he has enacted a similar wide-interval gesture using his thumb and first finger in reference to the marking pattern above 12 inches in line 78–Video frame 6.

We can infer that C became consciously aware of the possibility and/or usefulness of utilizing this form of interval gesture as a result of using it to describe the intervals above 12 inches because he then backtracked to elaborate on his previous description of the region of the measuring tape below 12 inches using this same form of gesture. The finding of this third form of iconicity–noticing and re-enacting parts of one's own semiotic activity in a new context–is a new contribution to the theory of knowledge objectification.

CONCLUDING REMARKS

The brief excerpt that is the focus of this paper is taken from the beginning of a tutoring session involving a pre-apprentice in the pipe-trades learning to read the mathematical meaning embedded within a measuring tape marked in inches with the researcher serving in the role of tutor. This analysis illustrates the sensuous and artifact mediated nature of mathematical thinking and knowledge objectification. Particular features of the theory of knowledge objectification were evident including: examples of semiotic means of objectification–for another as well as for oneself–and three forms of iconicity: re-enactment using matching semiotic actions, re-enactment using different forms of semiotic action, and a newly reported form of iconicity, re-enactment of one's own previous form of semiotic actions in a different context.

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MATHEMATICAL REFLECTION IN PRIMARY SCHOOL EDUCATION

Theoretical Foundation and Empirical Analysis of a Case Study

Cordula Schülke & Heinz Steinbring,
Universität Duisburg–Essen, Campus Essen

Abstract. *The paper presents the theoretical construct "mathematical reflection" and elaborates its specificity with regard to the epistemological conditions of mathematical knowledge. This construct of "mathematical reflection" is the key concept in a wider research project. A conceptual grid with fundamental categories is developed that serves to carefully characterize the important components of "mathematical reflection" and that is used as an instrument for qualitatively analyzing students' mathematical collaboration in clinical interviews and for identifying different types of "mathematical reflection" in interaction.*

Key words: reflection, mathematical interaction, qualitative analysis, epistemology

1. INTRODUCTION: THE CENTRAL CONCEPT OF THE RESEARCH PROJECT – *MATHEMATICAL* REFLECTION

In several primary schools in Germany – also in North Rhine-Westphalia – teaching in grades 1 & 2 is organised comprehensively in the frame of experimental trials. It is assumed that "learning in grade-comprehensive groups [...][offers] a lot of opportunities of using the different learning potentials for the mutual stimulation and support for the students as a whole" (North Rhine-Westphalia State Ministry for School, Youth and Children 2004)

The research project presented here refers to age-mixed mathematics learning and is oriented on the paradigm of interpretative instruction research. On the basis of the interaction-theoretic perspective (developed by Bauersfeld 1994) and the specific research approach of social epistemology of mathematical knowledge (developed by Steinbring 2005), this project deals in particular with the socio-interactive learning of mathematics in grade-heterogeneous learning groups in the flexible entrance phase of elementary schooling. The analyses of mathematical interactions, elaborated in this project, refer in a complementary way to individual-psychological and social processes and at the same time to the particularity of mathematical knowledge as the object of the interaction.

The fundamental concept of the analyses attempts to theoretically capture the *reflective mathematical thinking* of the children. We proceed on the assumption that, by means of the collaboration of younger and older children on mathematical problems, particularly the older children receive manifold opportunities of reflecting *mathematically*. With his concept of *observed mathematics*, Freudenthal characterized the (reflective) moment of thinking, where mathematics carried out and used on a lower level becomes observed mathematics on a higher level (cf. Freudenthal 1978, 64). In addition, Nührenbörger and Pust (2006) pointed out that, in the interaction with the younger children, the older children, already used to school, are challenged to "verbalize their own thoughts and insights. In this process, existing knowledge is reflected and newly organized before it is handed on to others, and becomes further differentiated during the explanation process. For the children who are already used to school, a possible retrospection onto a previous learning process opens up opportunities for reflection on the meta-level" (Nührenbörger/Pust 2006, 24).

But how can *reflective thinking* in *mathematical* interaction processes be identified and what can be understood by *reflective mathematical thinking* as a conceptual element of an epistemologically oriented interaction-theoretical point of view onto learning mathematics and the nature of mathematical knowledge?

An initial foundation of the concept of "reflection" took place on the basis of already existing descriptions of "reflection" within the existing research literature, particularly in (actual) mathematics education literature. The examination of the status of research clearly showed the necessity of a precision of the theoretical construct "*mathematical* reflection".

The elaboration of a broadened conceptual understanding of *mathematical* reflection is based on the (particular) epistemological nature and the conditions of the development of mathematical knowledge (cf. Steinbring 2005) as well as on the concept of reflection as a "change of standpoint", which Freudenthal has developed in his article "How does reflective thinking develop?": "The unfolding reflection shows different traits. One of them, I would like to call standpoint change – a mental standpoint change, where the standpoint itself can be local or mental, while the change can take place in space, time, or another, for instance mental, dimension" (Freudenthal 1983, 492).

Thus, by *mathematical* reflection, we understand a *cognitive activity, a process of thinking*, in the sense of a *change of standpoint* or *perspective*, on the basis of which *processes of re-interpretation* take place. Old, common mathematical knowledge and familiar ways of proceeding are thought through again intentionally, they are scrutinized and *newly* or *re-interpreted*. The construct "reflective mathematical thinking" corresponds with the epistemological character of mathematical knowledge as pattern-like, relational structures. With the assumption that stimulating reflective thinking aims at the development of mathematical knowledge, *mathematical* reflective thinking is not merely a repeated consideration, a remembrance, or a reference to familiar contents.

This specific characterization of *mathematical* reflection requires to take into consideration the following essential issues when trying to analyze whether one can observe within a mathematical interaction this kind of *mathematical* reflection. First, when analyzing a *change of standpoint* or *perspective* (in the sense of Freudenthal) within an observed mathematical interaction, we use the epistemological analysis and apply the epistemological triangle (see Steinbring 2005) to figure out whether one can speak in a *proper epistemological* sense of a change of standpoint that introduces new mathematical relations or that generalizes mathematical relations. The second analysis instrument is the “analysis grid“ that tries to characterize the specific *type* of change of standpoint; this basic instrument is developed in the following section.

2. A GRID FOR THE ANALYSIS OF *MATHEMATICAL* REFLECTIONS WITHIN INTERACTION PROCESSES

The analysis grid (see Fig. 1) is divided into four fields, labelled “trigger”, “response”, “reaction” and “reflective level” together with sub-categories. The two fields “trigger” and “reaction” are *descriptive* elements in the analysis grid, and the fields “reaction” and the central category of the “reflective level” are characterized as *interpretative* elements.

In an interaction sequence, the question to which extent a new or re-interpretation of a mathematical content on the basis of a standpoint change becomes apparent, can only be examined in an exclusively interpretative way. In the frame of a sequential analysis of the scope of possible interpretation hypotheses, the convincing possibilities of interpretation, which can be justified by the direct reference to the transcript, are elaborated.

<table>
<tr><td>descriptive element</td><td>interpretative element</td><td>descriptive element</td><td>interpretative element</td></tr>
<tr><th>Trigger</th><th>Response</th><th>Reaction</th><th>Reflective Level</th></tr>
<tr><td rowspan="2">Question Exercise Remark
(Interviewer /Teacher)</td><td rowspan="3">No Irritation</td><td>No Remark</td><td rowspan="2">Standpoint Change “foreign Perspective”</td></tr>
<tr><td rowspan="2">Imitation</td></tr>
<tr><td rowspan="3">Question Discovery Way of proceeding Remark
(Partner child)</td><td rowspan="2">Standpoint Change “Context”</td></tr>
<tr><td rowspan="3">Moment of Irritation</td><td rowspan="2">Recapitulation</td></tr>
<tr><td rowspan="2">Standpoint Change “Retrospection”</td></tr>
<tr><td>Own way of proceeding/remark</td><td>Construction</td></tr>
</table>

Fig. 1: Analysis grid

The allocation to the descriptive elements of the analysis grid is exclusively oriented on the linguistic format of a remark and has a purely *descriptive* character.

The grid serves for the purpose of being able to focus on the central research questions and it allows on the basis of an epistemological analysis to examine the interactive processes taking place during a partner interview in a purposive and careful way. Even if, at first sight, the analysis grid might seem to present a chronological sequence of the fields, it is expressly not the aim of the grid to simply be used for the description of a temporal sequence.

During the real interaction proceedings, different sub-categories can overlap. For instance, a mathematical remark, which on the basis of its linguistic format is allocated to the sub-category of recapitulation, can at the same time contain a hint towards a moment of irritation. The following more detailed explanations of the categories and sub-categories will further clarify the analysis grid.

The different elements of the analysis grid

• *The element “trigger”:* On a descriptive level, several possible triggers for reflection or thinking activities can be identified in the interviews. Examples: a question, a discovery or a way of proceeding can represent a triggering moment.

For the research it is important *which person* stimulates reflections. Is this rather true for the remarks by the interviewer, for one's own discoveries and ways of proceeding, or the remarks of a cooperating partner child? This relevant aspect is allowed for by the distinction of the three sub-categories.

• *The element “response”:* A first central research question concerns the identification of possible clues in the analysis of interactive processes, which suggest reflective thinking. When does a question or a mathematical problem not only initiate recapitulation or imitation, but a *reflective* process?

The research results up to now show that irritation or a moment of surprise is an important indicator in this context. If, for example, an exercise cannot be done spontaneously, if one does not agree with the previous proceeding of the answer or with the ways of proceeding, ideas or remarks by another participant, and if one shows irritation or surprise, that means that it is not possible to simply resort to common knowledge or familiar ways of proceeding. An irritating exercise can challenge to engage in a foreign perspective.

• *The element “reaction” (descriptive element):* Children can react differently to the different triggers. In this regard, we distinguish between the sub-categories “no remark”, “imitation”, “recapitulation” and “construction”.

Besides “not remarking”, a possible reaction is “imitation”, which means the literal repetition of one's own or someone else's remarks or the direct imitation of familiar ways of proceeding or the partner child's strategies.

By “recapitulation”, we understand resorting to knowledge or ways of proceeding already familiar from the previous context, or the reference to remarks and strategies of a partner child in one's own words.

If the children also refer to mathematical knowledge, which had not been introduced by any of the interaction participants in the previous contexts, the category of “construction” is fulfilled.

The allocation of the children's reaction to one of the given categories takes place depending on the format of the remark and is oriented on the linguistic elements used, on a purely descriptive level.

If the children only refer to common knowledge or familiar ways of proceeding in phases of cooperation, the interaction remains on the level of reaction. But if new or re-interpretations of old knowledge or new constructions take place, the level of "mathematical reflection" is addressed as well.

• "*Reflective level" (interpretative element)*: The question whether new or re-interpretations are carried out within interactions or if new mathematical knowledge is constructed, can only be examined interpretatively. In order to do so, the epistemological triangle (Steinbring 2005) is used in the analysis.

The identification of the standpoint changes, which might follow, takes place with the help of the developed characteristics and features of differentiation.

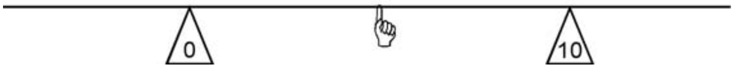
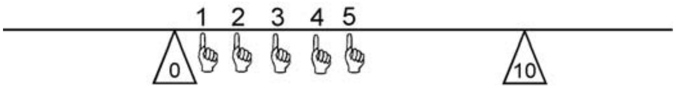
Three levels of changes of standpoint or perspective: The point of view developed by Freudenthal about reflective thinking as a standpoint or perspective change made it possible to characterize and distinguish three different forms of possible standpoint changes from the data material. Besides the theoretical clarification of the concept *mathematical* reflection, these represent an essential result of this research.

An important feature of the three levels of standpoint changes consists in the new or re-interpretation of a mathematical exercise, a mathematical content or a mathematical sign / symbol. A distinction is made with regard to the different possibilities or ways of changing one's own standpoint.

- Standpoint change "foreign perspective": The children take a foreign perspective, someone else's standpoint, for instance they relate the ways of proceedings, discoveries and views of their partner child to their own points of view and ways of proceeding, test and evaluate these and are stimulated to newly or re-interpret their own mathematical knowledge.
- Standpoint change "context": A mathematical challenge is put into and observed within another context and thus is subject to a new or re-interpretation. In contrast to the standpoint change "foreign perspective", no concrete possibility of interpretation is given, which then might be followed, but rather the change of context *allows for* a new point of view. If, by means of such a context change, one of the participants develops a new interpretation perspective, a mathematical reflection according to the standpoint change "context" has taken place.
- Standpoint change "retrospection": If there is an *intentional* resort to common knowledge and familiar ways of proceeding from a previous context in order to thus new or re-interpret a mathematical content, a standpoint change "retrospection" has taken place. Such a standpoint change can only be spoken of if a remark by a participant presents a way of proceeding or a mathematical context as familiar and relates this with the current exercise.

3. Analysis of an Exemplary episode: Gina & Sharon discuss a "Number line"–Problem

1	Int	(*places the number cards 0 and 10 at the number line*) I am placing the number cards at the number line, …
2	S	[incomprehensible]
3	Int	Put this card, (*places the number card "5" between Gina and Sharon onto the table*)

4	G	*(Gina takes the number card “5” with her left hand)*
5	Int	at the number line.
6	G	# (*leans forward / holds the number card “5” with both hands / looks at the number line*)
7	S	# In fact the zero belongs in front # (*places her left hand onto the left end of the number line*)
8	G	# (*holds the number card in her right hand / looks at Sharon”s left hand*)
9	S	or shall we now, well shall that now be like that the number line begins with this? (*puts the edge of her left hand on the left of the number card zero on the table*)
10	Int	Think about it together, how you can do that now.
11	S	You now certainly have (*looks at I.*) well. (…) (*turns to Gina*) She probably has chosen such a place (*points over the section of the number line which is marked by the number cards 0 and 10 / Gina looks at the number line*) where one could add that, so that we well that this, that this is supposed to be the beginning (*places the edge of her right hand to the left of the number card “0” on the table*) that this piece is then practically gone, (*moves her right hand in the direction of the left edge of the table over the number line*) in your mind, right? (*looks at Gina / Gina continues to look at the number line*) Well such a place, then the five would go here, right? (*puts a finger between the numbers 0 and 10 onto the number line, see below. / looks at Gina / Gina continues to look at the number line*) (…)
		
		because one two three four five. (*while counting the numbers, she points at the spots marked in the diagram, see below*)
		

This short episode originates from an interview about the topic “number line”, which was conducted with Sharon and her classmate Gina in the second project year. For Sharon, this was the fifth interview during the research project, for Gina it was the first.

Before the children were introduced to the number line, which they had never used as means of visualisation. This scene of positioning of “5” takes 5 minutes.

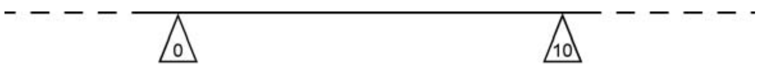


Fig. 2: Section of the number line

On the children's desk, a string was attached as a number line. The interviewer had positioned the “0” and “10” (cf. Fig. 2) when asking the exercise question.

Analysis of the interview sequence

The exercise is opened by the interviewer. She positions the “0” and “10” thus providing the initial situation. This *task of the interviewer* is emphasised by the remark (“I am placing the number cards at the number line” (1)).

Sharon directly reacts to this action or remark (2). Maybe she already shows a first reaction to the positioning of the number cards. As Sharon's remark is incomprehensible, therefore this guess cannot clarified definitively.

Gina immediately takes up the number card “5” and at the same time watches the number line (4, 6). While doing this, she shows that she is engaging with the exercise question and is considering where to put the number card “5”.

Sharon exclusively refers to the current position of the number card “0” and wonders about the position of the “0” and “10” at the number line in her following remarks (7, 9, 11).

Sharon's remarks are of essential importance for the central research question and the identification of reflective mathematical thinking and thus represent the main focus and the starting point of the following interpretations. The *clarification of the position of the number card “0”* as an element of the number line (by Sharon) is at the centre of analysis.

In her first remark after the exercise question, Sharon points at the left end of the number line and explains that the “0” should be placed directly at the beginning of the number line (7 “In fact the zero belongs in front”). The positioning of the “0” by the interviewer does not correspond with her idea of the “correct position”. Her remark suggests that, according to her previous point of view, the position of the “0” on the number line is fixed and cannot be chosen freely.

The possible previous consideration of changing the position of the number card in the frame of the work on the exercise can be seen in particular in remark (9) “or shall we now, well shall that now be like that the number line begins with this?”. This is supported by the use of the words “in fact”, which underlines the discrepancy between the current and Sharon's “correct positioning” of the “0”.

The interviewer gives the question raised by Sharon back to the two students (10 “Think about it together, how you can do that now.”).

Sharon's remark (11) suggests that she now assumes an intentional positioning of the “0” by the interviewer and is challenged to find an explanation for the “unusual position” of the number card at the number line (“She probably has chosen such a place where one could add that, so that we well that this, that this is supposed to be the beginning”).

Applying the analysis grid “mathematical reflection” to the episode

The element trigger: The exercise question given by the interviewer as well as the given positioning of the number cards 0 and 10 at the number line (1, 3, 5) represents the trigger for the following cognitive activities by the two students.

The element response: Sharon makes a remark about the current position of the number card “0” at the number line directly after the explanation of the exercise question by the interviewer. The position of the number card does not correspond with her idea and she is probably surprised or irritated by the interviewer's way of proceeding. A clue for a possible moment of irritation becomes apparent in her remark (7): “In fact the zero belongs in front”. Sharon points out an alternative possibility of positioning the number card. Her remark “In fact” can be seen as an indicator for her not agreeing with the current position of the number card.

The element reaction: In her reaction to the triggering moment, which is the exercise question and the localization of the section of the number line to be observed, Sharon refers to the positioning of the number card “0” and discusses this (not verbally expressed) action of the interviewer with her own words. Thus Sharon's reaction can be allocated to the sub-category recapitulation.

The levels of mathematical reflection

The question to which extent Sharon performs a change of view and carries out a new or re-orientation of her mathematical knowledge regarding the positioning of the “0” at the number line is examined with the epistemological triangle (Steinbring 2005) as an analysis instrument of relations between signs, reference contexts and concept.

If a change of standpoint or perspective can be identified, this will be allocated to one of the three levels of mathematical reflection on the basis of the characteristics described in the presentation of the analysis grid.

The analysis instrument “epistemological triangle”

Conventional interpretation: The *sign* to be clarified in the present interview sequence is the position of the number card “0” at the number line. In this first representation the original, conventional interpretation by Sharon regarding the position of the number card is made clear by referring to a familiar *reference context*.

In her remark (7) “In fact the zero belongs in front”, Sharon probably refers to the known “familiar” position of the number card “0” at the beginning of the number line. Maybe she remembers the positioning carried out previously to the interview and points at the left end of the number line as the only possible position for the number card up until this point. Two different aspects become manifest in her remarks. On the one hand, there seems to be a fixed position for the number card at the number line for Sharon, on the other hand the number card “0” belongs to the beginning of the number line, i. e. left of this number, neither does the number line continue nor can there be further number cards.

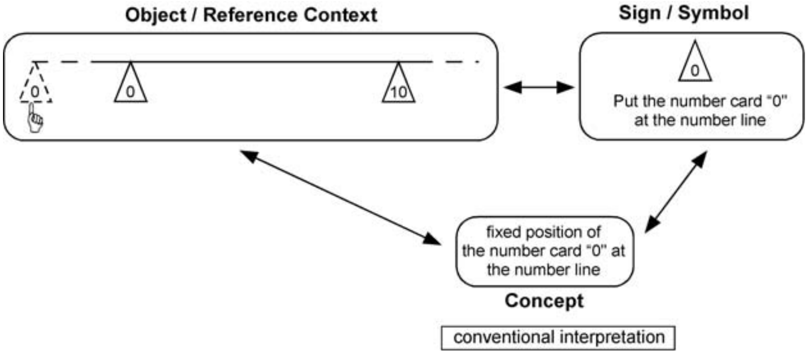


Fig. 3: Epistemological triangle: The original interpretation of the position of the number card “0”

Beginning of a relational interpretation: Besides the originally conventional view concerning the position of the number card “0”, a beginning mentally more flexible interpretation becomes apparent in this scene. Sharon tries to conciliate her previous point of view with the current position of the number card. In doing so, she refers to the reference context presented in Fig. 4. She explains the – for her point of view – still unfamiliar position of the number card “0” by placing her hand to the left of the number card and remarking in the one hand (9): "or shall we now, well shall that now be like that the number line begins with this?", on the other hand (11): ("She probably has chosen such a place where one could add that, so that we well that this, that this is supposed to be the beginning".

The mentally changed number line thus forms the reference context, i. e. the current position of the number card is interpreted by referring to the theoretical picture of the number line, which Sharon has developed and in which the sequence in front of the number line is mentally ignored.

In this interaction of sign and reference context the beginning of a detachment from a purely empirical point of view concentrated on the concrete, towards a stronger mental use and change of the number line becomes apparent. The following remarks by Sharon can serve as concrete indicators of this more flexible point of view "in your mind" (11) and "would" (11: "then the five would go here, right?"). The positioning of the "5" which she suggests takes place depending on the current position of the "0" and "10".

While at the beginning of the interview sequence Sharon still allocates a fixed position at the beginning of the number line to the number card "0", she ultimately takes a more flexible point of view about this: By means of the possibility of putting the number card "0" at a random position of the number line, sections of the number line can be realized variably.

Still, the number card "0" remains the first card for Sharon, however, thus left of this number card there can be no other number cards. Furthermore, her way of proceeding when positioning the number card "5" (11) indicates that she continues to pay attention to the sequence and distance of the number cards.

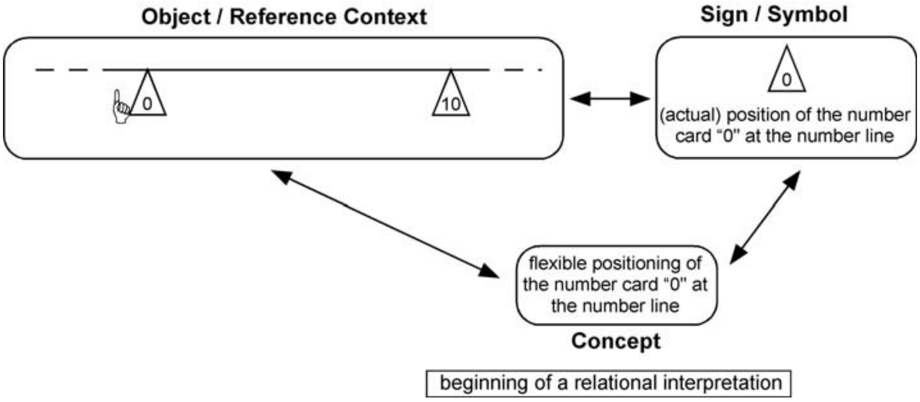


Fig. 4: Epistemological triangle: Beginning of a relational interpretation

Characterization of the standpoint change

As has already become clear in the first step of the analysis, Sharon performs a new or re-interpretation of the number line regarding the positioning of the number card "0" during the course of the interaction.

As previously to the present interview sequence, the number card was always placed at the beginning of the number line, its current position represents a *changed context* in this regard.

Concerning the position of the number card "0", Sharon develops a new interpretation perspective and thus carries out a *standpoint change "context"* on the basis of this changed context given by the interviewer.

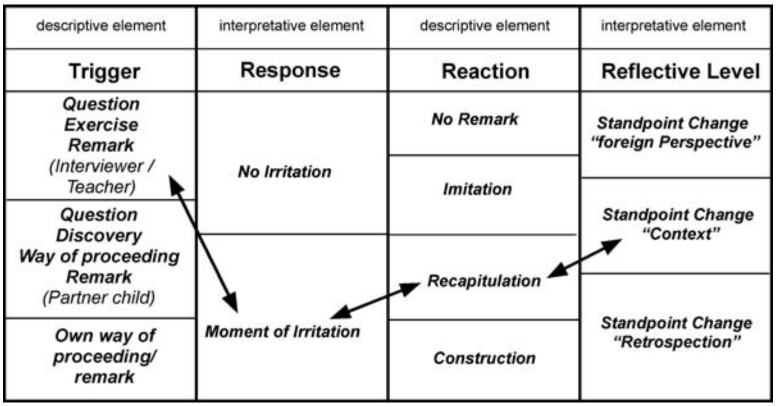


Fig. 5: Application of the analysis grid

4 SHORT RÉSUMÉ

The analysis grid developed in the course of the research project offers the possibility of presenting the results of the analyses and interpretations cohesively. The central element of the grid is the "reflective level". The distinction of the three categories of standpoint changes is a fundamental result of the research up until now and allows for the analysis to pursue the question which specific form of a standpoint change provokes and stimulates the process of new interpretation of mathematical knowledge, which is essential for the learning of mathematics.

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SURFACE SIGNS OF REASONING

Nathalie Sinclair, David Pimm

Simon Fraser University, University of Alberta

Abstract

In this paper, we explore forms of verbal expression undergraduate mathematics students employ while working in pairs on geometric tasks in a computer environment, focusing in particular on the connectives (notably 'because') they use as well as the modal expressions in their talk as they discuss ideas with their partner. We use this data to bring together C. S. Peirce's idea of abduction, the linguistic notion of hedging and Toulmin's argumentation scheme, and argue that in trying to identify abductions, the presence of hedges (of which Toulmin's 'modal qualifiers' are an instance) or a particular use of 'because' may provide some evidence.

> It is a commonplace of philosophical logic that there are, or appear to be, divergences in meaning between, on the one hand, at least some of what I shall call the formal devices—$\sim$, $\wedge$, $\vee$, $\supset$, $(\forall x)$, $(\exists x)$ (ιx) (when these are given a standard two-value interpretation)—and, on the other, what are taken to be their analogues or counterparts in natural language—such expressions such as *not, and, or, if, all, some* (or *at least one*), *the*. (Grice, 1975/1989, p. 22)

In this paper, we wish to explore some of the natural language markers (in English) that are employed in students' *spoken* mathematical reasoning. One motivation for doing so is a realisation of how different, on occasion, even experienced mathematical undergraduates *speak* when working on problems in pairs, from the conventional way formal mathematics is supposed to be *written* (e.g. Morgan, 1998). A second was the difficulty we had at times in identifying the nature of the reasoning from the speech of the participants. A third arose from our growing interest in the notion of *abduction*, which has been receiving attention in the past few years within mathematics education (e.g. Mason, 1995; Pedemonte, 2007; Reid, 2003; Rivera, 2008; Sinclair, Lee and Strickland, under review), as well as possible connections to the linguistic notion of hedging (see, e.g., Rowland, 1995) and Toulmin's argumentation scheme (see, e.g., Inglis et al., 2007).

In mathematical discourse, there are significant differences between speech and writing. We are not claiming that there are disjoint vocabularies, but there are some words that are usually only spoken (including a few that require invented spellings for transcription e.g. 'cuz', 'gonna', 'gotta') and some that are much more comonly written (hence, therefore, consequently). The formal written mathematical register is quite tightly specified in terms of particular conjunctions to be used in proofs, particularly at the beginnings of sentences to mark the relation between the preceding and subsequent comments (e.g., 'let', 'hence', 'therefore', 'if', 'since', 'conversely'). This is another level of difference beyond that to which Grice is drawing attention.

However, one linguistic challenge arises from the fact that mathematical purposes are not the only functions that these words encode. The language of 'if …, then …', for instance, so common in written mathematics, is also the language of threats. Many of the conventional connectives in other circumstances carry a space, time or sequencing connotation (e.g. then, since, when, hence) – for more on mathematics and time, see Pimm (2006). In conversation, the *then* of 'if …, then …' is often elided, and there are occasions when even the *if* marker can be absent.

In this paper we wish to go further than Paul Grice in differentiating logical operators from what he terms 'natural language', by distinguishing spoken from written natural language. Unlike Grice, however, we will offer attested speech data for consideration rather than invented data. In the opening chapter to his book *Text and Corpus Analysis*, linguist Michael Stubbs (1996) criticises the dominant tradition since Chomsky (and including Grice) for basing extensive theoretical arguments on no real language data. Nevertheless, Stubbs (see below) supports Grice's specific claim about the non-congruence between logical connectives and English words and goes further, paying close attention to the role of modality in verbal communication.

This paper draws on data collected within a larger study of mathematical reasoning in undergraduate students. The data consist of twenty videotaped episodes (ranging from ten to twenty minutes in length) in which pairs of students are working at computers, using *The Geometer's Sketchpad* (Jackiw, 1991) to solve geometric tasks. These tasks include, among many others, using *Sketchpad* to construct a parabola, to identify the particular transformation that relates two given shapes, to solve the Apollonius problem and to figure out the fractal dimension of given curves.

SPOKEN MARKERS OF REASONING

> A third case of the interaction of pragmatic and syntactic matters is provided by the so-called logical connectors (e.g. *and, but, or, if, because*). Their uses in everyday English are not reducible to their logical functions in the propositional calculus, but have to do with speakers justifying their confidence in the truth of assertions, or justifying other speech acts. (Stubbs, 1996, p. 224)

Any modal utterance contains both propositional information and the speaker's attitude towards the information. Echoing Grice, Stubbs uses modality to distinguish between different functions of connectives. He claims *because* is representative in having two distinguishable uses, which he terms *logical* and *pragmatic*: the first has the structure of 'effect plus cause', the second 'assertion plus justification'. Stubbs notes that the pragmatic use of *because* is often signalled by the addition of epistemic *must* ('he must have been drunk because he fell down the steps'). In addition, He provides a number of syntactic criteria to help distinguish the two uses. He claims these points are also true for the pragmatic use of *if, or, but,* and *and.*

An example of the logical use comes from Birkhoff and Mac Lane (1941/1956) "Because of the correspondence between matrices and linear transformation, we need supply the proof only for one case” (p. 227). Similarly, in Spivak (1967), we find: “Because this sequence varies so erratically near zero, our primitive mathematical instincts might suggest that $\lim_{n \to \infty} f_n(x)$ does not always exist” (p. 414).

There is no scope within this paper for a detailed corpus analysis of connectives in our data, though we wish to remark on the prevalence of ‘so’ and ‘which means’ as markers of deductive utterances. From our data, we find very few logical uses of because.

A: Well, **because** those two don't, for sure, lie in the circle, so if we rotate it around that point, it's not gonna be exact.

In A’s statement above, the cause is signalled by ‘**so if**.’ Far more often, the uses of because are pragmatic, as in the following two examples.

D: No, **because** the rotation point is gonna be over here.

E: Yeah, the original one **because** then O_1 will convert to a line and through … never mind. That didn’t work. We did it wrong.

In both these and other similar instances, what we find is students hypothesising or positing justifications for claims they are making. This connects in an interesting manner to the theme we turn to in the next section, namely abduction as a form of inferring, which is proving challenging to us to identify confidently. This brief look at ‘because’ suggests that one place to look for abductions is in pragmatic uses of the connective ‘because’.

TWO SHORT EPISODES OF STUDENT REASONING

Here are two episodes of student mathematical problem solving where we found the form of reasoning less clearly identifiable, less likely to be deductive, and replete with modal utterances. We provide a brief contextualisation of each episode in this section, and then offer two tentative analyses—one using Peircean abduction and the other Toulmin’s model of argumentation— of each episode in the following section.

Example 1

Two students (Lucie and Brad) are trying to solve the problem of geometrically constructing a parabola in *Sketchpad* given a focus point *P* and a directrix line *j*. The students have already constructed the envelope of the parabola by tracing the perpendicular bisector of *PB* where *B* is a point on *j* that can be dragged back and forth along the line. The students begin looking for ways to construct a point that depends on B so as they move B along *j* it will trace out the parabola.

At first, they place a point on the segment *PB* right where the segment first touches the envelope edge. When Lucie drags B, they both realise that this point does not always lie on the curve, so they delete this point. In turn 1 below, Brad notices that if the solution point is placed on *PB*, then it could never reach the upper parts of the parabola (given that *PB* is a segment). This seems to give rise to an anomaly for Brad

– that the point will have to be able to travel high up the sides of the parabola. Indeed, his expression is emphatic and strong-voiced and the modal verb 'can't' is also strong: "We can't have …". Indeed, he tries to convince Lucie of what he's noticing: "see that point". In turn 3, Brad makes a deductive inference, first using the word 'so' and then "which means" to indicate the implication that the point *cannot* be on *PB*.

1	Brad:	We can't have [..] [1] Well, like, [….] like, see that point has to be able to get up here, right? (He points to *j* with his pen and then points to the top left of the curve with his pen and then his finger.)
2	Lucie:	Uhuh.
3	Brad:	So, which means it can't touch the line.

Lucie then proposes that this point lies on a line passing through *P* perpendicular to *j*.

4	Lucie:	Yep [….] So then [….....] Let's say […………] (Constructs the line through *P* perpendicular to *j*, as in Figure 1.) Maybe that's the line […] 'cause um [..] the distance from like [..] here to here would be the same as that one? (Points to distance between the envelope of the curve on the left and her new line.) But I don't know if that's right. (Points to her new line and the curve on the right.)
5	Brad:	So what line did you just create?
6	Lucie:	The perpendicular line to the bottom through P. But I don't think it's right.

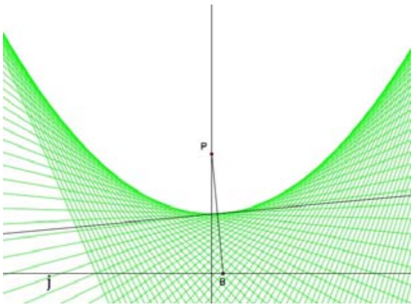

Figure 1: The envelope of a parabola with focus P and directrix *j*

Brad seems to think that Lucie's line "couldn't be" the right one, but acknowledges her statement about equidistance. At this point, the instructor intervenes and redirects the students' attention to the more pertinent equidistance relationship (to point P and line *j*). The students eventually figure out how to construct the point on the parabola as the intersection between the perpendicular bisector of *PB* and the line perpendicular to *j*, passing through B.

Example 2

Two students (Gloria and Peter) are trying to figure out which isometry maps a given shape on the computer screen onto another and then to construct the specific transformation. The students have studied the composition of reflections (and found that the composition of two reflections gives a rotation, unless the two lines of reflection are parallel). In turn 1, Gloria has already identified two corresponding segments of the shape and asks "can we continue these two lines?"

1	Gloria:	Rotation right? [..] Which is two reflections but I don't know how to do that. (Points to the right edge of top figure and top edge of bottom one – see Figure 2 below.) OK, can we continue these two lines?
2	Peter:	Probably two reflections.
3	Gloria:	Can we, yeah, or a rotation, same difference.
4	Peter:	[inaudible]. (Gloria draws a straight line extending the right-hand vertical edge of the top figure.)
5	Gloria:	Can we make this a straight line and find out what this angle is, and then rotate it that much? [……….] Um […..] That'd work, wouldn't it?

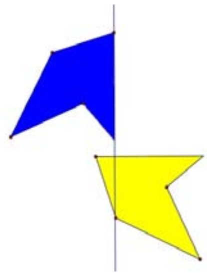

Figure 2: Line extending one side of the top shape

In turn 4, Gloria extends the line and then, in turn 5, infers that the intersection of the line and the horizontal side of the lower shape will form an angle that corresponds to the angle of rotation necessary between the two shapes.

INTERPRETING THE EPISODES

In each episode, we see mathematical reasoning that plays an important role in the problem-solving process of the pairs, but that does not fall easily into the two most commonly-discussed categories of inductive and deductive reasoning. We thus begin by interpreting the two episodes described above in terms of Peircean abduction. We then interpret the same episodes using Toulmin's (1958) structure of argumentation.

Focus on Peirce's different types of inferences

> Deduction proves that something *must* be; Induction shows that something *actually is* operative; Abduction merely suggests that something *may be*. (Peirce, 1931/1960, 5.171)

Peirce's description of the three forms of inference, as quoted above, marks a shift in interpretation away from the logical *form* of a given inference (how it might be characterised through syllogistic propositions) toward its *use*, by the inquirer, in the process of inquiry. While researchers such as Reid (2003) and Cifarelli (2000) claim to have identified student abductions based on these logical forms, Mason (2005) cautions, "The tricky part about abduction is locating at the same time the appropriate rule and the conjectured case" (p. 5). In many cases, neither of these propositions will be uttered out loud in spoken conversation – they must be inferred from context.

While logical forms are sometimes easy to identify in written language (especially in mathematics texts), they can be much harder to identify in speech, which is frequently less planned and more emergent in real time, especially in the context of pairs jointly co-constructing the talk. While some students will state that something "must be" (or 'has to be' or 'gotta be') true, others may choose to express their certainty through other means, both verbal and non-verbal. Peirce's emphasis on the *uses* of deduction, induction and abduction invites attention to the intentions of the inquirer, but these intentions, about what must be, what actually is, and what may be, can't always be clearly identified either. Thus, one challenge facing researchers is how to work with the surface elements of language in order to make interpretations about the type of inference demonstrated in particular conversational exchanges. The short list given by Grice in our opening quotation, which includes clear, propositional terms of inference, is completely insufficient when looking at real people reasoning in conversational pairs about mathematics.

Considering episode 1, we can see Brad's inference that the point cannot lie on PB as a deduction, since he states what *must* be the case. Here, the logical form is quite easy to identify, as are the linguistic features. By contrast, Lucie's proposal that the point lies on the perpendicular to *j* through P can be seen as an abduction, since it indicates what *may be* true, as exemplified by her own words "Maybe that's the line" and her later hedged statement of hesitation "But I don't know if that's right." Lucie's inference satisfies two additional characteristics of abduction: (1) it involves the generation of a new idea (the line she constructs did not exist before, and stands as a genuinely new and plausible solution); and (2) it is not logically derivable from true statements (and, indeed, the line she proposes is not the right one). Further, the use of "'cause" is a pragmatic one, in Stubbs's sense as described above.

We might also attempt to interpret Lucie's abduction in the following logical form, where the case is the only thing Lucie knows to be true, and the result has been hypothesised as a plausible situation in light of the novel rule.

case: The (solution) point has to go up

rule: If it's on that line, it would go up

result: The point is on that line

In contrast with the linguistic interpretation offered above, the logical form fails to capture the interlocutor's degree of conviction when she hedges her proposal both with 'maybe' and "I don't think that's right'. Additionally, there is a close link between this formulation of abduction and Stubbs's pragmatic category of connective use, as noted above in relation to "'cause". Curiously, Stubbs's term 'pragmatic' seems to evoke Peirce's work on pragmatism.

We turn now to episode 2, where Gloria and Peter are trying to identify the isometry relating two shapes. In turn 1, Gloria asks, after pointing to the two lines in question, "can we continue these two lines?" She has not explicitly stated that she is trying to identify the angle of rotation (or the angle between the two lines of reflection), but this becomes explicit in turn 5, where she asks (again): "Can we make this a straight line and find out what this angle is?" We see this as an abductive inference, since it follows the use of what *may be* true, as evidenced by her questioning tone of voice, her use of the hedge tag phrase “can we” and the final, doubtful, tagged utterance “That would work, wouldn’t it?”

We find further evidence of this as an abductive inference by the fact that it introduces a new idea (the technique of extending lines had not been previously used in class), which, in this case, turns out to be fruitful. Once again, we could offer an interpretation based on the ‘underlying’ logical form of the inference, but the preceding analysis seems to offer an identification consistent with Peirce’s conceptualisation of abduction in its pragmatic function.

Focus on Toulmin’s forms of argumentation

In work on forms of argumentation and informal logic, Toulmin’s (1958) scheme has had its place. But, as Inglis et al. (2007) clearly point out, it is a reduced form of Toulmin’s scheme that has been commonly used in mathematics education, one which leaves out two of the six components: the rebuttal and, of greater relevance for us here, modal qualifiers. Inglis *et al.* worked with the production of individual oral arguments of graduate students in mathematics, exploring a range of mathematical conjectures. We were struck in their paper by the fact that modal qualifiers are precisely hedges, those statements of propositional attitude concerning the degree of conviction the speaker is willing to express. This made us wonder about the connection between overt hedging and abduction, which suggest that the student was to some extent aware of the making of an abduction that consequently required a more tentative assertion.

Inglis *et al* (2007) give a visual summary to illustrate Toulmin’s model of argumentation (Figure 3). The argument would read: based on the data (D) given, the warrant (W) – which is supported by the backing (B) – justifies the connection between D and the conclusion (C), unless the rebuttal (R) refutes it. The modal qualifier (Q) qualifies the certainty of the conclusion by expressing degrees of confidence.

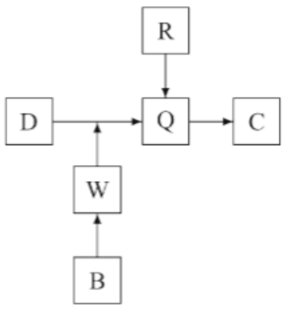

Figure 3: Toulmin’s model of argumentation

We now run the first episode above through Toulmin’s model to obtain Figure 4. The data include the point P, the directrix *j*, the point B on *j*, as well as the segment *PB.* Lucie’s conclusion, that the point lies on the line perpendicular to *j* and passing through P is qualified by her hedged utterances "Maybe" and "But I don't know if that's right". We see her statement regarding the equidistance of the line to each side of the parabola functioning as the warrant, even though it is offered *after* the argument – following some hesitation and speculate that it is the presence of her partner that makes her verbalise this at all. The backing includes the fact that the point must be on some line (instead of a line segment like PB), but one that should somehow involve both P and *j* (the givens in the situation). The rebuttal is not evident in her argument and may not exist at all.

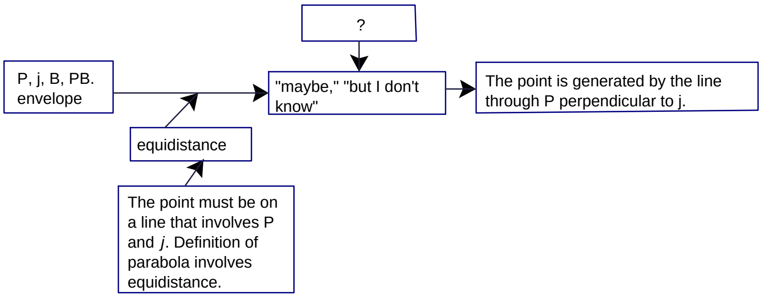

Figure 4: Lucie's argument expressed using Toulmin's scheme

Turning now to the second episode, we can also run Toulmin's scheme on Gloria's argument (in Figure 5).

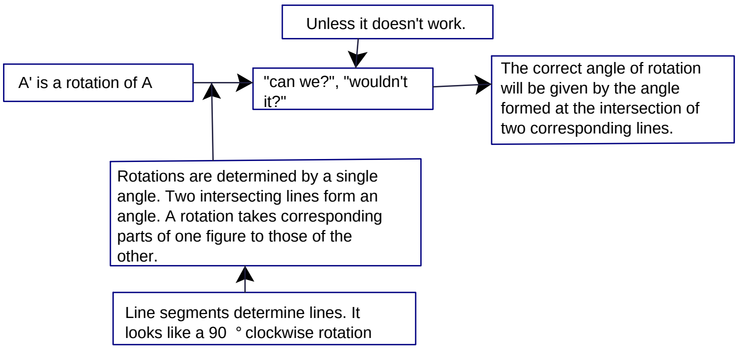

Figure 5: Gloria's argument expressed using Toulmin's scheme

This time the modal qualification is not expressed through specific words, such as 'maybe' or 'probably', but instead in the intonation of Gloria's statement, which is made in question form: "Can we […]?". In this episode, we also find no evidence of a rebuttal, though presumably Gloria had an immediate and pragmatic rebuttal in mind, which was to actually see whether the angle of rotation created by intersecting the line and side segment would work to rotate the pre-image to its image. Filling in the scheme, Gloria's conclusion is that the angle of rotation between the two shapes is the angle created by the intersection of two corresponding sides (one extended).

CONCLUSION

The above analyses show that it is possible to interpret the two excerpts of paired student reasoning in conversation using either Peirce's idea of abduction or

Toulmin's model of argumentation. Both are challenging to use as interpretational frameworks, and this is so for several reasons. First, both Peirce and Toulmin tended to work with made-up examples to illustrate their inferences or arguments; and, as we have seen, real speech is much messier – some phrases are omitted, others are communicated non-linguistically, and so on. Second, and especially for abduction, we have already noted that the most important component of the abductive inference – the stating of the general rule – must often be inferred from context. However, even in Toulmin's case, what counts as data, warrant, and backing is not always obvious, and certainly not objectively knowable. Third, neither Peirce nor Toulmin has conversational reasoning in mind when articulating their theories. In some senses, Toulmin's emphasis on argument is *post hoc*, given that the interaction between two students (in our own data) frequently involved negotiation of meanings, and subsequent attempts to explain and/or convince.

The analyses we conducted reveal interesting similarities and differences. Most remarkable of the former related to the importance attached to the degree of confidence held by the reasoner. Toulmin includes modal qualifiers in his model in order to account for the variety of certainty that one might have about a claim. Pierce's abductions are seen as hypothetical *may be*'s. Their attention to uncertainty might seem strange in the context of mathematics, where one frequently seeks precisely the opposite. Yet both Peirce and Toulmin seem to care about how the reasoner can make advances in inquiry, and take it as given that many advances will be tentative. A particular resonance such a perspective has in mathematics education can be found in the work of Rowland, who has studied the notion of hedging in the mathematics classroom. We suggest that this notion could be used productively to help identify and analyse and interpret student reasoning in terms of Toulmin or Peirce. Lastly, the pragmatic use of 'because' also appeared as a surface marker in one of the two episodes that may help identify abductions in some cases.

Toulmin is concerned with trying to identify the structure and form of an existing argument, whereas Peirce is more concerned with examining the process of scientific discovery. Peirce draws attention to the way in which problem solving may require abductive 'leaps of faith', where one is reasoning ahead of more explicit or acknowledged deductive or inductive means. This seems to us an important awareness in educators involved in supporting and eliciting mathematical problem solving. Toulmin's analysis of an argument acknowledges the qualification involved in any emergent complex argument, and serves to draw attention to argument structures and resources that may not have been apparent in the more 'logical' literature analyzing the form and nature of mathematical arguments.

By juxtaposing the results of each analysis of the same two mathematical episodes, as well as identifying hedging as one surface linguistic phenomenon common to both, we have attempted to highlight how one might ground each theoretical account in the specifics of moment-to-moment conversation, as well as thereby drawing attention to commonalities across the two accounts that have not been made before.

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A TEACHER'S USE OF GESTURE AND DISCOURSE AS COMMUNICATIVE STRATEGIES IN CONCLUDING A MATHEMATICAL TASK

Raymond Bjuland, Maria Luiza Cestari & Hans Erik Borgersen

University of Agder, Kristiansand, Norway

An experienced teacher has been observed in dialogue with her sixth-grade pupils when summing up their solutions to a mathematical task. The pupils have worked in small groups on this task, which is related to a transposition of data (age and height) from a figure to a Cartesian diagram and to a written text. The teacher's discourse has been analysed, using the dialogical approach to communication and cognition. Analyses of gestures are based on McNeill's classification expanded by Edwards, using the concept of embodied cognition and complemented by the work of Goodwin, taking into account the contribution of the environment to the organisation of the gesture. Some communicative strategies used by the teacher have been identified, for example, questioning (who, how, why, asking for other suggestions). Pointing gestures are used, but they are not prominent. Our findings suggest that gestures are more used and connected to the teacher's explanations than to other procedures.

INTRODUCTION

Gesture and discourse have, for a long time, been seen as two distinct ways of conveying meaning. The tendency today is to conceive these two modalities of expression of meaning as complementary. In teaching-learning situations, gestures can be considered as carriers of meaning having the function to *locate* ideas in space, to make them visually perceived. Meanwhile, discourse has the function of *transforming/making* ideas in words. These are privileged tools used by teachers when communicating, explaining, and discussing mathematical concepts in the classroom. The aim of this paper is to focus on a teacher's communicative strategies while summing up, in dialogue with her pupils, the solutions from the pupils' small-group discussion on a mathematical task (called the diagram task), emphasising the transition between three semiotic representations: figure, diagram and written text.

This study is related to the research and developmental project, Learning Communities in Mathematics[1] (LCM) which was designed at the University of Agder (UiA) in Norway. The project was implemented in the period from 2004-2007, and the theoretical framework for it was presented at Cerme 4 (Cestari, Daland, Eriksen, & Jaworski, 2006). The project aimed to "create *inquiry communities* of teachers and didacticians to both develop and explore the development of mathematics teaching and learning" (Jaworski, Fuglestad, Bjuland, Breiteig, Goodchild, & Grevholm, 2007, p. 7).

Inspired by ideas and discussions at workshops in the LCM project, the experienced teacher in focus (about 35 years in service, spring 2005) organised workshops in the classroom with her pupils during one lesson a week. It is in such a workshop context that the diagram task was used in the classroom with the following structure in three parts: 1. Introduction of activities (00:00-04:28), 2. Working in groups of two and three (04:28-13:47), and 3. Summing up with the whole class (13:47-18:47). In Bjuland, Cestari and Borgersen (2008c) we identified the teacher's communicative strategies while presenting the task in a dialogue with her pupils (part 1). The teacher used both speech and gestures when focusing on the transition from the two different semiotic representations, figure and diagram. More specifically, she posed open *questions* while simultaneously "*pointing* to the diagram followed by a gradually decreasing *circular sliding* between the diagram and the picture" (op. cit., p. 190).

We were also concerned with the difficulties the pupils met in the solution process. One group (two girls) made incorrect suggestions without being attuned to each other, and they had difficulties in focusing on two dimensions in the diagram. The teacher visited the girls twice during the solution process (part 2). She posed different questions (yes-no, open, specific) in order to help them to express their difficulties. The teacher gave verbal explanations simultaneously with using gestures like *pointing* and *circular slidings* to make connections between figure and diagram (Bjuland et al., 2008c).

After having reported from the first two parts of the work on the diagram task, we are now concerned with the way the teacher sums up and concludes the mathematical activity (part 3). This paper addresses the following research question: What kinds of communicative strategies does an experienced teacher use in her dialogues with sixth-grade pupils, while summing up the pupils' solutions to a task that involves moving between different semiotic representations? In Bjuland et al. (2008c), we have illustrated that gesture and speech are natural mediating devices when this teacher introduced the diagram task and when she visited the girls' group. It is therefore important to ask how gestures are used in connection with speech in part 3.

THEORETICAL FRAMEWORK

Gestures and discourses are fundamental modalities in the interpretation of communicative strategies used by teachers in the classroom. According to Roth (2001), teachers employ many gestural resources crucial for understanding a concept. So, pupils need to attend to both their teachers' speech and their gestures in order to access information presented in a lesson. In Bjuland et al. (2008b), we have revealed how the multimodal components of expression, speech, gesture, and written inscriptions develop synchronically. These major components of the objectification process (Radford, 2003) have stimulated the pupils to come up with a solution. We have in our work mostly observed deictic gestures. These are defined by Mc Neill as "pointing movements, which are prototypically performed with the pointing finger" (1992, p. 80). This kind of gestures has an important function of locating in space the referent of the discussion. Likewise, Edwards (2005) reported that almost all gestures produced in the solution of a problem, related to fractions, by prospective teachers were deictic. According to Edwards (2009), they constitute a particular modality of embodied cognition.

In this paper we take a complementary approach, inspired by the work of Goodwin (2003), and include the analysis of the structure of the task. He has introduced the concept of *symbiotic gesture* when investigating how gesture is related to the physical, semiotic, social and cultural components of the context where it is embedded. An example provided by Goodwin (op. cit.) refers to archaeological analysis related to patterns of earth. He explains that the finger of the archaeologist pointing to the ground shows the graphic structure in the dirt, and, at the same time, that structure provides the context, the place, for the precise movement of the gesture. Another example of a football player is a classic one: if taken in isolation, it is not evident what he is doing. However, if the player is placed in the context of the game, the meaning emerges naturally. According to Goodwin (op. cit.), the nature of embodied practices which promote the competence to act as a member of a community is basically interactive. So, instead of taking as an analytical focus the gesture and discourse by themselves, we include the object which gestures are referring to as part of the analysis. We include as well the activity where this object is inserted in a sequential organisation, taking into account contributions from participants assuming different roles at different moments in the lesson. We illustrate how the teacher makes use of these components in the dialogues with her pupils.

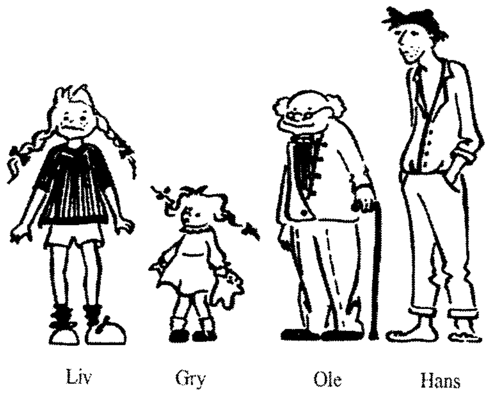
METHOD

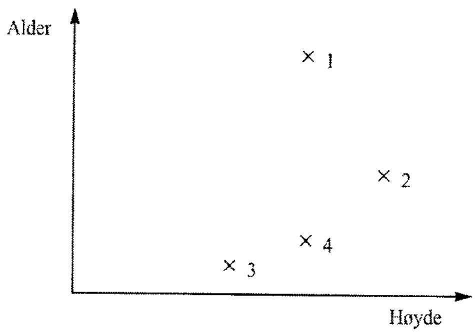
For analysing the discourses we have used a dialogical approach to communication and cognition (Bjuland, 2002; Cestari, 1997; Linell, 1998; Marková & Foppa, 1990) in order to identify an experienced teacher's communicative strategies used in the dialogue with her pupils. In this approach, there are some important principles: the *sequentiality*, *joint construction*, and *act-activity interdependency* (Linell, 1998). As far as the *sequential organisation* of discourse is concerned, "each constituent action, contribution or sequence, gets significant parts of its meaning from the position in a sequence. That means that one can never fully understand an utterance or an extract, if taken out of the sequence which provides its context" (op. cit., p. 85). In this case we have to take into account how a particular utterance is related to the *previous* utterance as well as to the *subsequent* one. The teacher's gestures are identified within a theoretical framework that considers cognition as an embodied phenomenon (Edwards, 2009) and as an interactional process (Goodwin, 2003). Further details about this multimodal approach can be found in Borgersen, Cestari, and Bjuland (in press) and in Bjuland et al. (2008b).

The dialogues presented in this paper are situated in a particular instructional context where the teacher, in dialogue with her pupils, sums up the mathematical solutions (part 3). In our analysis, we focus on the teacher's speech and gestures embodied and situated in the lesson. Part 3 of the selected 19-minutes video clip has been transcribed line by line, and we have divided the transcribed material into numbered utterances/turns. “An utterance lasts as long as a speaker holds the floor” (op. cit., p. 281). The gestures are described in italics inside brackets [ ] within the utterances/turns where they occurred.

The task

The following task was given to the pupils: <u>Write down which person corresponds to each of the points in the diagram</u> (the Norwegian words *alder* and *høyde* mean age and height respectively).





Liv corresponds to point ………………….

Gry corresponds to point ………………….

Ole corresponds to point ………………….

Hans corresponds to point ………………….

In earlier papers (Bjuland et al., 2008a; Bjuland et al., 2008b) we presented a detailed analysis of the proposed task, emphasising the characteristics of the three mathematical representations *figure, diagram* and *written text* respectively. Here, we only present the task as a background for understanding the dialogue between the teacher and her pupils while summing up the mathematical solutions. The teacher-pupil dialogues therefore focus particularly on the third representation (written text), including questions asking for the number in the diagram corresponding to every person in the figure.

SUMMING UP IN THE CLASSROOM

The plenary discussion (part 3) could be summarised in one ongoing episode, consisting of five thematic sequences:

Sequence	Communicative Strategies	Time	Turns
1. The location of Ole – explanation	Open question: Who is number one, two, three and four respectively? Two how-questions, trigger pupil explanation. The answer is visualised on the overhead projector. One further how-question, and the pupil repeats his explanation. Question asking for other suggestions. The teacher uses gestures by pointing to point 1, 2, 3 and 4 on the transparency.	1.13 min	162–172a
2. The location of Gry – explanation and justification	Open question: What about the other points? How-question – triggers an explanation. The answer is visualised on the overhead. Why-question related to the two variables, height and age. Gestures are not identified.	0.43 min	172b–179
3. The location of Hans – explanation and justification	Open question: Other answers? The answer is visualised on the overhead. How-question – triggers an explanation. Question asking for other suggestions in combination with gestures, pointing to point 1. Why could Hans not be point 1?	1.21 min	180–200a
4. The location of Liv – explanation and justification	Question directed to a pupil, Do you have the last solution? The answer is visualised on the overhead. How-question – triggers an explanation. One further question, Was it just a guess or should it be like this? Gestures are not identified.	0.40 min	200b–206a
5. Teacher summing up	Do all of you agree with these answers? Other solutions? Give praise to the pupils. Focus on the unusual – height at the horizontal axis. Recapitulation of the two dimensions, height and age. Gestures are not identified.	0.54 min	206b

Table 1: Plenary discussion after the small-group work

In our analysis we have focused on the first sequence of the dialogue since it illustrates how the teacher initiates the discussion. We have also chosen an extract from the third sequence since this dialogue shows how the teacher focuses on the pupils' argumentation, emphasising the connection between the two dimensions, height and age in the diagram. This third sequence also shows how one of the pupils (from the group with the two girls) that seemed to have most difficulties in understanding the task (Bjuland et al., 2008a; Bjuland et al., 2008b) responds to one of the teacher's questions, giving us some impressions of her understanding of the problem at this moment.

These sequences show the direction of the mathematical discussion between the teacher and her pupils, from a discussion of the location of Ole to the location of Gry and so on. This is based on the pupils' responses to the questions posed by their teacher.

The location of Ole

The dialogue below illustrates the first utterances in the teacher-pupil discussion of the mathematical solutions which have resulted from the collaborative small-group work. The teacher (Tea) initiates the dialogue, inviting her pupils of both sexes to be attentive to the task:

162	Tea:	Girls and boys [*Turns on the overhead projector*]. What I wonder about, what I actually wonder about, where are the different persons? Who is number one? [*Points at point 1, diagram*], who is two? [*Points at point 2, diagram*], who is three? [*Points at point 3, diagram*], and who is four? [*Points at point 4, diagram*] Per?
163	Per:	We think Ole is one.
164	Tea:	Ole is number one. How can you be sure of that? How did you think that out?
165	Per:	Since he's oldest, and then he is tallest [Hans] (…).
166	Tea:	Yes.
167	Per:	[Ole is] as tall as Liv.
168	Tea:	Okay. But Ole he's then number one. Can you write it on [the transparency], so we know it? [Per goes to the overhead projector and writes "1" on the transparency] … Ole is number one. [Per gives the pen/Indian ink to his teacher and goes down to his seat] But what did you think when you found out that Ole was number one?
169	Per:	Since, when he is [oldest]
170	Tea:	[Ssss]
171	Per:	and then he is on the picture, then he is as tall as Liv. No one else is as old as him [Ole].
172	Tea:	Okay. Mm. Did anyone think differently? Since he is oldest, okay.

The teacher initiates the discussion by using the same open questions as she did when she presented the task before the collaborative small-group work (Bjuland et al., 2008c). However, her gestures are a bit different. In Bjuland (op. cit.) we observed that she focused on the transition from the figure to the Cartesian coordinate diagram by making four consecutive *pointings* to the diagram with a gradually decreasing *circular sliding* between the diagram and the figure. The interplay between the teacher's gesture and her questions seemed to be a mediating device in her presentation, showing the relationship between figure and diagram. She is here using the four *pointing gestures* to the diagram in connection with her questions without moving between the two representations (162). We observe from the dialogue that the teacher's use of gestures in part 3 is far less prominent than in the presentation of the task (part 1) and in her small-group dialogue (part 2) with the two girls (Bjuland et al., 2008b). This indicates that the teacher uses more gestures in connection with her explanations to the pupils than in relation to pupils' explanations. In the dialogue between the teacher and the pupil Per (162-172), he comes up with the group solution for Ole as a candidate for point 1 (163). This response guides the direction of the discussion, showing that the teacher-pupil dialogue begins to focus on one of the extreme locations. The two questions from the teacher (164) stimulate Per to give an explanation (165) by making a comparison between Hans and Ole related to both age and height and a comparison of Liv and Ole related to their same height (167).

After having been concerned with the third representation (*written text*), showing the written solution on the transparency, the teacher poses a third how-question (168), provoking Per to repeat his explanation (169), (171). The teacher invites the pupils to make other suggestions (172), but she does not wait for a response. It seems that the teacher has observed that her pupils are satisfied with the solution putting Ole at point 1.

The location of Hans

The dialogue below contains a particular extract from the third sequence.

194	Tea:	But you [singular you], what did you [plural you] think when you found out that Hans should be number two?
195	Odd:	We thought that he was tall, and he [Hans] was much younger than Ole.
196	Tea:	Mm. Yes, so therefore he should be there. Is there anyone else that thought about it? [Silence, 6. sec.] Leo, what did you think?
197	Leo:	Eeh, no I (…)
198	Tea:	Eeh, yes, Is there anyone else that thought about it? Let's see, Hans is number two. He had to be there. Why couldn't Hans be there [Points at point 1, diagram] Why couldn't Hans be there, Eli?[The teacher chose Eli among several pupils who raised their hands]
199	Eli:	Since he, or if Ole, he is the oldest and then couldn't he [Hans], since he [Hans] is the youngest [of these two].
200a	Tea:	Mm. Yes.

In the second sequence of the episode, one of the girls chooses Gry at another extreme location in point 3 and gives an explanation for the location of Gry (see Table 1). One of the boys has responded to the teacher's open question and told the class that Hans corresponds to point 2, the third extreme location. This answer has also been visualised on the transparency.

In the continuation of the dialogue, the teacher poses a question that stimulates the pupils to explain how they come up with this particular location for Hans (194). The pupils were not only to produce an answer, but they are also challenged to explain their thinking. Odd's response, starting with *we*, (195) shows that he explains the group's thinking. In his explanation Odd is concerned with the two variables, age and height, making a comparison between Ole and Hans. Since they have already discussed the location of Ole (first sequence), it is natural for Odd to explain how his group has discovered the relationship between the placement of Hans and Ole respectively.

After having evaluated this response, the teacher goes on to pose another question that provokes other suggestions (196). The pause indicates that the teacher allows a waiting time of six seconds, giving the pupils opportunities for individual considerations. Since the pupils do not respond to this initiative, the teacher repeats her question and directs it to the individual pupil, Leo (197). His response and the teacher's next question (198) show that the pupils do not have other suggestions. They seem to be convinced that Hans corresponds to point 2. We might wonder why the teacher is so focused on bringing other suggestions into the dialogue. One possible explanation could be that she wants to focus on possible misconceptions. The teacher seems to be aware of how complex it could be for pupils to realise how the two variables, height and age, are connected in the Cartesian coordinate system. By focusing on point 1 as a possible location for Hans, the teacher also triggers the visual misconception: the tallest person corresponds to the point, located highest in the diagram. In connection with this question she also uses gestures to make the pupils aware of the possible location of Hans at point 1. In the analysis of the dialogue of the two girls (Bjuland et al., 2008b), we identified this misconception.

When the teacher poses the challenging why-question twice, provoking the pupils to consider the wrong location of Hans, the pupil Eli (pupil 4 from our girl group) responds to the teacher's initiative (199). Eli makes a comparison of Hans and Ole due to their ages. In one respect, it is possible to argue that Eli is still just focusing on one dimension, the variable of age. However, if we situate the response in this particular context based on the teacher's way of posing the question and also the teacher's evaluation of the response (200), it seems as if Eli has given a proper explanation and developed her understanding from the group work.

CONCLUDING REMARKS

Through the analysis of dialogues from the teacher-pupil discussion of group solutions on the diagram task, we have identified the teacher's communicative strategies. Her use of *questioning* (who, how, why, other suggestions) is the most prominent strategy. The analysis has also revealed that her use of gestures is more restricted in part 3 compared to gestures used in connection with her explanations while presenting the task and in a small-group dialogue with the two girls (Bjuland et al., 2008c). We could wonder why this restriction happens in part 3. When the teacher plays the role as a *presenter* (part 1) and as a *supervisor* (part 2), she uses gestures as a mediating device in combination with verbal explanations. In part 3 she uses mainly gestures, pointing to the diagram without circular slidings between representations, to initiate the discussion. Here (in part 3) the teacher plays the role as a *coordinator*, opening the floor for the pupils to write their answers. The teacher-pupil discussion focuses on the mathematical representation, *written text*, in which the pupils show their group solutions on the transparency, making explanations and justifications.

Concerning the contribution of the environment, supported by the concept of *symbiotic gestures* (Goodwin, 2003) we have observed that the nature of the task is influencing the different pointing gestures. It is indeed the pupils' responses that guide the direction of the mathematical discussion. Gestures and discourses are conceived as *meaning translators* between different mathematical and pedagogical ideas used by the teacher as communicative strategies.

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A TEACHER'S ROLE IN WHOLE CLASS MATHEMATICAL DISCUSSION: FACILITATOR OF PERFORMANCE ETIQUETTE?

Thérèse Dooley

University of Cambridge and St. Patrick's College, Dublin

In the improvisation that occurs in a jazz ensemble, a soloist rarely develops a completely new idea but, instead, elaborates and builds on the previous player's input. From an emergent perspective, classroom mathematical practice is akin to such improvisation. How this might happen in a whole-class situation is unclear. In this paper, a description is given of a whole-class discussion that took an unplanned trajectory. The teacher did not impose a particular structure on the lesson but focused pupils' attention on productive mathematical ideas that emerged from the group. In the concluding discussion, it will be shown that the improvisation metaphor, while useful for describing mathematics as a socio-cultural activity, may have a different application in a whole-class situation than in small group settings.

INTRODUCTION

Although plenary sessions are common to mathematics lessons, they are often characterized by traditional approaches that endorse the position of mathematics as a kind of received knowledge and the teacher as sole validator of students' contributions (See, for example, Boaler, 2002; Cobb, Wood, Yackel, & McNeal, 1992) While research shows that whole-class discussion can be fertile ground for higher-order mathematical thinking (Cobb et al., 1992; O'Connor, 2001), the fast pace with which it is usually associated means that there is little scope for students to make comments and build on each others' mathematical ideas (Hodgen, 2007). One consequence of this is that students become disengaged from the subject, perceiving it to be one in which they have little opportunity for participation (Boaler, 2002). However, the orchestration of inquiry-based discussion in mathematics is challenging for teachers. Sherin (2002) alludes to two key tensions whereby teachers, on the one hand, are expected to encourage students to share ideas and, on the other, have to ensure that the lesson is mathematically productive.

In this paper the improvisation metaphor is used to show how a teacher and her pupils co-constructed new mathematical ideas in the context of a whole-class discussion in a primary school. In particular, attention is paid to the way provision can be made for different levels of understanding within the class. In the concluding discussion, reference will be made to limitations of some tools that are used to analyse such research.

THE IMPROVISATION METAPHOR

According to Lakoff and Johnson (1980), metaphors not only help us to understand one kind of thing in terms of another but they can also create a reality and thus act as guides for future action. In relation to the teaching of mathematics the improvisation metaphor is one that serves both of these purposes. Consistent with a view of mathematics as a socially and culturally situated activity, the point of reference in mathematics education is the classroom mathematical practice, a perspective that has been described by Cobb (2000) as *emergent*. Sawyer (2004) maintains that this perspective implies that teaching must be improvisational and 'that the most effective learning results when the classroom proceeds in an open, improvisational fashion, as children are allowed to experiment, interact, and participate in the collaborative construction of their own knowledge' (p.14).

In theatrical improvisation, a group of actors creates a performance without using a script. Because it is characterized by a high level of unpredictability, the performance has associated with it what Sawyer describes as a 'moment-to-moment contingency' (Sawyer, 2006: p.153). As the actors play their parts, several potential possibilities are brought into the frame. What emerges is not decided by any one person but rather is a phenomenon that is produced by the group. In jazz improvisation, each soloist is assigned a number of measures to play before the next soloist takes over. Due to the rapidity of the transition, a player rarely develops a completely new idea but rather responds to and builds on the previous player's input (Berliner, 1994).

Sawyer (2004) maintains that like the improvisation that occurs in theatre or in a jazz ensemble, creative teaching is both emergent and collaborative. It is emergent because the outcome cannot be predicted in advance and it is collaborative because the outcome is determined not by any one individual but by the participants of the group. Martin, Towers and Pirie (2006) used the improvisational lens to analyse collective mathematical understanding. They describe collective mathematical understanding as the kind of learning and understandings that occur when a group of any size work together on a mathematical activity. Central to their analysis is the idea of *co-acting* which they define as

> …a process through which mathematical ideas and actions, initially stemming from an individual learner, become taken up, built on, developed, reworked, and elaborated by others, and thus emerge as shared understandings for and across the group, rather than remaining located within any one individual. (p.156)

They make a distinction between co-actions and interactions. While in interactions there is an emphasis on reciprocity and mutuality, co-actions concern actions that are dependant and contingent upon the actions of other members of the group (Towers & Martin, 2006). Through this co-acting, an understanding emerges that is the property of the group rather than any individual. It is not that all individuals bring the same understandings to the scene but rather that individual contributions will result in something greater than the sum of the parts. Neither does it preclude an individual making his or her own personal advancements.

In a more fine-grained analysis of the improvisational metaphor, Martin and Towers (2007) have introduced the notion of performance etiquette. In jazz terms this refers to a situation where players drop their own ideas in deference to a better (in the view of the collective) idea if that works. It means that due attention and equal status have to be given to all players' ideas and intuitions. According to Martin and Towers, '(in) mathematics, 'better' is likely to be defined as a mathematical idea, meriting the attention of the group, which appears to advance them towards the solution to the problem' (p.202). Although much of the work done by Martin et al. concerns small groups there is evidence that the metaphor is also applicable to whole class discussion (See, for example, Dooley, 2007). King (2001) contends that in lessons where students and teachers co-create classroom discourse, 'one can view students as other participants in [the] improvisation, following the direction of the lead improviser, the teacher'(p.11). She proposes that the teacher is rather like the soloist who must modulate her performance to her instrumentalists and audience. There is some danger that this analogy leads to the teacher's role being perceived as centre of (as opposed to central to) the learning process. Sherin (2002) suggests that, in order to achieve a satisfactory balance between process and content, the teacher engages in *filtering* by which is meant a narrowing of ideas generated by students so that so that there is a focus on mathematical content. An implication for whole class discussion is that the teacher is more facilitator of group etiquette than lead improviser. This idea is pursued further in the account below.

BACKGROUND

The aim of my research is to investigate the factors that contribute to the development of mathematical insight by primary school pupils. The methodology is that of 'teaching experiment' which was developed by Cobb (2000) in the context of the emergent perspective and in which students' mathematical development is analysed in the social context of the classroom. For a period of six months, I taught mathematics to a class of thirty-one pupils (seven girls and twenty-four boys) aged 9 - 10 years. The school is situated in Ireland in an area of middle socio-economic status. Although I taught the lessons, the class teacher played an active role as co-researcher, advising on the suitability of lesson content, clarifying any confusion that arose in whole class discussions, working with pupils during group work and making observations in post lesson discussions. Many lessons took place over two or three consecutive days, each period lasting forty to fifty minutes. I visited the class on a total of twenty-seven occasions. All phases of the lesson were audiotaped. When children were working in pairs, audio tape recorders were distributed around the room. Each pupil maintained a reflective diary. Follow-up interviews were held with students who had shown some evidence of reaching new understandings over the course of a lesson.

Forman and Ansell (2001) contend that analysis based on isolation and coding of individual turns is too limited to bridge the individual and social. Therefore, I conducted ethnographic microanalysis, which according to Erickson (1992) is especially appropriate when the character of events unfolds moment by moment. The approach adopted was top-down starting with the molar units (lessons) and moving to progressively smaller fragments. I transcribed all lessons and isolated those in which pupils showed evidence of constructing new mathematical insight. Thereafter I identified constituent parts of the lesson, starting with major events and moving progressively to the actions of individuals. A comparative analysis of lessons was also undertaken.

The lesson described here took place on a third consecutive visit to the class during a week of the Spring term. On the previous two days, the pupils had been working on a lesson entitled ‘Chess’, the object of which had to find the minimum number of games that could be played by participants in a competition where each competitor had to play all other players. At the conclusion of this lesson some pupils had found the answer for one hundred players (i.e., the sum of 1 - 99) by using a calculator while others had latched onto the discovery made by one pupil, David[1] that the solution could be found ‘by multiplying by the number less than it and halving it’ ((100 x 99) ÷2). It was my intention on the third day to begin a new lesson but first told the story of Gauss (the mathematician who, as a boy, had amazed his teacher by his rapid calculation of the sum of integers from 1 to 100) in order to see if the pupils would make any connections between it and the chess problem. I expected that talk on this problem would last no longer than five or ten minutes. However, a rich discussion followed in which I truly had to improvise. Although this lesson is not being promoted as exemplary, I learnt from it something about the power of ‘letting go’ and ways in which group etiquette might be facilitated.

The focus of this paper is on the discussion that took place after I first related the story of Gauss. Although space does not allow the full transcript to be presented, an effort is made to give as full as possible a sense of the lesson trajectory (a problem described by O'Connor (2001: p.144) as ‘the competing requirements of data reduction and interpretive explicitness’). The following transcript conventions are used: T.D.: the researcher/teacher (myself); Ch: a child whose name I was unable to identify in recordings;…: a hesitation or short pause; […]: a pause longer than three seconds; ( ): inaudible speech; [ ]: lines omitted from transcript because they are extraneous to the substantive content of the lesson.

THE IMPROVISATIONAL CREATION

On telling the story, some pupils suggested that Gauss may have found his solution by adding fifty and fifty or five twenties, considering addends of rather than the sum to a hundred. When I focused their attention on the problem conditions, Barry had this idea:

18	Barry:	Eh, you add up all the numbers that are in ten like one, two, three, four, five, six, seven, eight, nine, ten…

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[1] Pseudonyms are used throughout the paper.

19	T.D.:	Hmm.
20	Barry:	and then multiply by ten.
21	T.D.:	Ok, so you would add up as far as ten and then multiply the answer by ten?
22	Barry:	Or nine, I'm not really sure.
23	T.D.:	Ok, why do you think it might be nine?
24	Barry:	Eh, because you have already counted up to ten and it's ten tens in a hundred.

Here he was making an assumption that the sum of numbers between 1 and 10 would be the same for all decades. Brenda then asked if she could check the answer on the calculator which was interesting given that she had thus correctly established the solution for forty players in the Chess activity.

Anne and Fiona then built on the idea proposed by Barry:

48	Anne:	I think it's thirty multiplied by ten.
49	T.D.:	Sorry?
50	Anne:	Thirty multiplied by ten.
51	T.D.:	Thirty multiplied by ten, why would you say it's thirty? [ ]
54	Anne:	Because if you add from one up to ten it's thirty.
55	T.D.:	How do you know if you add one up to ten it's thirty?
56	Anne:	If you add one to five, that's fifteen…
57	T.D.:	Hm, hm
58	Anne:	and then fifteen and fifteen is thirty so then if you multiply that by ten.
59	T.D.:	Ok, possibly that would get it for you. Fiona?
60	Fiona:	Well, could you em, oh, em, do, eh, you could do one plus two and up to fifty and then double it...

I chose not to correct misconceptions at this point but wrote the suggestions on the blackboard. This proved a good judgement in this instance because a short while later two pupils commented on Anne's input:

66	Alan:	Em, well, I don't think Anne's one is right.
67	T.D.:	Why?
68	Alan:	Cos ninety-nine plus ninety-eight plus ninety-seven plus ninety-six to ninety would be around over five hundred and when…
69	Ch:	Oh!
70	T.D.:	Ok, [ ] you are thinking ninety plus ninety one plus ninety two plus ninety three would give you approximately how much?
71	Alan:	Em, I don't know.
72	T.D.:	But it's…
73	Alan:	But it would probably be over five hundred.
74	T.D.:	It would be over five hundred, so in that section, if you are thinking about all those numbers there that would give you about, even just adding ninety to a hundred so you are thinking that would give you about five hundred. [ ]. Barry?

75 Barry: Eh, well, I disagree with Anne as well because, eh, I counted, I counted up all the numbers up to ten and I got fifty-five.

Enda then said that multiplying five by twenty or adding fifty plus fifty (both ideas were written on the blackboard) didn’t ‘actually have much to do with this’. Anne now corrected her earlier idea:

91 Anne: I don’t think…my answer wouldn’t work.

92 T.D.: What were you thinking your answer was?

93 Anne: I thought it would be thirty multiplied by a hundred.

94 T.D.: Why would it not work?

95 Anne: Em, because you would have to, cos I did eh one plus two plus three plus four plus five and then em I got fifteen and then I added fifteen and fifteen equals thirty but then it would be more because you would have to add six, seven and that.

Anne seemed to have reached a new understanding about the addition of a series of numbers. It is possible that she began to reflect on her thinking because Barry and Alan disagreed with it. Colin then arrived at a new approach to the problem:

97 Colin: It could like eh add the, say you could have ninety-nine, add the closest and the furthest and then the second closest and the second furthest.

98 T.D.: So give me an idea what you are talking about now. Tell me, elaborate a bit on that. [ ]

101 Colin: Eh if it was ninety-nine, you add one, if it was ninety-eight you add two, if it was…

102 T.D.: Ok, so you are thinking - very interesting because that’s - you could have ninety-nine plus one, go on!

103 Colin: Ninety-eight plus two, ninety-seven plus three, ninety-six plus four, eh, ninety-five plus five, ninety-six or ninety-four plus six (*teacher records on blackboard*)…

104 T.D.: Ok, so what’s that giving you, why are you putting those numbers together?

105 Colin: They all go up to a hundred.

106 T.D.: So what’s that telling you then, what do you think it might be, have you any idea what the answer might be?

107 Colin: Eh, no.

108 T.D.: Do you see what Colin is doing there? He is matching up numbers, he is taking the numbers at the very beginning and he is matching them up with the numbers at the end.

I was quite excited when I heard this input as this was the method used by Gauss as a young boy, hence my remark, on line 102, ‘very interesting because..’. I wrote his suggestion on the blackboard but also ‘revoiced’ his input (line 108), a teacher strategy that serves to repeat or expand a student’s explanation for the rest of the class (Forman & Ansell, 2001; O'Connor, 2001). Enda then proposed a different way of grouping the numbers. However, I did not grasp his idea:

113	Enda:	Eh, well, I think one possible way it would probably would be just as hard, it would be harder than one plus two plus three, it’s probably not going to help us, what I was going to say is eh adding…when adding ninety plus ninety-one plus ninety-two and all that sort of stuff…
114	T.D.:	Hm, hm.
115	Enda:	It’s the same every time, you would just, all you would probably, eh, you would probably need to go backwards and just take way ten from the answer above every time. That would ( ) if you took away ten from the answer every time.
116	T.D.:	Hm, hm
117	Enda:	So add up the numbers going from a hundred backwards. [ ]
120	T.D.:	If you went a hundred plus ninety-nine plus ninety-eight plus ninety seven…
121	Enda:	Yeah
122	T.D.:	all the way back as far as one, would you still get the same answer?
123	Enda:	The same answer, even though it would just be easier to do it backwards with that way em you just need to take ten away from it every time. If you were on ninety, if you got a hundred back to ninety and you were on eighty, just take ten away from the answer above.

Enda had found an interesting solution method, that is, adding from 100 to 91 and then finding the solution for the sum from 90 to 81 by subtracting ten. In fact this is a very viable method (if one hundred is subtracted each time). I had assumed he was talking about commencing the addition from a hundred rather than one. It is very possible that I did not comprehend his approach because it was one I had never considered. I did, however, ask him to pursue his idea in his diary.

Liam then made another observation about Colin’s list:

135	Liam:	I don’t think like if you go back to Colin’s way…if you go back, you wouldn’t be able to do it, if you go back to one then you might double it, the whole thing.
136	T.D.:	Sorry?
137	Liam:	If you go all the way to one, then you double the whole thing.

Neal then suggested that the list should terminate at 50 + 50 and I urged pupils to think about the number of ‘hundreds’ there might be. Anne then proposed that the answer would be a thousand and this led to an interesting contribution by Brenda:

166	Anne:	I think the answer would be a thousand.
167	T.D.:	You think it’s going to be a thousand. Do you agree with Anne that it’s about a thousand? Brenda?
168	Brenda:	Eh, no cos when I em added up forty for it and, em, I got more than a thousand.

This is the first time in the lesson that a direct reference has been made to the chess activity. Fiona confirmed that the answer for 40 children (i.e., the sum from 1 to 39 although this was not as yet clear) was 780. Anne picked up on this idea:

183 Anne: Well, in the one we did yesterday, when the number of children was a hundred, then the number of games was four thousand, nine hundred and fifty so that there would be the answer.

I wrote 4950 on the blackboard as one other possibility. Hugh however noticed the error:

197 Hugh: I think it would be, em, five thousand, nine hundred and fifty.

198 T.D.: Where are you getting that from?

199 Hugh: Em, because eh yesterday we didn't add on the hundred.

200 T.D.: Ok […] so

201 Hugh: So then it would be …five thousand…and fifty.

Liam now saw that 50 + 50 should not be included in the list:

209 Liam: Well on the last one in Colin's one you have to do a triple sum kind of ( ) because it would be forty nine plus fifty one and then add fifty on to it.

David confirmed that the solution was 5050 and explained his reasoning as follows:

213 David: Em, well if you do Colin's way and then, em, you get, em fifty ( ) and then when you get to forty nine plus fifty one and you have to add the fifty on and that gives you about five thousand and fifty.

At this point in the discussion the class teacher indicated that a small group of pupils had taken out their diaries and were working on solution methods in them. In particular, Declan seemed to be very keen to complete the listing suggested by Colin. The pupils embarked on paired/individual work during which the class teacher sat with Declan. In the plenary session that was held at the conclusion of the lesson, Fiona and Clare discussed possible answers for the sum of numbers up to 200 (they proposed 5050 x 2). Some pupils spoke about the solution they found on the calculator. Declan described how he solved the problem using Colin's method. Miles began to consider that the answer might be obtained by multiplying a hundred by a hundred and then halving it 'to take way the pluses that you add on to get one hundred'. David, however, did not use the formula he had found for the chess problem to add the numbers from 1 to 100.

DISCUSSION

There is evidence that co-acting took place in this lesson. For example, in the early part of the lesson, Fiona and Anne picked up on Barry's idea of adding a section of numbers and applying proportional reasoning (albeit incorrectly). Later Anne reconsidered her reasoning on the basis of input by Alan and Barry. Colin's idea may well have emerged because of the discussion around addition of numbers between 1 - 10 and 90 - 100 (see lines 68 and 75). Enda's method could be an elaboration of that proposed by Colin. Brenda made the explicit connection with the previous day's lesson which prompted solutions by Anne and Hugh. However, the co-acting is not as linear as might be the case in small group discussion. Rather there is a weaving in and out of ideas. Lines 135 and 209, where Liam broke the flow of conversation to transform Colin's listing, are instances of this. It also seemed that some students who made no contribution to the dialogue reported above were nonetheless actively engaged. For example, Declan, a student who is not confident about his mathematical ability, pursued Colin's idea with great zeal. An implication of this is that tools used to analyse whole class discussion must extend to include those who are silent but participating in the enquiry.

O'Connor (2001) ponders the difficulties of looking objectively at transcriptions and attempting to discern the motives of the teacher in taking certain actions. As the researcher/teacher on this lesson, I am in a position to say, at least to some extent, why I took certain courses of action. A primary concern was keeping things, to continue with the jazz metaphor 'in the groove', for the group while at the same time respecting the input of individuals. Enda's idea (lines 115 and 123) did not become part of the collective because I did not understand it. Recourse to a diary allowed him to pursue his own investigation, however. My position in this lesson was not that of lead improviser because the lesson took an unexpected trajectory, but I feel that I facilitated group etiquette by drawing attention to ideas that would lead to solution to the problem.

With regard to the future direction of this research, the ways in which whole class discussion can impede or facilitate pupils' mathematical insight will be further analysed. In particular attention will be paid to the ways in which the making public of ideas by writing them on the blackboard and the revoicing of pupils' input stimulates the filtering process.

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USE OF WORDS – LANGUAGE-GAMES IN MATHEMATICS EDUCATION

Michael Meyer

TU Dortmund

This article focuses on the introduction of new concepts in mathematics classrooms. A theoretical framework is presented which helps to analyse and to reflect on the processes of teaching and learning mathematical concepts. The framework is based on the theory of Ludwig Wittgenstein. His language-game model and especially its core, the primacy of the use of words, provide insight into the processes of giving meaning to words. The theoretical considerations are exemplified by the interpretation of a scene, in which students are introduced to the concepts of "perpendicular", "parallel" and "right angle".

INTRODUCTION

> "Mathematics education begins and proceeds in language, it advances and stumbles because of language, and its outcomes are often assessed in language." (Durkin and Shire, 1991, p. 3)

A lot of research has been done on communication in the mathematics classroom. Mathematical interactions have been analysed from many different perspectives (cf. Cazden, 1986). This text will focus on the teaching and learning of mathematical concepts in classroom communication. The importance of introducing mathematical concepts is underlined by the multitude of theories used for analysing concepts. In this paper only a few of them can be taken into account: de Saussure (1931), Peirce (CP 2.92) and Steinbring (2005).

By his concept of "language-game" Wittgenstein offers us an alternative view on the introduction of concepts in mathematics classrooms. His perspective has often been used to discuss problems concerning communication in the mathematics classroom (e.g., Bauersfeld, 1995; Schmidt, 1998). Sfard (2008) is using Wittgenstein's theory within her "commogitive model".

Wittgenstein presents considerations we can use to analyse language and especially the meaning of words. His theory of language-games and the construction of meaning will be considered in this paper, which presents first results of scientific research in progress. According to Wittgenstein, the expression of words does not constitute their meaning. Words have another function in the process of constructing knowledge. The main aim of the research is to analyse whether Wittgenstein's theory is useful for reconstructing and thus understanding communication. In spite of the multiple Wittgenstein references, I only know a few examples of using Wittgenstein's theory for analysing communication in the mathematics classroom (cf. the examples of Sfard 2008). More specific aims will be described in the course of this article. The core of the theory, the primacy of the use of words, will be exemplified.

USING WORDS IN LANGUAGE-GAMES

In his later philosophy (cf. the "philosophical investigations" and the "remarks on the foundation of mathematics") Wittgenstein describes a pragmatic theory of language and meaning. He denies every fixed relation between language and objects. Also Wittgenstein is no longer searching for anything, which could be taken as something basically shared by all linguistic acts. Language is not an objective mediator between human beings and objects given. Nevertheless, he considers knowledge – and thus mathematical knowledge – not to be transmitted objectively:

> "Language is a universal medium – thus it is impossible to describe one's own language from outside: We are always and inevitable within our own language […]. Knowledge appears as knowing, and knowing is always performed in language games. Language as languaging or playing a language game is equal to constituting meaning and, thus, constituting objects. There are no objects without meaning, and meaning is constituted by a special use of language within a respective language game" (Schmidt, 1998, p. 390).

For Wittgenstein the construction of knowledge takes place by playing language-games. The term "game" does not imply an option for those who are involved. We cannot choose in the first place whether we want to play the game or not. The problem is that Wittgenstein does not explain in detail what he means when speaking of "language-games". As we will see, this is not because he does not care. Rather it is due to his theory of giving meaning to words.

Words have neither a consistent nor an objective meaning. In different language-games various meanings of a word can occur. Following Wittgenstein there is no direct transformation from a word to its meaning: "[…] experiencing a word, we also speak of 'the meaning' and of 'meaning it.' […] Call it a dream" (Wittgenstein, 1958, p. 216). Moreover, it is the use of a word which determines its meaning:

> "For a large class of cases – though not for all – in which we employ the word 'meaning' it can be defined thus: the meaning of a word is its use in the language" (Wittgenstein, PI, §43).

The term "use" is not limited to the application of words (e.g., in order to solve problems). If we exemplify a word, we also make use of it. One research-guiding problem will be to identify different forms of uses of mathematical words.

A word does not mirror objects and the meaning of a word cannot be observed while looking at its association with a specific object. The meaning of a word is nothing but the role it is playing in the specific language-game and accordingly can be observed only by looking at the use of words. This central thesis might be the reason why Wittgenstein does not define what he means using the term "language-game". He stays consistent: He exemplifies the words he makes use of [1]. Language-games can be different in character. So Wittgenstein (PI, §23) presents the following examples among others:

- "Giving orders and obeying them",

- “Forming and testing a hypothesis” and
- “Solving a problem in practical arithmetic”.

These examples may indicate that language-games are little “passages” or specific situations in our daily communication, but Wittgenstein also presents a larger field:

> “I shall also call the whole, consisting of language and the actions into which it is woven, the ‘language-game’.” (Wittgenstein, PI, §7)

Language is constituted by a “multiplicity” (Wittgenstein, PI, §23) of language-games. And all these language-games bear a temporal dynamic:

> “And this multiplicity is not something fixed, given once for all; but new types of language, new language-games, as we may say, come into existence, and others become obsolete and get forgotten. (We can get a *rough picture* of this from the changes in mathematics.)” (Wittgenstein, PI, §23)

The temporal dynamic indicates once more that there is no specific meaning for words fixed forever. Changing the meaning of a word is accompanied by a change of the language-game. Learning also means to realize changing meanings of words. Learning includes learning how to play different language-games. Thus, learning implies partaking in changing and new language-games.

USING WITTGENSTEIN

In mathematics education there has been a lot of research to consider and to analyse concepts and how students get used to them. Some work (e.g., Duval, 2006) is based on de Saussure’s (1931) relation between signifier and signified (fig. 1). The theory of de Saussure provides a subject-object dualism and thus implies some problems:

> “If there would be a correspondence between language and reality, then, surely, one could arrive at true verbal statements about the world. Descriptions (and teaching), then, would become a case only of an adequate selecting and of providing for sufficient precision of the verbal means (denotations), as well as an adequate fit of these means with the object” (Bauersfeld, 1995, p. 277).

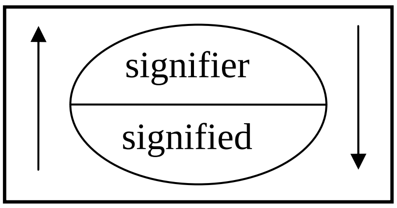

Figure 1: De Saussure’s (1931) relation between signifier and signified

Peirce (CP 2.92) offered a more detailed framework. His triadic relation between the sign, its object and its interpretant (fig. 2) has been used to analyse and to describe verbal or non-verbal interaction (e.g., Hoffmann & Roth, 2004; Presmeg, 2001; Sáenz-Ludow, 2006; Schreiber, 2004). The reconstruction of classroom interaction based on this framework has to deal with the difficulty that it is problematic to determine the object to which the sign is related. Contrarily, Wittgenstein's theory is a more pragmatic one. He does not regard any ontology of a sign. According to his theory words only get their meaning by their use and do not transport any given meaning. There is no fixed relation between words and objects.

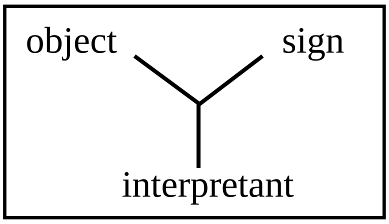

Figure 2: Peirce's triad

By his epistemological triangle (fig. 3) Steinbring (2006) provides a way to analyse static moments in the process of giving meaning to words. He presents a triadic relation between "sign/symbol", "object/reference context" and "concept":

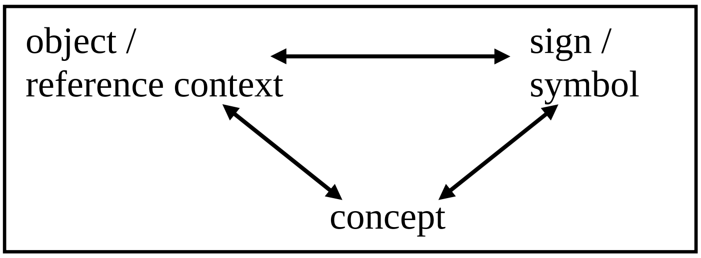

Figure 3: Steinbring's epistemological triangle (2006, p. 135)

The importance of the context can also be observed in Wittgenstein's writings, as he is considering the use of a word in the specific language-game. And language-games are depending on the situation:

> "Here the term 'language-game' is meant to bring out into prominence the fact that the speaking of language is part of an activity, or of a form of life." (PI, §23)

Wittgenstein points out that there is no direct transport of meaning from the teacher to the student, nor a direct understanding. We only can analyse the meaning of a word by looking at the use of that word in a specific language-game, which is at the same time influenced by other language-games. If we take a look at the language-game "mathematics education", we are also confronted with influences of every-day language-games of the students (and the teacher) and, all the more, of the rather mathematical language-games the teacher is able to participate in with mathematics experts outside of the classroom.

Words can be used in more than one language-game and thus each word can exhibit different meanings. If the teacher is going to introduce a concept in mathematics education, the children might immediately associate some meaning to it – due to the use of that word in another language-game the student took part in. This might be an every-day language-game or a language-game of mathematics education of a previous era (e.g., subtraction means to remove things, which does not work for negative numbers).

Words could be used in more than just one way. Accordingly, they can convey different meanings or meanings, which cannot be grasped only by knowing one form of their use. Thus, the use of a word in a specific situation must not lead to the whole range of possible meanings. Also, some concepts are restricted or expanded in the course of mathematics education (e.g., the concept of numbers). Therefore, this study is going to focus on the introduction of new concepts in the mathematics classroom and their development during following lessons. Some research-guiding questions are: How do students make use of words? What might be the meaning of a word for them? How do teachers influence the play of another language-game?

METHODOLOGY

The empirical data emerged from classroom observations in different grades (1-10) in Germany. Classroom communication has been videotaped by teacher students acting as researchers. The videographed units comprised 4-8 lessons of 45 minutes each. The teacher students were observers; they were told to exert no influence on the classroom communication and on the teachers' way to introduce the concepts. Altogether eight classes were visited.

The qualitative interpretation of the classroom communication is founded on an ethnomethodological and interactionist point of view (cf. Voigt, 1984; Meyer, 2007). Symbolic interactionism and ethnomethodology build the theoretical framework which is going to be combined with the concepts of "language-game" and "use".

According to Wittgenstein we should not ask: What is the meaning of a word? Rather we should analyse what kind of meaning a word gets in the classroom. Therefore, we have to analyse social processes. Thus, we have to follow the ethnomethodological premise: The explication of meaning is the constitution of meaning.

Analysing students' languaging for mathematical concepts, the development and the alteration of meaning by the use of the according words, we are able to reconstruct the social learning in the mathematics classroom [2].

The main aim of this study is to get a deeper insight into the processes of giving meaning to words in the mathematics classroom. Therefore, alternative ways of introducing concepts are going to be considered. Comparing possible and real language-games can help to understand the special characteristics of the actual played language-game.

THE USE OF WORDS IN CLASSROOM COMMUNICATION

The following scene emerged from a 4th grade classroom in Germany (students aged from 9 to 10 years). It is the first time that these students get in contact with the concepts of "parallel", "perpendicular" and "right angle" in this mathematics class.

The teacher starts the lessons by writing the words on the blackboard and asking the students to associate anything coming to their mind about these words. Afterwards a painting by Mondrian (cf. fig. 4) is presented on the blackboard [3].

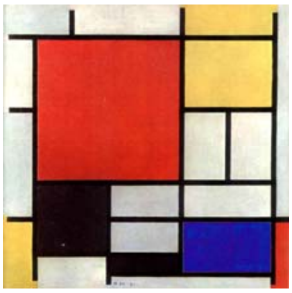

Figure 4: Painting by Mondrian on the blackboard

Teacher: Why do I fix such a picture on the blackboard? And why are these concepts written down on the blackboard? I have a reason to do so. Jonathan, it is your turn.

Jonathan: Because the painter has done everything in parallel, perpendicular and in right angles.

Teacher: You are right. You seem to know what parallel, perpendicular and right angle means. Maybe you can show it to us on the picture.

Jonathan: Perpendicular is this here (points first at a vertical, afterwards at a horizontal line). Parallel is this here (points at two vertical lines). A right angle is this (pursues two lines he former would have called perpendicular).

By pointing to different things on the blackboard, Jonathan makes use of the words "perpendicular", "parallel" and "right angle". He must have been in contact with practices of using them and thus with meanings of these words in a language-game outside this classroom. In this situation the words get a meaning by him pointing at something. This use can be described as an *exemplaric use*: An example is used to show the meaning of a word.

The use Jonathan makes of the words need not imply that those words could also be used in different ways, but *this* use and respectively *this* meaning get established in this classroom communication.

The teacher does not have any further questions. The teacher accepts the use of the words Jonathan must have known from another language-game. Thus, it seems that the exemplaric use is an acceptable one and that the meaning of the words is "taken-to-be-shared" in the classroom (cf. Voigt, 1998, pp. 203).

Certainly, in another language-game the meaning of the words "perpendicular", "parallel" and "right angle" can be different. They can be defined by using other concepts. A right angle can be defined as an angle of 90 degrees. Also the word "right angle" can be used in coherence with Pythagoras' theorem or in relation to the shortest distance of parallel lines. Perpendicular can be described by using the concept of “right angle”. All of these uses describe other language-games and not all of them can be played in a 4th grade classroom. Altogether, the words can have different uses and, thus, different meanings. In this classroom the words are used in order to represent things (cf. de Saussure’s model).

In the next few minutes the students had to create a mindmap, which should contain “something which can fit to the picture”. Then, afterwards “perpendicular” gets exemplified on the picture again. Now the classroom communication goes on with “parallel” and “right angle”:

Teacher: Now we just have two problems: parallel and right angle.

Sebastian: Right angle is easy (holds the set square at the blackboard).

Teacher: Can you show it here (points at two lines on the painting by Mondrian which have been used to show “perpendicular”). (After five seconds) Doris just say it. Wait! Before you go ahead, let –

Doris: You can make out four right angles out of it.

Teacher: This is the sign for the right angle (draws ⊾ on the blackboard). Maybe you can just draw it into the picture? (After three seconds) You can also choose another one.

Doris: John

Teacher: John and Tim come here. Doris said you would be able to find four right angles.

John: You two, me two (speaks to Tim while pointing at two lines).

Teacher: That is not right. No. Doris, show him were they are.

John: There is a right angle.

Teacher: Ah, yes!

The class is going to consider the last two “problems” (parallel and right angle), which have not been exemplified a second time. Doris identifies four right angles on those lines, which had been used before in order to show the meaning of the word “perpendicular”. John shows an example for a right angle. Again we can speak of an *exemplaric use*. The meaning of the word “right angle” is connected to the examples on the blackboard. Now and again, it seems that the meaning of “right angle” is “taken-to-be-shared”, but the students do not yet express characteristics of right angles, they only have examples.

Now the scene is going on:

Tim: Ah, this corner which is coming from the right side (marks the angle with the teachers’ sign)

Teacher: Correct! Just make it a little bit thicker, so that the other ones can see it.

Tim: This is a left angle. (points at the opposite side of the vertical line)

Teacher: No!

Lisa: That is always a right angle.

Tim recognizes the examples as examples for the use of the word “right angle”. He explains why John’s example can be called “a right angle”. Thus, he abstracts from the concrete example and presents a use of the word “right angle” by a kind of definition: The word “right angle” can be used, if a line for the angle comes from the right side. Tim tries to give an *explicit-definitional use* (cf. Winter, 1983) of the word: The student describes a general characteristic when and how the word “right angle” has to be used. He relates the word “right angle“ to other words. Contrarily to the former use of the word “right angle“, Tim uses another ethnomethod to constitute meaning.

The concept of the word “left angle” is used by an implicit reference. It is implicit, because the pair of concepts “left-right” indicates that an orientation in space is considered – a relation between observer and object. Thus, the word “left angle” gets an *implicit-definitional use*. The exemplaric use Tim makes of the word “left angle” can be seen as a test of his proposal. It is a probable consequence of his first definition. In other words: It is a hypothetic-deductive approach of verification (cf. Meyer, 2008).

Tim’s use of the word “right angle” can be explained only because there is use of the word “right” in common practice. Here the word “right” can be used to show a certain relation between observer and object. So Tim was able to combine the two uses of the words “right” and “angle” to establish a constructive meaning of the conglomerated word “right angle”. The comment of the teacher harshly shows that the new language-game is not acceptable.

Tim’s use shows that the former meaning of the word “right-angle“ only seemed(!) to be “taken-as-shared”. It has not been shared. Tim has been trying to give a theoretical fixation of the concept. The language-game he initiated is not an acceptable one. Lisa does not take part in the new language-game. She seems to play the former game and to explicate a routine: We need to have more examples to grasp the meaning of the word “right angle”.

FINAL REMARKS

The episode shows that de Saussure’s model is not sufficient to analyse classroom communication. Mathematical concepts are in need of a fixation by other concepts (a theoretical fixation). An empirical way can be used to introduce words, but the language-game has to change afterwards. In this scene a student initiates another language-game, which is condensing in (not acceptable) theorems.

The use of Wittgenstein’s theory shows that concepts can be observed by looking at the way teacher and students make use of the words at hand in the specific language-game. In this scene we have seen an *exemplaric*, an *explicit-definitional* and an *implicit-definitional* use. The exemplaric use consists of pointing at examples to illustrate the words. The explicit-definitional use consists in giving an explanation for the word in relation to other concepts. Thus, it provides a deeper insight in mathematical coherences: Characteristics of the underlying concept get expressed. The concept gets a general character, not being linked to special examples any more. An explicit-definitional use is also in need of a deeper mathematical insight, as it has to be known what counts as a definition. The implicit-definitional use in this scene requires a common pair of concepts ("left-right") and an explicit-definitional use of the other word.

Wittgenstein's theory itself is not a theory of interpretation. Rather he presents a theoretical framework, which can be used on top of a theory of interpretation. Symbolic interactionism and ethnomethodology fit to Wittgenstein's considerations of social processes in languaging. Future analyses have to show the fruitfulness of this framework.

NOTES

1. " 'The meaning of a word is what is explained by the explanation of the meaning.' I.e.: if you want to understand the use of the word 'meaning', look for what are called 'explanations of meaning'." (Wittgenstein, PI, §560).

2. As proposed by Bauersfeld (1995) I will speak of "languaging" to accentuate the connotation of language use.

3. Many thanks to Johannes Doroschewski and Philipp Heidgen for the video. The translation has been done and simplified by the author of this article. The original transcript will be sent on demand.

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SPEAKING OF MATHEMATICS – MATHEMATICS, EVERY-DAY LIFE AND EDUCATIONAL MATHEMATICS DISCOURSE

Eva Riesbeck

Department of Behavioural Sciences and Learning

The aim of this paper is to describe and analyze how discourse as a theoretical and didactical concept can help in advancing knowledge about the teaching of mathematics in school. The collection of empirical data was made up of video and audio tape recordings of the interaction of teachers and pupils in mathematics classrooms when they deal with problem-solving tasks. Discourse analysis was used as a tool to shed light upon how pupils learn and develop understanding of mathematics. The results underline that a specific and precise dialogue can contribute towards teachers' and pupils' conscious participation in the learning process. Teachers and pupils can construct a meta-language leading to new knowledge and new learning in mathematics.

INTRODUCTION AND AIM OF THE STUDY

This research deals with teachers and pupils discussing with each other in different situations within and about mathematics in school. The theoretical point of departure is first and foremost an in-depth study of the meaning of and relationships between concepts, words and signs in order to demonstrate how mathematical discussions can be understood. The concepts of context, mediation and artefacts are central to the socio-cultural perspective chosen and thus play an important role in this research, (Vygotsky, 1978, 1934/1986, 2004). The concept of context can be described as being the environment where our actions take place and thus create and re-create the context as such. Mediation implies that human beings interact with external tools in their perception of the world around them. Linguistic as well as physical artefacts are created by mankind to perform actions and solve problems. They are cultural resources which contribute towards maintaining and developing knowledge and abilities in society (Vygotsky, 1978, 1986). Using semiotic tools one can demonstrate how a linguistic element is connected to its meaning, (Ogden and Richards, 1923; Melin-Olsen, 1984; Johnsen-Hoines, 2002). We can picture a semiotic triangle made up of concept, expression and reference. If we look upon language as a medium for communication based on conventional signs it is by applying language that the reference to the world at large is created.

The relationship between thought and symbol is, like the one between thought and reference causal and direct in a semiotic triangle. The relationship between symbol and reference, on the other hand, is indirect and attributed. Concepts within a socio-cultural perspective which may be applied to the semiotic triangle are expression, content and reference. These three functions of a sign can only be understood when they are applied simultaneously. Thus we can see signs such as words, numbers, symbols, diagrams, equations and letters. The sign expresses something separate from the sign itself. Signs, objects are related to the meaning or conception of them. Mathematical knowledge must be actively constructed in relationship to signs, words and symbols.

I have chosen to describe mathematical discussions out of a discourse perspective. The concept of discourse can be understood in different ways. It can be interpreted as a set of conventional rules for discussing, understanding and conceiving the world and its different phenomena (Winther-Jörgensen & Phillips, 2000; Sfard, 2002). A discourse can be understood as a linguistic system which delineates issues of exclusion and inclusion, borders on what is excluded and inner standardization (Gee, 2005; Börjesson & Palmblad, 2007).

Foucault (1972/2002) wants to clarify how we are caught up in and blinded by lines of reasoning without really being conscious of what we say. We can refer to this as an invisible discourse. In the discourse on teaching mathematics there is an invisible element which is difficult to affect unless we make ourselves aware of its existence.

From a socio-cultural perspective discourse is defined as the language which gives and is attributed meaning in various contexts and which excludes and includes things to be understood (Säljö, 1999, 2000). In this work I have chosen to metaphorically regard discourse as a network where signs, concepts and references make up the nodes. Nets can be chosen or created in such a way that meaning is constructed in situated action as well as socio-cultural practices which transgress defined situations. Thus, a discourse can also be a set of rules for talking, writing and thinking about a specific content. Many discourses are mixed in school which both teachers and pupils must learn to become involved in, understand and master. This includes knowing when borders between different discourses are crossed. Mathematical instruction means that teachers and pupils are placed in different discourses, ranging from those applied to every-day life to purely mathematical ones. This means that they move over borders and between registers all the time. An example of this occurs when pupils work with concrete materials and are to express themselves using numbers and symbols. In doing so, they will move over different borders. When working in school we must learn to understand when we are situated in a specific discourse.

A mathematics lesson contains a number of words and expressions from every-day life. The language applied is rich and we talk departing from many different perspectives and towards many different aims. To be able to conduct conversations in a context as specific as school mathematics we have to develop a meta-language which makes it possible to put what we want to express into perspective. In every-day life we build models in order to understand reality and we use every-day methods for solving problems in order to describe connections to mathematics. We seek the history of mathematics to be able to see how every-day application developed into pure mathematics. This paper mirrors how teachers and pupils apply different types of discussions to deal with problem-solving tasks in and about mathematics. In these discussions we develop our thinking and our methods for learning and it is in the same discussions that we shed light on the transitions required in order to move from concrete to abstract activities. A knowledge rendered in linguistic terms is required. This is something that I aim to disclose in my empirical studies. In the discussions in and about school mathematics an oscillating movement between reality and mathematical concepts and expressions is to be seen.

Communication in a mathematics classroom can be described in terms of learning a mathematical register, (Duval, 2006). It can also be looked upon as a situation where there are two parties involved – two individuals who speak, think, write, read and listen. It is therefore highly interesting to study what learners and teachers have to say in and about mathematical practices.

The over-riding aim here is to raise this issue: “How can discourse as a theoretical and didactical concept contribute towards further developing mathematical teaching?”

Method

I have for many years been interested in communication and interaction within and about mathematical teaching. In my studies I have chosen to monitor how teachers and pupils have generated knowledge in discussions on mathematical concepts, problem-solving and formal mathematics. I did so in order to be able to establish what happens in interaction between teachers and pupils and between pupils.

In these studies I have made use of video and audio recordings. Video recordings were applied in order to make sure that it became clearly visible what went on in the interaction within a classroom. It also proved to be fruitful in that the activities on both teachers’ and pupils’ part became evident. The audio recordings were used as a means of analyzing the discussions as interactive situations. Group interviews are a well-chosen strategy for trying to capture discourse as regards what they include and exclude. The table below describes the environment used to acquire data in the respective studies.

Design of the Empirical Studies

Study I	**Study II**	**Study III**	**Study IV**
7 teachers	14 groups	26 groups	68 groups
Teacher-pupil interaction Classroom	Pupil interaction Classroom	Group talk Three pupils	Group talk Three pupils
Video	Video	Audio tapes	Audio tapes
Mathematical content			
The Area of a Triangle	The Area of a Triangle	Problem-solving	Rational Numbers

Table 1. Data acquisition in the empirical studies I-IV

Seven teachers took part in my first study. They were assigned to plan and carry out an introductory lesson on the area of the triangle in year 5 in compulsory school. Choosing mathematical content was a regular concept to the teachers who took part. Focus for these video recordings lay on documenting the public and the teacher-led interaction in the classrooms involved. Each recording lasted between forty and sixty minutes. Twenty-five occasions were recorded and focused on interaction between teachers and pupils. The study further describes how teachers cross discourse borders in teaching on the area of the triangle and in what ways they carried out their lessons as regards interaction between teachers and pupils, as well as what types of questions they used in their talks with pupils.

The introductory lesson on the area of the triangle is carried on into this second study but here focus is on pupils' interaction in a laboratory situation, where the teacher gives explicit directives to the groups of pupils. Varying directives from the teacher in the classroom lead to different trains of action and linguistic concepts on the pupils' part. In total the interaction of fourteen groups has been recorded and analyzed in the classes involved. The groups were made up of five to six pupils. The laboratory situations are described as regards activity and linguistic interaction. The pupils are active in that they draw, cut and fold pieces of paper. Every-day language is used to a great extent and retains its every-day character.

The point of departure for the third study was to monitor 26 groups of pupils when they set about a written mathematical task. The task is of an open variety and contains different pieces of information that the pupils are to decide on. One of the concepts which stay in focus for the pupils is the word fairness. Pupils seek, talk, make guesses, test and calculate an answer. There is, however, no evident way to go about solving the task. On the one hand the pupils end up in an every-day discourse and on the other hand in a mathematical discourse. They have difficulties making judgments as they reason with each other. Each group has been recorded on audio tape which has then been transcribed and analyzed. The pupils were put into groups on the basis of their mathematical skills as deemed by their teachers. The recordings took place in a small room next to the classroom.

For the fourth study one of the assignments from the National Test of mathematics for year five was used. The assignment deals with rational numbers. Five different partial studies were carried out. Sixty-eight groups of three pupils each and 120 individual pupils took part in the different studies. The first partial study was carried out with 30 pupils in year five who solved the assignment on their own and were asked to provide a written explanation. The second study took place in three classes of 30 pupils each. For the third partial study I used five schools from different parts of a large municipality. Thirty-one group interviews with pupils in year five were carried out, each group consisting of three pupils. When the pupils solve their assignment they rely on an every-day discourse. The next study involved 31 new groups of pupils. They were allowed to use a pocket calculator and they engaged in a solely mathematical discourse. The last part of this study was carried out with six groups of three pupils each and it deals with the issue of reasoning with the help of a numerical line. The results show that, depending on what tools are applied and what situation the pupils are in, the outcome turns out differently in different discourses.

I have used a discourse analysis to analyse the group discussions and the discussions in the classroom, (Wertsch, 1985, 1998; Kozulin, 1998; Fairclough, 1992, 1995, Gee, 2005). A discourse analysis is based on details in what is written and said in a particular situation. In the restricted discourse language can be seen as “language-in-action” which is always an active process in constructing knowledge. My study focuses on the interaction between individuals and in what ways knowledge, language and mathematical skills develop.

Results

Discourse analysis can be used as a tool with help of which descriptions of how pupils learn and develop their understanding of mathematics can be made clear. Looking at my empirical material I have come to discern the discourse in school mathematics which can provide the bridge upon which teachers and pupils can meet and become mutually involved.

In school mathematics teachers and pupils talk using every-day concepts and mathematical concepts, signs and words. This intercourse demands that a mutual understanding takes place. The analysis of what is said in the different groups shows that the discussions are situated somewhere on a scale between two extremes – on the one hand every-day concepts, on the other hand purely mathematical concepts. Words such as “put on” and “put together” are based in every-day practice whereas words such as “add/addition” and complex numbers are situated in a purely mathematical discourse. Any individual is to be found somewhere in this continuum depending on how far this individual has come in the process of developing an understanding of abstract reasoning. If we consider signs and expressions the same thing can be said for them.

In my empirical data where teachers talk to pupils in whole-class discussions and in group talks, teachers utilize different signs and change registers in their teaching. They go from geometrical into arithmetical/algebraic discourse and back. Analysis of these talks clearly reveals how pupils talk about and understand the concepts. Most pupils use every-day language and it demonstrates that teachers are situated in one discourse and pupils in another. The same thing can be seen when pupils work with concrete materials, performing acts but not acquiring the mathematical concepts which the teacher had planned. Pupils find themselves in a distanced discourse rather than an inclusive one as the teacher had intended. In one of my excerpts the pupils are engage in a group discussion of how to move from a rectangle made of red paper to a triangle. The teacher has told the pupils to prove that the triangle’s square is half of the rectangle. Here we can follow their discussion:

Måns: Mine is so smeary. Nobody can think about that it is so smeary.

Kalle: We can fix this so it will be the half.

Beatrice: It’ll be a square.

Stina: Do you know how to fold all pieces of papers. I can’t fold anything.

Måns: You can learn how to fold if you know how to fold.

Kalle: The fundamental form to fold frogs, but I can’t, they don’t jump like this.

Stina: I can fold aeroplanes.

Here you can see pupils being in an every-day and distanced discourse. They try to follow the teacher’s goal to prove but they got into another discourse.

In another assignment of a problem-solving character about decimals the pupils first had to work with an every-day picture as a point of departure and their talks are thus carried out in an every-day discourse. Some pupils do not arrive at the mathematical terms and an understanding of them. Other groups are given a formal assignment to be solved using a pocket calculator and they remain there, locked up in the system of signs and decimals. Yet another group of pupils draw lines together in order to understand the decimals and can accommodate the mathematical signs and words, which makes them involved in the discussion and solving of the assignment. They start to speak, think and write “Mathematish”.

I: Now I want you to explain why you think that this is right.

H: Nine is a whole number, it’s one smaller, only a whole number. 9,12 is nine whole and one tenth and two hundredths, I think, 9,2 then there is nine whole number and two tenths.

E: Nine is such a whole one. 9,12 there is a tenth smaller than two tenths so then 9,2 will be bigger than 9,12.

N: Nine is a whole number the second number in 9,12 is a hundredth and 9,2 the second is a tenth.

The connections are created between every-day references and mathematical concepts and expressions and it becomes easier for pupils to leave the idea of “doing”. Meaning has been attributed to mathematical concepts and signs and these have been created for defined ends. But the meaning can only be understood by those who are able to take part in a mathematical discourse.

By analyzing how teachers and pupils talk about mathematical phenomena in different situations I can use the concept of discourse to establish that connections are often not created between every-day concepts and their mathematical counterparts. If pupils cannot interact and thus form networks of concepts which assist them on their path to conscious mathematical thinking this becomes a major problem for them. Consequently teachers and pupils must develop their mathematical language in concord with every-day language.

Discourse analysis can thus be used as a tool where descriptions of pupils’ learning processes and understanding of mathematics can be made clear. I have displayed the results of my documented discussions and will place discourse in focus and further develop it as a means of establishing a direction.

Discussion

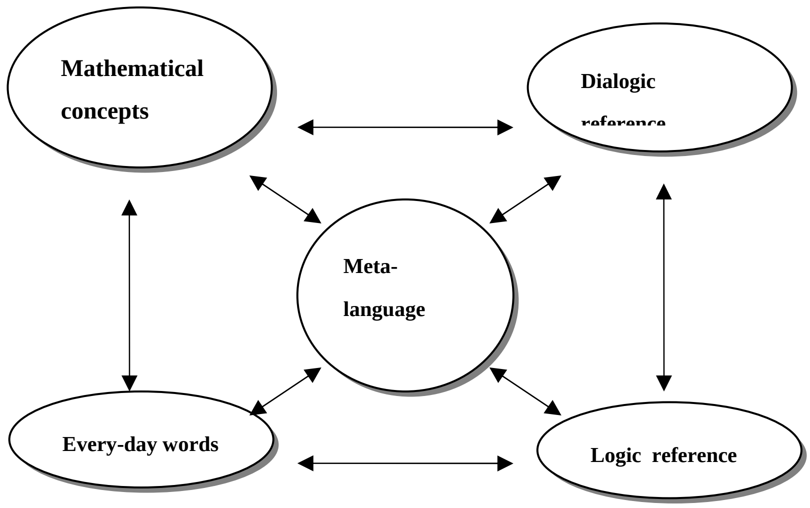
If the discourse is viewed as a distinct means of establishing the direction for teaching mathematics, it becomes the teacher’s task to bring to a conscious level the different ways pupils use for passing borders between different discourses, so that pupils become aware of the nature of mathematical concepts. A discourse is made up of artefacts and products created by mankind for specific ends and the language used can be understood only if the discourse itself is understood (Säljö, 2005). Teaching should invite pupils to become participants in a mathematical discourse.

The words *speak*, *think* and *write* can be viewed as parts of a discourse and when teachers and pupils apply them in the teaching and learning process, it can reinforce consciousness and participation in mathematical thinking. This could constitute the formative discourse. Furthermore, teachers and pupils must learn to realize what is changed when going from one discourse to another in mathematics. To be able to discern whether the discussion is carried out in an every-day or a mathematical discourse, to be able to recognize whether one is situated in a geometrical or an algebraic discourse and how the movement between registers manifests itself in mathematics is important knowledge for teachers, student teachers and pupils. When an individual speaks the way language is applied can develop qualitatively by the process of learning to value, scrutinize and put forth arguments in both every-day and mathematical discourses. In these, thinking is developed and by using linguistic and concrete artefacts in interplay thinking is further prompted. We can thus create a connection between every-day life and mathematics. Since mathematics started in a culture which used conventional signs and written language it has also developed texts and thus reading is a part of mathematics. The concepts of listening and reading should also be entered into the discourse, leading onto the concept of interpreting. In this perspective pupils will actively form and interpret their knowledge.

Discourse can be defined as a "way of speech" but I would prefer to widen the definition in so much that I view discourse as a network where teachers and pupils acquire knowledge by moving between and utilizing mathematical and every-day concepts, expressions and situations by talking, thinking, writing, listening and reading.

It has been my ambition to put the concept of discourse into perspective in the following manner. By adopting a discourse perspective we can direct attention to linguistic dimensions of mathematics teaching. It would also assist us in letting individual, silent calculation interact with a communicative aspect. By formulating and interpreting their mathematical knowledge pupils can acquire new knowledge. We will create a recognizing nearness through experience and distancing, fostered in a development and a familiarity with the system of mathematical signs. Through quality in the discussions which arise in a learning process we can develop the language concerned and thus improve understanding. In this context quality means that teachers and pupils use words, signs, concepts and situations in awareness of the specific discourse. We should also keep in mind that a mathematical discourse is something that develops over time.

Current research presents many images of the existent situation – "this is what it is like". My discourse perspective, however, focuses possible changes. I want to present a discourse theory which recognizes qualities in language and knowledge from both the every-day world and the mathematical sphere and in doing so clarifies both every-day and mathematical concepts. In this context quality means that we communicate around a concept, a sign, a reference and a situation by looking critically at it, putting forth arguments for and against, and eventually arriving at understanding what I take with me from this learning process. It is absolutely clear that the further our acquisition of new knowledge develops into an issue of learning to apply abstract and complex intellectual and practical tools, the more essential it becomes to engage in communicative practices. Thus we can learn how to apply and co-ordinate these tools, both linguistic and physical, with an outside world to reach new mathematical knowledge. Models and symbolic representations can be tested critically as regards their connections to the every-day world and other concepts as well as their logical consequence and explanatory value. The table below reinforces discourse as a theoretical and didactical concept.



Model describing the passing of borders between discourses.

By placing focus in learning processes on the concept of discourse our teachers and pupils can grow to master a meta-language for school mathematics. This will then constitute a specific and precise language in and about mathematics. Language is constructed in our actions and how we express ourselves using the appropriate signs. By putting forth arguments and making interpretations in a dialogical environment we can acquire knowledge as regards knowing when borders between discourses are passed, as well as regarding the interplay between thought and experience in mathematics.

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COMMUNICATIVE POSITIONINGS AS IDENTIFICATIONS IN MATHEMATICS TEACHER EDUCATION

Hans Jørgen Braathe

Oslo University College, Norway

Student teachers positioning related to own emotions and experiences, the mathematics and the teaching and learning of mathematics, and the classroom, teachers and others are theorised, and exemplified, as aspects of identifications as becoming mathematics teachers.

INTRODUCTION

As a teacher educator I have searched for signs of how the student teachers in the pre-service mathematics courses change from seeing themselves as students of mathematics to seeing themselves as teachers of mathematics. That is negotiating identities as mathematics teachers.

Teaching is not a knowledge base, it is an action, and teacher knowledge is only useful to the extent that it interacts productively and dynamically with all of the different variables in teaching. Therefore connecting the act of teaching and teacher identities focuses on identities as something people do which is embedded in social activities, and not something they are.

Identifications as teachers of mathematics, through acting, or performing, as teachers in mathematics, are closely associated with meaning making in mathematical contexts. In this paper I will outline descriptive devices in order to analyse the properties in texts and the technical skills of mathematical communication that are employed in the service of mobilizing teacher identities by student teachers.

Dewey (1916) examined the purpose of education in a democratic society. He writes: "society not only continues to exist *by* transmission, *by* communication, but it may fairly be said to exist *in* communication" (p. 4, emphasis in original). He further holds that "This transmission occurs by means of communication of habits of *doing, thinking* and *feeling* from the older to the younger" (p. 3, emphasis added by Ongstad 2006).

Conceiving teachers' knowledge as part of a complex set of interactions involving *action, cognition and affect*, places teaching as a complex practice. A main perspective then is a view of teaching and learning as communication (Braathe, 2007; 2009; Ongstad, 2006; Sfard, 2008).

POSITIONING THEORY

"Positioning Theory" has been discussed and developed among others by Harré and van Langenhove (1999). Their concept of positioning is offered as a dynamic replacement of the more static concept of role. Role identity theory views society as made up of roles, and explains how roles are internalised, as cognitive schemes, as identities that people enact and try to live up to (Stryker and Burke, 2000). "'Position' will be offered as the immanentist replacement […] of transcendentalist concepts like 'role'" (Harré and van Langenhove, 1999, p. 33).

Harré argues that during communicative interactions, people use narratives, or "storylines", to make their words and actions meaningful to themselves and others. They can be thought of as presenting themselves as actors in a drama, with different parts or "positions" assigned by the various participants. Positions made available in this way are not fixed, but fluid, and may change from one moment to the next, depending on the storylines through which the various participants make meaning of the interaction.

> In positioning theory, the concept of positioning is introduced as a metaphor to enable an investigator to grasp how persons are 'located' […] as […] participants in jointly produced storylines.
>
> One mode of positioning of particular interest to us […] is the intentional self-positioning in which a person express his/her personal identity (Harré and van Langenhove, 1999, pp. 61-62).

IDENTITIES

Identities have been used as a strategic concept in research addressing the relationship between individuals and society, and, related to this, in formulating how selves are socially constituted, and in explaining how social structures or processes affect individuals' lives.

The kind of questions asked in traditional social science are *what* identities people have, what *criteria* distinguish identities from each other, and what *part* identities play in the maintenance of society and in enabling the functioning of social structures and institutions. In this respect social identities are assumed to have an overarching relevance (Stryker and Burke, 2000).

Underlying most of these approaches, whether sociological or social psychological, are concepts of identities that can be characterised as essentialist and realist. The concepts are essentialist in the sense that identities are taken to be properties of individuals or society; and realist in the sense that it is assumed that there is some kind of correspondence between identities and some aspects of social reality (Sfard and Prusak, 2005).

Across the social sciences, the main criticism of, and alternatives to, traditional models of identities are found in a variety of social constructionist approaches. The concept of identity produced is designed in part to deal with variability and flexibility and how even the most obvious identities are negotiable. Although they are various, these approaches share in common an emphasis on the multiple ways that social identities are constructed, negotiated and performed. Contrary to the use of identity for the purpose of classification, or as a causal variable related to other phenomena, this view of identities, it is argued, enables a social constructionist to provide a more dynamic view of individual-social relations.

A social constructionist approach also draws on the idea that symbolic or cultural resources influence identities, and how identities are constructed through historical, political, cultural and discursive practices. It is argued that the symbolic or linguistic resources available in the discourses provide possibilities and constraints on identities individuals can take. Methodologically this is used empirically to identify the linguistic resources or repertoire available in a culture for individuals to construct their self-understanding. In other words, they aim to show how cultural narratives become a set of personalised voices and positions. This offers alternative ‘texts of identities’.

IDENTIFICATIONS

The positioning theory developed by Harré and van Langenhove (1999) is based on social constructionism. They see positioning in terms of a triad of interrelated concepts: storyline, positions and actions/acts. The storyline is the narrative that is being acted out in the metaphorical drama. Within it, the positions are the parts being performed by the participants. The actions of the participants are given meaning by the storyline and the positions available, and once given meaning become social acts. This positioning can be seen as interactors identifying themselves as actors, and being identified by others, in a metaphorical drama.

The focus on identifications as a participant’s resources generates different questions and a different focus. Thus, instead of asking what identities people have, the focus is on *whether*, *when* and *how* identities are used in social acts, for example performing as teachers of mathematics.

In their pre-service teacher education student teachers have to produce texts answering different tasks and reporting from group works and from practicing teaching in practice schools. Text in this connection will also include mathematical text. These texts can be seen as utterances in a dialogic relation to their teachers in the teacher education, or as social acts within the storylines of mathematics teacher education. These social acts are seen as positionings, or identifications as becoming teachers of mathematics.

I investigate student teachers’ identifications relative to the three aspects of action, cognition and affect. Instead of methodologically trying to identify available positions in these storylines as categories following a social constructionist methodology, I will use another related dynamic concept of communicative positioning derived from Bakhtinian thinking searching for these three aspects. This concept of positioning is used as an analytic tool to analyse the student teachers texts as they are seen as struggling for making meaning of teaching and learning of mathematics.

POSITIONING AS A TRIADIC DISCURSIVE CONCEPT

The communicative positioning developed and used by Ongstad (2006) is partly generated from Bakhtin's essay "The problem of speech genres" (Bakhtin, 1986, pp. 60-102). Ongstad identifies Bakhtin's communicative elements necessary for an utterance to communicate in dialogic relations. One of these is how the utterance is positioning, and positioned, as such by addressing someone, referring a semantic content, and expressing feelings and intentions.

Methodologically the utterance is seen as the unit of analysis. We communicate through utterances. Utterances are any sufficiently closed use of sign that makes sense. All utterances are uttered and interpreted related to expectations of genres, i.e. contexts that helps us to understand the utterance. Genres are ideological, i.e. they give tacit premises for the utterances' positioning in the communication (Bakthin, 1986). Ideology is broadly defined as unspoken premises for communication (Braathe and Ongstad, 2001). It is something we think *from*, not on. Genres can be described as kinds of communication.

The genres are to be seen as triadic in the same sense as the positioning of the utterance, that they simultaneously give potential for the addressing, referring and the expressing. The three aspects are seen as parallel, inseparable, reciprocal, simultaneous processes (Ongstad, 2006).

In the mathematics teacher education context the three aspects are seen as positioning related to addressing the classroom, teachers and others, referring the mathematics and the teaching and learning of mathematics, and expressing own emotions and experiences. Students' different texts relate to different components of teacher education. Consequently they are positioned differently with dominance either on the expressive, referential or the addressive aspect. However, as utterances, all three aspects are simultaneously present, and consequently identifying the student as becoming teacher of mathematics related to all three aspects. This identifying process focuses identities as something the student teachers do, as communicative positioning, which is embedded in the social activity of teacher education.

MATHEMATICS AS GENRES

Seeing mathematics and mathematics education as a kind of communication will be to see mathematics and mathematics education as genres. I will hold the view that in their pre-service training student teachers are parts of different genres, kinds of communication, including mathematical, and potentially experiencing different ways to act as a teacher. It is helpful to call this process 'learning'. This will theoretically be connected to seeing learning as semiosis in the field of teaching mathematics. This connects to seeing learning as communication. This shifts seeing development from a psychological to a semiotic perspective so as to locate developmental principles in the making of meanings. As I see learning, or developing of identities, as being positioned in communicational genres, I locate identities as dialogically situated in, negotiated and formed by genres, and so can have many expressions dependent on the context. Identity can then be seen dynamically combining the personal, the cultural and the social (Braathe, 2007).

Sfard (2002; 2008) takes a similar "communicational approach to cognition" (2002, p.26), where she holds that "[t]hinking may be conceptualised as a case of communication" (2002, p. 26), and even constructs the concept of "commognition" (2008, p. 296) to emphasise the necessary connection between the two. She further holds that "[l]earning mathematics may […] be defined as an initiation to mathematical discourse, that is, initiating to a special form of communication known as mathematical" (2002, p. 28).

Furthermore Sfard holds that "[c]ommunication may be defined as a person's attempt to make an interlocutor *act, think or feel* according to her intentions" (Sfard, 2002, p. 27, emphasis by me). Discussing factors that give discourses their distinct identities Sfard identifies meta-discursive rules as

> usually not something the interlocutors would be fully aware of, or would follow consciously, […] there are special sets of meta-rules involved in regulating interlocutors' mutual positioning and shaping their identities (ibid. p. 30-31).

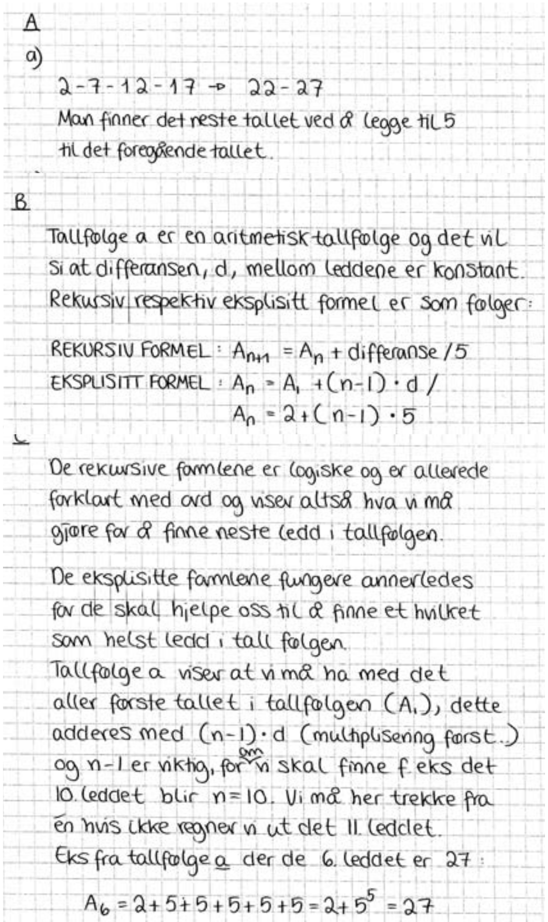
TELLING IDENTITIES

In Braathe (2007) I discuss the theoretical framework presented in Holland et al (1998), especially their use of the Bakhtinian diverted concept of "the authoring self". I relate this Bakthinian concept to Sfard and Prusak (2005) and their conception of identity (Braathe, 2009). They define identities as stories about persons. In a communicative and dialogic sense they adhere to that "[i]dentity […] is thought of as man-made and as constantly created and re-created in interactions between people" (Sfard and Prusak, 2005, p.15). Stories about persons, the term identifying, is in their context to be understood as "the activity in which one uses common resources to create a unique, individually tailored combination" (p. 14). From seeing the processes of identifying as discursive activities, the activities of communication, they suggest that "identities may be defined as collections of stories about persons or, more specifically, as those stories about individuals that are *reified, endorsable and significant*" (2005, p. 16, emphasis in original). This definition is an attempt to avoid the problem of essentialism, the extra-discursive existence that often is either implicit or explicit in the use of the concept of identity in educational research.

Discursive acts of positioning, identifying, are seen in my context as communicative acts for establishing meaning. In the teacher education students' produced texts can be seen as utterances that communicatively position the student teacher dynamically combining the personal, the cultural and the social.

These texts/stories are not about persons, but about the explorative mathematics activities in their pre-service training, where the students have to explain mathematical patterns, connections and reasoning. These texts are seen as utterances in the genres of teacher education, told by the students of "themselves" to their teacher. Sfard and Prusak (2005) call these stories the student teacher's first-person identity. On the other hand my analysis of positioning of these texts will be called stories about stories. These stories about stories can also be seen as the student teacher's third-person identity told by me as the researcher. In teacher education the resources, voices, used by the student teacher when writing in the different genres of mathematics educational texts, are found in dialog both with practice, theory and experience, and as such seen as influencing the negotiation of their semiotic identifications as teachers of mathematics.

The analysis of positioning, applying the triadic discursive concept to these texts, explores how the students position themselves in relation to 1) own emotions and experiences, 2) the mathematics and the teaching and learning of mathematics and 3) the classroom, teachers and others.



A

a)

2-7-12-17 → 22-27

Man finner det neste tallet ved å legge til 5 til det foregående tallet.

B

Tallfølge a er en aritmetisk tallfølge og det vil si at differansen, d, mellom leddene er konstant. Rekursiv respektiv eksplisitt formel er som følger:

REKURSIV FORMEL: $A_{n+1} = A_n$ + differanse /5

EKSPLISITT FORMEL: $A_n = A_1 + (n-1)\cdot d$ /

$A_n = 2+(n-1)\cdot 5$

c

De rekursive formlene er logiske og er allerede forklart med ord og viser altså hva vi må gjøre for å finne neste ledd i tallfølgen.

De eksplisitte formlene fungere annerledes for de skal hjelpe oss til å finne et hvilket som helst ledd i tall følgen.

Tallfølge a viser at vi må ha med det aller første tallet i tallfølgen (A_1), dette adderes med $(n-1)\cdot d$ (multiplisering først.) og n-1 er viktig, for om vi skal finne f eks det 10. leddet blir n=10. Vi må her trekke fra én hvis ikke regner vi ut det 11. leddet.

Eks fra tallfølge a der de 6. leddet er 27:

$A_6 = 2+5+5+5+5+5 = 2+5^5 = 27$

Analysis of positioning

To illustrate the analytical tool, I give a short extract of a text produced by a student teacher. The text is translated into English by me.

The student teacher, Ina, is solving a task on finding and describing the pattern of a given number sequence. This text is produced in her second semester in her teacher training.

The number sequence is given: 2, 7, 12, 17,….

The student teacher is asked to:

A: Find the next two numbers in the sequences.

B: Find the recursive and the explicit formulae for the sequences.

C: Explain why the formulae are correct.

The written text in A is:

a) One finds the next number by adding 5 to the previous number.

In B: The number sequence a is an arithmetic sequence and that means that the difference, d, between the terms is constant. Recursive respectively explicit formulae are as follows:

In C: The recursive formulae are logical and are already explained in words and shows what we must do to find the next term in the number sequence.

The explicit formulae functions differently because they shall help us to find any term in the number sequence.

> The number sequence a shows that we must include the first number in the number sequence (A_1), this is added to (n-1)·d (multiplication first..) and n-1 is important, because if we shall find f. ex. the 10. term then n=10. Here we must subtract one if not we are calculating the 11. term.
>
> Ex from the number sequence a where the 6. term is 27:

The expressive aspects of utterances are related to form and what this form symptomatically can express. One can read how Ina uses the arrow connecting the next two numbers in a) either as a (rough) draft she does to help her own thinking, and/or it can be read as a communicative utterance where she explains how the next number in the sequence is constructed. In both cases Ina uses an informal, illustrative, nearly oral, genre. The written text in a) is referring to an impersonal “one”, which is quite familiar in mathematical texts in textbooks. We can read it as a “rule giving” genre; written in an impersonal voice, in present tense and in general terms (it is about “the next number”).

In B Ina lists the two formulae. In her writing of the recursive formula she writes /5 to indicate that the difference is 5 in this case. The / is kept in the explicit formula, but “difference” is replaced with the variable d. This form may be a symptom of insecurity in the mathematical terminology. It could be read as if the difference in meaning, expressed with written symbols, is not quite clear to her yet. In both cases, writing formulae, she is writing in what can be identified as from a technical genre, as in her mathematics textbooks. Ina seems to have grasped the ideas, but I read this as she has not yet acquired the genre as a cultural tool, and have difficulties in expressing these ideas in writing. This mix of genres could be seen as voices from her earlier school experiences and also from the lectures at the teacher college.

The referential aspects of the utterance are related to the mathematics in her text. She has got the answers correct. The notions of pattern and generalisation, in particular generalisation expressed in formulae, plays an important role both in the immediate context of situation through the instructions given in the statement of the task to “Find the […] formulae” and to “Explain why the formulae are correct” as well as through the assessment criteria and more generally through the genre of investigation in which ‘spotting’ and generalising patterns is highly valued.

Her explanation of the recursive formula refers to what she has written in a), and she uses ‘logical’ as a self-explaining argument. Both formulae are given an authority as mathematical objects that can perform activity. The recursive formula *“shows what we must do”*, and the explicit formula *“help us to find any term”*. However when Ina presents the process she is also including actors in addition to the mathematical objects, as inclusive *“we”* and *“us”* respectively. This is also expressed in: *“One finds the next number”*, *“The number sequence a shows that we must include”*, *“because if we shall find”*, *“Here we must subtract one if not we are calculating the 11. term”*. These actors can also be read as a general “one” or “we”, rather than specific persons. Thus, the process of varying values in the problem is not shown as something done by the author herself. It shifts from being a process that may be carried out by any mathematician, to a process performed by mathematical objects themselves or by some unspecified agent, and finally, using the grammatical metaphor of nominalization, to an object which may itself have properties and variations. This expression of agency in the utterance serves as construction of a picture of her mathematical world.

The addressive, or relational, aspects of the utterance are related to normativity, here in the sense of usefulness related to role of mathematics teacher in the primary school. Usefulness here includes ethical values concerning teaching and learning. Her explaining text in a) can be identified as "rule giving" genre within mathematics, and as such as part of the repertoire of the becoming teacher. In C she has included in brackets "*(multiplication first..)"*. This can be read as addressing the reader as a reminder of the rules for the priority of the numerical operations.

The normative claim can be understood as part of an instrumental view on teaching and learning mathematics. This can be seen as an element of Ina's experience and praxis as part of her stories of mathematics as a subject where she has to learn the rules, and where you have true or false answers. That is an ideology within the genres of teaching mathematics.

In the utterance Ina uses a mix of genres. However, one genre seems dominant, the "Explaining" or "Introduction" genre. This is demonstrated by the explicit formula in C as she is both explaining the general by an example and by the nearly tactile metaphor she uses in explaining the explicit formula. This is a genre which is frequently used in the mathematics texts in her study. Explaining by examples is used frequently both in educational texts and also in teaching sessions, both at the college and in the practice schools. One could see this as a sign on her appropriating the voices of mathematics educational genres. This appropriation, making meaning of mathematical communication, is seen as the negotiation of identity as becoming teacher of mathematics. This shifts seeing development from a psychological to a semiotic perspective so as to locate developmental principles in the making of meanings.

THEORIES FOR RESEARCHING TEACHERS IDENTITIES

In this paper I have presented Positioning Theory as Rom Harré and associates have developed it. Their concept of positioning has been interpreted as persons' identifications in a social psychological sense. From seeing teaching and learning as communication I have inserted a semiotic related concept of positioning based on Bakhtinian dialogism. This triadic discursive concept of positioning is then used as an analytic tool in analyzing identities according to the definition of identity proposed by Sfard and Prusak (2005). Here the utterance, as student's text, in the genre of mathematics teacher education is used as the unit of analysis.

I see development of identities as learning, and theoretically investigating negotiation of identities from a semiotic perspective, not a psychological one. Therefore I explain identifications exposed in student teachers' utterances as meanings within the genres, and the underlying ideologies, of teacher education. In the Norwegian mathematics classroom there are different ideologies simultaneously represented by different actors (Braathe and Ongstad, 2001). Essentially these are ideological conflicts within which the student teachers are struggling to create and negotiate their teacher identities. Going back to Dewey and seeing education as communication of doing, feeling and thinking from the older to the younger, has given me support for searching within theories of communication for a triadic understanding of learning to become mathematics teacher. Becoming a mathematics teacher includes building professional identities. This again includes knowledge of and identification with both mathematics and teaching and learning of mathematics.

The concern then is to focus on identities and the settings in which those can change, as a way of conceptualising mathematics teacher development as learning processes including the personal, the social and the cultural. Seeing development from a semiotic perspective, and learning as semiosis, all these aspects will have to be taken into consideration simultaneously.

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TEACHERS' COLLEGIAL REFLECTIONS OF THEIR OWN MATHEMATICS TEACHING PROCESSES

Part 1: An analytical tool for interpreting teachers` reflections

Kerstin Bräuning & Marcus Nührenbörger

University of Duisburg-Essen

Abstract. *The research presented in this paper offers a theoretical approach to the analysis of teachers' professional development by collegial reflection. The analysis of the reflections is applied to teaching episodes observed by videos and transcripts. The communication processes of constructing interactive mathematical knowledge with regard to develop together a more and more professional reflection of the student/ teacher mathematical interactions are seen here from a complementary perspective: (1) The construction process of an analytical tool for describing the reflection process of teachers; (2) The reflection process of mathematics teachers on the videos and transcripts of a diagnostic episode showing their own interviewing. This paper as the first of two papers concentrates on the first perspective.*

1. INTRODUCTION: THE RESEARCH PROJECT AND ESSENTIAL RESEARCH PERSPECTIVES

The presented research frame deals with discussion and results of the epistemological analyses of mathematical interactions in different social contexts (cf. Nührenbörger and Steinbring, 2009). In this article, we will concentrate on the development of teachers' professional learning by reflecting together their own teaching episodes. We will discuss an analytic tool for describing the reflection process with regard to a professional development of a more and more sensible interpretation and analysis of the students' mathematical interactions in the course of the teaching episodes observed. This research focus is one important element besides other research questions of two broader projects dealing with questions of the mathematical teaching and diagnosis of students' mathematical abilities in grades 1 and 2.

a. ›*M*athematics teaching in multi-*a*ge *l*earning groups – *i*nteraction and interventio*n*‹ (Malin). The question of this larger research report is: In which way do the teachers professional perspectives on their own role of teaching develop during the interactive lesson process with regard to the collegial reflections? For two years, eleven teachers from four elementary schools participate in the research project with their multi-age classes (grades 1 & 2). All teachers have been introduced to mathematics instruction in multi-age groups (cf. Nührenbörger and Pust, 2006). Each school year the partner work of two children (of different age) is video graphed in five lessons. The children work in pairs on open or structure-analogue tasks, which are supposed to permit an interaction and reflection from different points of view for both of them. After each term (four times over two years), the teachers of each school meet for a collegial reflection, in which video graphed episodes are watched out of their own instruction and analysed with the help of corresponding transcripts. The objects of their critical analyses are video episodes from their mathematical classroom that contain two types of mathematical communication in two different social contexts: “A short episode of two students interaction without the teacher's presence” and “A following short episode of the two students interaction with the teacher's participation”.

These interaction settings are taken as a productive opportunity for making sense of the students' processes of mathematical understanding within these two sub-settings and of constructing mathematical knowledge in view of their own interventions (cf. Nührenbörger and Steinbring, 2009).

b. “*Math*ematics tal*k*s with children – *i*ndividual *d*iagnosis and supporting” (MathKiD). The question of this research report is: In which way do the teachers` professional perspectives on their own role of talking with one child develop during a diagnostic interview by means of structured talks of reflections? For one year, five teachers from two elementary schools participate in the research project with their children (grade 1 or 2). All teachers have been introduced to diagnostic situations in mathematics instruction. In one year, the interaction between the teacher and one child of his class is video graphed about six times. The teacher and the child talk about “pure” math situations or playing situations with implemented math situations. They are supposed to permit diagnostic findings about the mathematics abilities of the child. In one year, the teachers of each school meet three times for a structured talk in which video graphed episodes out of their own diagnostic talks will be watched and analysed with the help of belonging transcripts and the intervention of a moderator (project leader). The objects of their critical analysis are video episodes from their diagnostic talks that contain interesting situations under three different analytic perspectives: “Analysing the understanding of the child”, “Analysing the intentions and actions of the teacher” and “Analysing the interactions between the teacher and the child.”

The cooperative reflection of mathematics teachers constitutes a practice-orientated discourse for constructing professional teacher knowledge. This research approach aiming at the analysis and reflection of the teachers’ *own* teaching activities in the course of their professional development differs from those approaches that offer exclusively theoretically elaborated patterns of teachers’ activities for reflection and imitation. The main focus of this paper is on the problem of developing an adequate tool for describing the process of collegial reflection with regard to the construction of a more professional knowledge for the learning and teaching process of mathematics. This leads directly to the research question of this contribution:

In which way teachers become aware of and understand carefully the students’ interactive mathematical interpretation processes in relation to their own intervention possibilities for stimulating students’ mathematical understanding processes?

In the last decades, research studies on mathematics teachers’ professional development have more and more emphasized the importance of video graphed episodes of mathematics teaching and interactions for sensitizing the teachers for their own teaching and talking activity in and about math (i.e. Maher, 2008; Benke et al., 2008). In this frame it is important to recognize that teaching itself is not a mere routine task of transferring more or less finished mathematical knowledge, which the teacher has prepared, to the students. Steinbring (2008, 372) points out that "school mathematics, as finished given knowledge, is not the *actual subject of teaching* in an unchanged way. Mathematical knowledge emerges and develops only in an effectively new and independent way within the instructional interaction with the students. Thus, finished, elaborated mathematics is not an independent input of the teacher into the teaching process which could then become an acquired output by means of students' elaboration processes."

During the process of teaching, the teachers are involved directly in the interaction with the student(s) and cannot play the role of a distanced observer of the events. The teacher has to draw directly a conclusion of the situation. "Normally, whenever we hear anything said we spring spontaneously to an immediate conclusion, namely, that the speaker is referring to what we should be referring to were we speaking the words ourselves. In some cases this interpretation may be correct; this will prove to be what he has referred to. But in most discussions which attempt greater subtleties than could be handled in a gesture language this will not be so" (Ogden & Richards, 1972, p. 15). But the development and change of the activity of teaching requires a critical consideration and thus a distance of ones own activity (cf. Krainer, 2003). Collegial reflections offer the teachers an "unusual" view of interaction processes. Possibly they will be irritated, they observe greater subtleties and thereby view the situation in another way (cf. Gellert 2003).

Otherwise one cannot see a typical dilemma of mathematical teaching routines: Mathematical teachers know, on the one side, of the importance of interactive learning processes during a learning environment, supporting the active-exploring work of students. But on the other side, the talk of the teachers during the teaching is affected by an attitude that mathematical knowledge is a *complete and clear product*, which can be developed directly by the students (cf. Steinbring, 2005). Hence, it might be the danger that teachers act on the assumption to support the students` learning processes with open learning environments. But due to the direct involvement in the mathematical teaching process, teachers tend to their personal views on knowledge. Their spontaneous work bases on own experiences and routines: Their talk to students is characterized by leading, funnelling and product-orientating, so the students have no choice to develop active own mathematical interpretations (cf. Bauersfeld, 1995). The teachers involved in the teaching process cannot see this dilemma. It is only noticeable in the distance and in a critical-reflected talk with colleagues observing by a video of their teaching. The distanced observation of a communication process in the classroom can highlight causal relations between the learning and teaching process. "During the common systematic reflection in a group of teachers about their own teaching processes with students thus emerges a further communication system, which again has to deal with the necessary interrelation between one's own consciousness and common communication. This communication now has communication processes as its subject and it is supposed to animate a professional consciousness" (Steinbring 2008, 379). However, the reflection of one's own activities that temporally separates from the teaching situation looks to future teaching activities. These future teaching processes can relate to the results of the distanced reflection (cf. Krainer 2003; Sherin and Han, 2004).

As a basis of professional teacher development we see an active, self-responsible and reflective elaboration of one's own practice with colleagues (cf. Altrichter, 2003, Krammer et al., 2006). „Systematic reflection on mathematical interactions that focus on the students' learning and understanding processes, as well as on one's own interaction behavior, represents an essential professional competence of teachers" (Scherer & Steinbring, 2006, p. 166, cf. Mason, 2002).

The growth of new insights refers to the active process of reflecting ones own teaching and learning. „If mathematics education is to be influenced in a positive way and ameliorated, the teachers have to be the ones who initiate these changes, and their reflection on their own activity is crucial" (Scherer and Steinbring, 2006, 165). Professional development needs to talk with the professional group about the own practice. In this sense, we mean with "collegial reflection" the common discussion and negotiation of teachers watching a video of a teaching episode and reading the transcript.

In this article, we will discuss the question, how the collegial reflections support teachers with the help of videos and transcripts to be sensitive to the power of the mathematical negotiating process of students: In which way teachers develop in the course of collegial reflections differentiated mathematical interpretations and interrelations? In which way teachers look to the possibilities to attend the students with open, mathematical focused and interactive orientated interventions?

2. THE DESIGN OF THE COLLEGIAL REFLECTIONS

In the context of the two research projects, the teachers take part on distanced collegial reflections of their own or of known (this means known lessons hold by colleagues) teaching lessons. In this sense, the projects do not focus on the imitation successful teaching and learning strategies. Both projects aim at the commonly constructed reflection of interaction processes with the focus on the understanding of the students' mathematical thinking, on the role of interaction for constructing mathematical knowledge, and on the patterns of the interactive teaching and learning process. The collegial reflection focuses on classroom cases (Malin-Project) or diagnostic talks (MathKiD-Project).

Teachers can be encouraged to reflect their own talking activities and to make conscious decisions by learning how to "read" and interpret a episode of talks in a complex classroom situation or in a diagnostic situation. In addition, the collegial reflection follows some guidelines for initiating joint analyses:

Continuity: The teachers meet more then one time a year. The long-term meetings are necessary to grow into and to stabilise the reflection process of exemplary cases. Furthermore, each teacher of the group of 3 to 5 teachers should be one or two times a year in the focus of the reflection.
Collegiality: The teachers work together and reflect their view of the real teaching episodes in a new way.
Familiarity: It is necessary to integrate the collegial reflection process in a trustful atmosphere to experience a positive learning community. A concentrate altercation of the teachers with the episode relates to the familiarity of the video episodes.
Concentration on teaching and learning: The analyses focus is on the teaching and talking activity, not on the teachers (cf. Stigler and Hiebert, 1999) - the teachers do not want to evaluate the teacher, they want to understand the teaching process and the practice of instructing - they give only alternative teaching offers (cf. Seago, 2004).
Concentration onto the teachers: The teachers will and should not analyse the transcripts like researchers. They have their own interests in working with the transcripts, just like the socio-cooperative possibilities of learning or the everyday constitutions of their practice.

The teachers can take different roles in the course of the analyses. The results discussed in this article bases on the research project “Malin”. The researcher takes the role of a cautious moderator to initiate the collegial reflections.

***Cautious* moderator**

After an empirical analysis the researcher chooses one video episode of the classroom teaching lessons of one participant. The video episode contains a potential for discussing the interactive knowledge construction of the children in relation to the intervention of a teacher. At the beginning the teachers get an orientation of the teaching episode by the teacher involved. The researcher offers the video episode and the corresponding transcript. Furthermore, the teachers discuss different perspectives for the interpretation process – such as special features of the mathematical understanding of a student, of the interactive construction of mathematical knowledge, or of the teachers` attitudes and verbal interventions and their consequences of the students` behaviour and knowledge construction (cf. Scherer et al, 2004). The video episode is structured in three sequences and each sequence is an “object” of the teachers’ cooperative and joint reflection:

a. Mathematical interpretation processes of two cooperating students
b. Mathematical interpretation processes of the intervening teacher
c. Mathematical interpretation processes of the two cooperating students after the leaving of the teacher

Firstly, the teachers see and discuss only the first sequence with the help of the transcript without knowing the teacher intervention. The researcher as a moderator has mainly the task to choose and structure a comprehensive teaching episode and to moderate cautiously the collegial reflection. At the end, he animates the teachers to a short review – in form of a "flashlight" – on the collegial reflection and on their learning process. The cautious moderation guarantees a negotiation of deep structures that seems to be important for the professional development process of the teachers` group. Furthermore, the teachers have the opportunity to adopt the collegial reflection as a school-internal way of professional learning. In this sense, we hope that this may guide the teachers to understand their school as a place where also teachers can learn.

3. THEORETICAL COMPONENTS OF ANALYSING TEACHERS` COLLEGIAL REFLECTION

In this report we concentrate exclusively on exemplary cases in order to elaborate the particularities of collegial reflections that were analysed in the Malin-Project. The qualitative data is carefully evaluated in an interpretative way and analysed with regard to the classification of specific interpretation dimensions (for the research approach of qualitative and interpretative analyses of mathematical interaction processes see e.g. ZDM (2000)).

The collegial discourse creates a new context, in which the teachers talk in a different way of teaching mathematics as during the lessons. The teachers` interpretations during the different collegial reflections of their own teaching episodes can be compared with the reconstruction of a "case". Their discussions are effected by the search for evidences to clarify the case. The results of the analyses lead to the assumption that the teachers construct an understanding of the interpretation to an agreed case – likewise teacher and students negotiate common mathematical interpretation during the lessons. For a collegial reflection, we will differ three main analysing aspects, which relate to the professional development of the teachers:

- The constructing of a case (What teachers are talking about the empirical event?)
- The reading (How teachers are speaking about the case?)
- The generation of case knowledge (Which knowledge teachers are expressing to make sense to their case?)

The constructing of a case: The teachers watch a video episode of a teaching sequence and read the corresponding transcript. Their discussions differ from spontaneously reflections in or after a teaching episode. The teacher involved in the case gives a lecture of his thinking of the named case. In the collegial reflection, the teachers frame firstly the empirical event in different ways. Here, we can mainly distinguish between three frames, which seem to be important for a professional development of mathematical teaching:

- An interactional frame containing utterances to the social learning of students, to their cooperative activities, to the dialogues between students or between students and teacher depending on their social roles (cf. Nührenbörger and Steinbring, 2009, e.g.: "*The starting situation, that [the student] Klaus decides and Sönke is* *in the role of working and writing, is changed, when a teacher comes to the students. Klaus is very orientated to the teacher telling him what they have already done*")

- An epistemological frame containing utterances to interactive construction of mathematical interpretations of the students and to the mathematical understanding of the teachers themselves in the distanced situation of the collegial reflection (e.g.: "*Ah, these four plus four idea.*" "*I think also this crux of the matter. Well, I mean, with six plus two and two plus six it is obvious, that they are exchange exercises which have the same result, but which are the other way round. And with four plus four. (…) It is in fact also an exchange exercise…*" "*But Ben, with your theory, well I am considering right now. If one puts them into a line and then you would have one plus seven, but also two exercises*.")
- An organisational frame containing utterances to the conditions of teaching (i.e. presentation of a task, time management etc.) and to the development of their own teaching (i.e. the effects of diagnostic questions etc.)

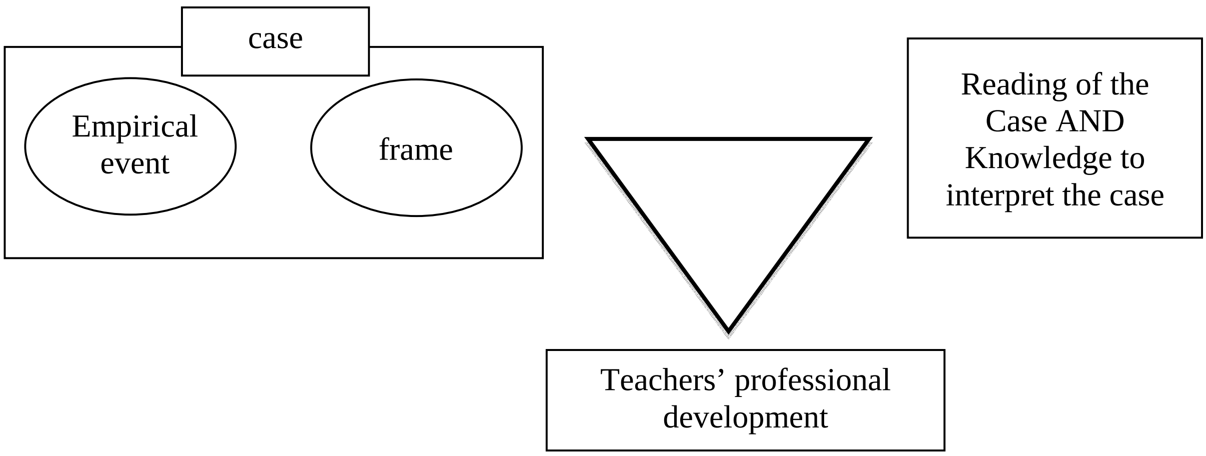
The relation between the empirical event and the frame of the teacher describes the case which the teachers construct in their collegial reflection and which is the focus of their understanding. The teachers pick different cases as a central theme during the active reflection of the different sequences. Five main cases can be differed: learning of mathematics with focus on results and algorithmic or on arithmetical and geometrical processes, social learning of the students, teaching of the teachers, mathematical context, diagnose of competences.

However, the teachers construct a case in the collegial reflection, they do not discuss a staged case. The constructed case must be proved (on) by the empirical event.

The reading of the case: The teachers can articulate the constructions of the cases in different ways. If teachers – after reading the transcript or watching the video - think to know and understand the interaction process, they *narrate* and *evaluate* the text in a biased-spontaneous way. A more open-reflected approach contains different *paraphrase* and *interpretations*. What will we mean with these notations indicating the access of the teachers to the case?

Description: The teachers concentrate on aspects of the episode and give a detailed or a short description. If the teachers illustrate the attitude or the talks as a clear and understandable learning episode, they tend to *narrate* the scene in a short way. But if the teachers illustrate different phenomena of the teaching and learning process in a neutral and accurate way, they tend to *paraphrase* the scene.
Evaluation: The teachers link their descriptions with personal views on the situation to evaluate the attitudes and talks in the teaching and learning process.
Interpretation: The attempt to clarify the teaching and learning episode must not go along with an evaluation. When the teachers describe the scene in a detailed way and try to analyse the different acts and utterances, they begin to interpret the scene. The interpretation leads to different explanations without regard to own experiences.

The readings of the case interrelate to a different case knowledge of the teachers. The analysis of the collegial reflection in the Malin-Project shows three different types of practice case knowledge (knowledge by observation, by experience, by transfer, by interrelation) that the teachers activate to clarify the case. However, in this sense the case relates to the common professional knowledge. The following diagram shows the coherences between the case and the construction of professional knowledge.



The generation of knowledge: During the reflection process the teachers bring in their knowledge to construct and understand a case. On the one hand, they use their common experiences and observations to clarify an utterance or an act of the students or of the teachers. This case knowledge relates to old knowledge (e.g.: "*I think it is typical. The older guy tells the younger one what to do. Klaus says to Sönke, how it will go*."). In this sense, the interpretation of the case is used to confirm one owns pedagogical and mathematical beliefs. A teacher will use his case knowledge by *observation* to describe and reconstruct the empirical event. When teachers use experiences of their own teaching practice that relates to the empirical event observed by the video, they activate case knowledge by *experience*. This means that they construct retrospectively an adequate perspective to give a plausible explanation for the colleagues.

On the other hand, teachers can pick the case as a central theme for constructing new relations dynamically. If the case provides a basis for a productive irritation, it can inspire the previous knowledge of mathematical topics (e.g. see the discussion of the teachers above, if there exist an exchange task to 4 + 4: The way of the students` interpretation of a mathematical task can lead to a new discussion about mathematical patterns), mathematical interpretations of children and mathematical interactions (e.g.: "*The schizophrenic thing is, I as a teacher have given them a partner work, but I do not lead the student-teacher-conversation as a partner-work-conversation*"). If a teacher reproduces the ideas of the other teachers in relation to his old knowledge, he constructs new case knowledge by *transfer and interrelation*.

4. CLOSING REMARKS: THE PROFESSIONAL DEVELOPMENT OF TEACHERS` IN RELATION TO THE COLLEGIAL REFLECTIONS

The teachers construct and negotiate different cases in different ways if they have the opportunity to reflect together their own teaching process. The analyses of the reflections in the Malin-Project (cf. Nührenbörger and Steinbring, 2009) showed that teachers activate different types of case knowledge to interpret the empirical events. We described a professional development of the teachers as a growth of the reading of a case in an open and reflected way (paraphrase and interpret). Likewise, one can see a growth of professional practice by the construction of relations between the case and the knowledge by transfer and interrelation based on a productive irritation by the teachers. Besides the organisational frame, the conditions and the trustful willingness of the teachers to open up for the exchange with their colleagues, it seems to be essential that the collegial reflections were founded on scenes from one's own teaching. But which role has the moderator?

The analysis of the collegial reflections showed that many times, the teachers discussed a scene without a mathematical orientated frame. They used the empirical event to talk about common pedagogical and organisational topics. What will happen if the moderator leaves the cautious role and takes a more active role? We have the hypothesis that the role of the moderator can focus on the discussions of the teachers on one case and can provoke a more open and reflected reading of a case with the use of knowledge by transfer and interrelation. An *active moderator* looked for special features which he wants to discuss with the teachers and which they shall notice. We will discuss a collegial reflection structured by an *active moderator* in the second part of this paper with regard to the MathKiD-Project.

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TEACHERS' REFLECTIONS OF THEIR OWN MATHEMATICS TEACHING PROCESSES

Part 2: Examples of an active moderated collegial reflection

Kerstin Bräuning & Marcus Nührenbörger

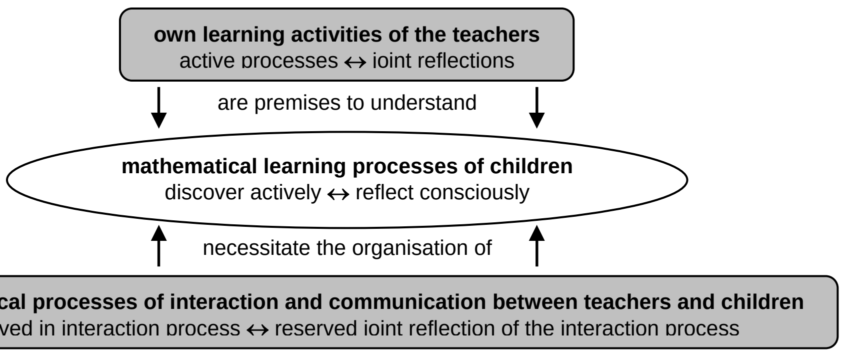
University of Duisburg-Essen

Abstract. *The research presented in this paper offers a methodological approach to the analysis of teachers' professional development by collegial reflection. Collegial reflections are professional development meetings in which teachers watch and discuss excerpts from talking with their pupils. We'll present an example of collegial reflection based on a diagnostic talk between a teacher and a 2nd grade child. The instruments presented in the first part of this paper will be used for the analysis of the collegial reflection. Investigating the case knowledge participants' construct in professional development can further our understanding of how teachers interact to influence one another's learning. We'll see how participants make inferences about the events they noticed and how they use videos as evidence for their interpretations.*

1. INTRODUCTION: THE RESEARCH PROJECT AND COLLEGIAL REFLECTIONS

The presented research deals with the development of teachers' professional learning by analyzing video episodes. In this article we will concentrate on one example of a collegial reflection process and we will use the analytic tool presented in the first part of this paper for describing the reflection process.

Teacher professional development seems to be short-term, individualized and disconnected from practice (Ball & Cohen, 1999; McLaughlin & Mitra, 2002). An important aspect of teacher learning groups is that they engage in long-term collaboration with their colleagues, focusing on issues that relate to their daily teaching activities (Little, 2002). To promote and support teachers in attending to and interpreting students' mathematical thinking there should be interplay between activity and reflection (figure in: Steinbring, 2003, p. 217/218).



Lesson study provides such a possibility for teachers where they examine systematically their instructional methods, teaching content and also their students' processes of learning and understanding (Yoshida, 2008, p. 85). A small group of teachers plan together a research lesson, implement it and the other teachers observe this lesson. Afterwards they discuss about this research lesson. With the collegial reflection we try to offer the teachers of our projects a possibility to deepen and broaden their understanding of the teaching episode by an unusual view of the situation.

Our interest is to find out what kind of readings the participants use in the collegial reflections and what kind of case knowledge they develop when talking about the video episodes. In the first part of this paper we explained the different kind of readings: *biased – spontaneous* (narrate, evaluate) than *open – reflected* (paraphrase, interpret). The teachers construct knowledge by observation, experience, transfer and interrelation. If the teacher just refers to his own thinking, he will develop knowledge by observation or experience. If he takes account of the other participants' utterances, he will construct knowledge by transfer and interrelation. We also want to find out what impact the moderator has on the readings and the case knowledge the teachers develop in the structured talk. A structured talk is a collegial reflection with a moderator attending the meeting.

Sherin and van Es use a related approach for analysing their video clubs (Sherin & van Es, 2005) which are similar to our collegial reflections. They examine the teachers' role in the video club setting. In contrast to our research they do not identify the case knowledge the teacher construct when talking about the video episode. They analyse speaking turns along the dimension *specificity* (general or specific) and *focus on video* this means that they explore if the comments grounded in the events that occurred in the video or based on events outside of the video episodes.

This article is based on two research projects ("Malin" and "MathKiD"), which both deal with collegial reflections, but which differ in the way of support and moderation (see also first part of this paper).

- *Cautious* moderator („Malin-Project") (Nührenbörger & Steinbring, 2008): The researcher chooses one video episode and provides the teachers with the video episode and the belonging transcript. Furthermore he introduces the methods of collegial reflection and presents a paper with analytic perspectives, which the teachers can use during the reflection process. The researcher moderates the reflection process in a *cautious* way. The teachers can discover and discuss independently the basic structures of their teaching. In the long-term they can adopt the collegial reflection as a school-internally way of professional learning. We hope that this may guide the teachers to understand their school as a place where also teachers can learn.

- *Active* moderator and *no* moderator („MathKiD"): The researcher chooses one video episode of a diagnostic talk, which one participant conducted. In every meeting the chosen episode will be discussed from a different analytic perspective. The teachers are provided with the video episode and the transcript to the chosen episode. In the structured talk, where the project leader is an *active* moderator, the teachers first get a short introduction about the following meeting. They receive a paper with several stimuli to the specific analytic perspective, which they can use in the interpretation process for their orientation (Scherer, Söbbeke, & Steinbring, 2004). The project leader is an *active* moderator in the structured talk because she analysed the whole transcript sensitively before the meeting and looked for special features to be discussed with the teachers and which they shall notice. The structured talk is like a supervision where the external moderator is the supervisor (Lippmann, 2005, p. 10 ff.). In the informal talks the teachers meet each other without the project leader. You can compare the informal talk with intervision. If people meet each other without a moderator it is called intervision (Lippmann, 2005, p. 12). The structured talks and the informal talks are both audio taped. The informal and structured talks take place in an alternating fashion. In every meeting new transcript will be discussed.

In the following we will look at one structured talk of the project MathKiD. The influence of the informal talk prior to the structured talk will not be discussed in this article.

2. THE COMPOSITION OF THE STRUCTURED TALK

The composition of the structured talk is the following:

1. The teachers' feedback on the informal talk.
2. Analysis of the video episode with the belonging transcript from a specific analytic perspective:
    a. Understanding of the child (first structured talk)
    b. Intentions and actions of the teacher (second structured talk)
    c. Interaction between the teacher and the child (third structured talk)
3. Flashlight to the new insights, which resulted from the analysis of the video episode.

Different stages of the structured talk are:

1. The teachers' feedback on the informal talk.

The moderator listens to the teachers and they report on the contents they discussed in the informal talk.

2. Analysing the video episode with the belonging transcript from a specific analytic perspective (understanding of the child, intentions and actions of the teacher, interaction between the teacher and the child).

First, the moderator asks the teacher who talked to the child in the video, what she expected from the child of her class before the diagnostic talk and what kind of feelings she had at the beginning of the diagnostic talk. Then all the participants watch the video episode and after that the teacher from the video has the possibility to express her first impressions of it. Then the other teachers can also express their impressions. In the analysing process the moderator structures the discussion, 1) she encourages the others to express what they think about a statement of one teacher, 2) she tries to find out what every participant wants to express, 3) she points to different possibilities to interpret a situation and look deeper on special issues in the transcript, 4) she refers to the given stimuli on the paper the teachers got, 5) she focuses the conversation on mathematical interactions, 6) she reminds the teachers to talk about the transcript and 7) she remarks the teachers to provide an evidence from the transcript for their interpretation. The moderator is not assessing the interpretations of the teachers, is not changing her role into the didactical expert and is not insisting on her stimuli, which she offered to the teachers.

3. Flashlight to the new insights, which resulted from the analysis of the video episode.

At the end of the structured talk the moderator asks every participant to express their own new insights after analysing the video episode and what kind of new information they got about the mathematical abilities of the child and the possibilities to support the child.

3. THE FIRST STRUCTURED TALK ABOUT AJDIN AND MRS. WHITE

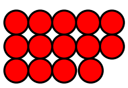
The MathKiD project started in August 2007 and five teachers from two different primary schools are participating. One group consists of three teachers, the other of two teachers. Each of the three teachers conducted one to three diagnostic talks with grade 1 or 2 pupils before the first structured talk in November 2007. The first informal talk was in October 2007 and is not audio taped.

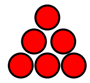
The structured talk is the first meeting of the three teachers with the project leader to analyse a video episode and the belonging transcript under the analytic perspective “understanding of the child in the observed situation”.

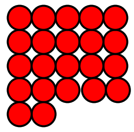
Content of the video episode Ajdin and Mrs. White

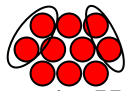
The content of the chosen video episode is the talk between Ajdin (grade 2) and Mrs. White about a pattern of coins at the beginning of the second grade. On one side the coins are red and on the other side they are blue. They are playing the game “Collecting coins” (Hengartner, Hirt, Wälti, & Lupsingen, 2006, pp. 27-30). In this game you throw your dice and move forward the shown number on the playing field. On special fields, where you see a structured or unstructured amount of coins, you can win coins. The goal of the game is to structure the won coins in a way that you always find out very easily and quickly how many coins you already won and to be able to compare your coins with the amount of coins your partner won.

Ajdin and Mrs. White play the game “Collecting coins” the second time. At the beginning Mrs. White told Ajdin that he should display his coins so that they would not have to count a lot to find out who has already won more coins. They have already talked about 13 minutes. Mrs. White won 14 coins and she structured them in 5+5+4.



Ajdin is winning his first 6 coins and he structures them like that: 

Mrs. White wins 5 more coins. Ajdin tells her that she now has 19 coins and she structures it like 5+5+5+4. She first asks him how he saw this and then how he calculated it. He tells her that 14+5=19, because 4+5=9. After that Mrs. White wins 3 coins and structures them like that 5+5+5+5+2: 

Ajdin wins four coins and structures the coins like that:  Mrs. White says that it is a "strange" pattern and asks what he thinks about it. He first tells her 3+4=7 and 7+3=10 and later he says 3+3=6 and 6+4=10 while pointing on the lines of his pattern.

Epistemological analysis of the video episode Ajdin and Mrs. White

For the interpretation it is important to notice that "Collecting coins" is on the one hand a game and on the other it is dealing with mathematical contents. The arrangement of the coins is different for Mrs. White and Ajdin. She refers to five and ten as the base of our counting system when arranging her coins. She is not changing her pattern after winning some more coins. She continues her pattern (Nührenbörger & Steinbring, 2008).

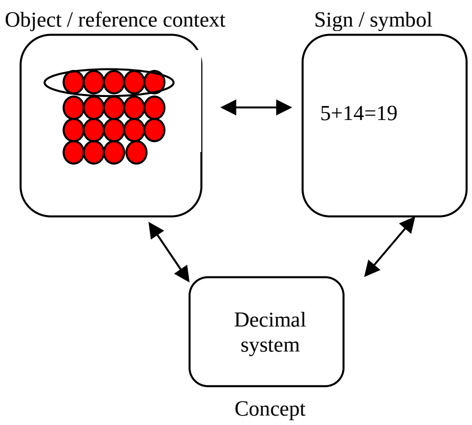


Fig. 1: Epistemological Triangle Mrs. W

Ajdin's first pattern would be called triangle number. He is "continuing" his pattern to the second pattern. There is no (geometric) label for this pattern like square or triangle or something else. It is not clear in which way he would continue his second pattern. The second pattern seems so complex for Ajdin that he gives two different calculations as interpretations: first 3+4=7 and 7+3=10 and later 3+3=6 and 6+4=10. With the calculations Ajdin does not explain his actions when arranging the coins to the first pattern. The second calculation explains the pattern in a symmetric way, but Mrs. White is not dealing with it.

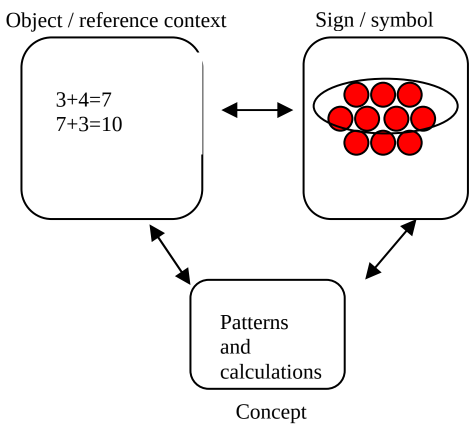


Fig. 2: Epistemological Triangle Ajdin task 1

Mrs. White uses the term “strange pattern” for his second pattern. Perhaps she uses it, because in her thinking her pattern is mathematically correct and not comparable with the pattern of Ajdin. For Mrs. White it is probably important to be able to “see” the amount of coins quickly and for Ajdin it is important to find an easy calculation for the pattern.

The moderator wants to discuss with the teachers about the different patterns of Ajdin and about the term “strange pattern”, which Mrs. White used.

Content of the structured talk about the video episode Ajdin and Mrs. White

The whole structured talk lasted 2 h and 15 min. Two different episodes were selected dealing with the first and the second pattern of Ajdin.

Content of the first episode of the structured talk

In the first episode the moderator tells the teachers that the first pattern of Ajdin is still a pattern even if it is not structured in rows of five or ten coins. This is meant as a stimulus for the others to discuss this statement. The participants are not discussing the first pattern. Through a statement of Mrs. White all the participants discuss the continuation from the first to the second pattern of Ajdin. The teachers discuss their own different interpretations of continuing the first pattern if they had won four additional coins.

Analysis of the first episode of the structured talk

The first episode deals with the continuation from the first to the second pattern of Ajdin. The teachers talk about patterns as a mathematical content and the working process of Ajdin. They do not differentiate between these two topics.

Each teacher talks about the cases in different readings, as specified below.

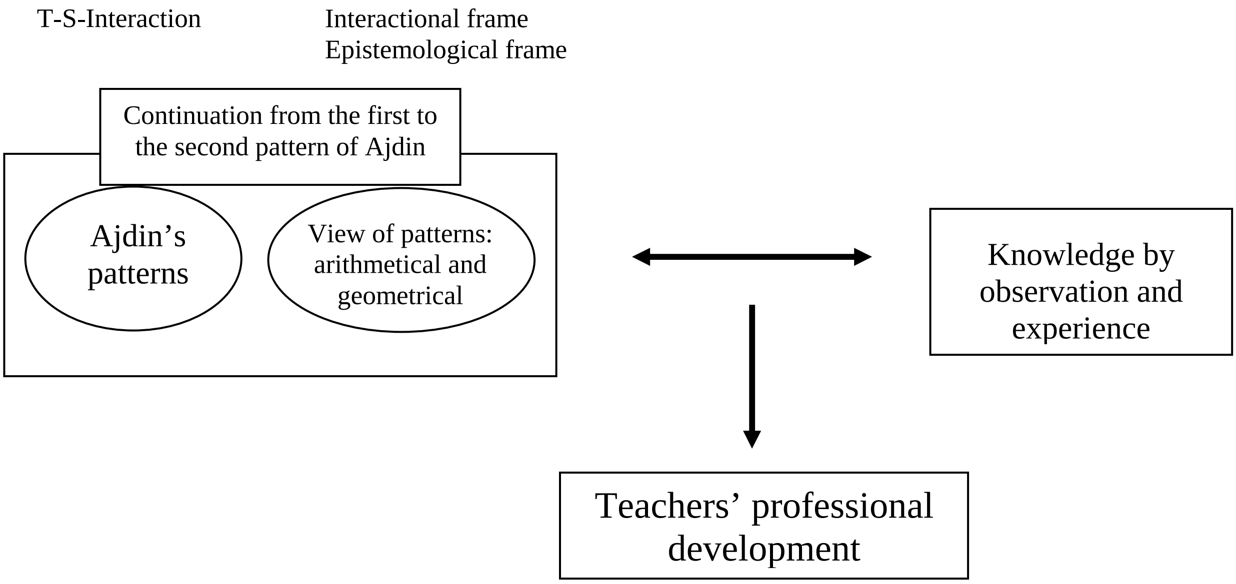
Mrs. White talks more than half of the time and dominates the discussion. She explains her understanding of patterns and what she believes how Ajdin is thinking. Probably Mrs. White has the feeling that she has to justify and to defend her actions in the diagnostic talk. On the one hand she is telling about her own thinking (“I would have” / “I put” / “for example I would” / “I would do”) and on the other hand it is presumable that she tries to get a sense of Ajdin’s statements (“I don’t know what he” / “I think” / “I believe” / “I find this unexpected” / “I can imagine”) (line 65 ff.). She describes her working process when she builds patterns, which is mainly based on her experiences. In this episode Mrs. White **narrates** and **evaluates** the continuation from the first to the second pattern of Ajdin (l. 69).

Mr. Peter talks about the structure of Ajdin’s first pattern, which Ajdin loses in the eyes of Mr. Peter when he creates the second pattern. Mr. Peter assumes that Ajdin followed the sequence of natural numbers in his first pattern (l. 71, 73, 75). Mr. Peter **evaluates** the situation in this episode.

Mrs. Dieter reacts to the stimulus of the moderator (l. 77, 79) by creating a pattern different from Ajdin’s second pattern. She neither refers to the transcript nor the episode. She connects the pattern with geometrical shapes like a square (l. 83, 85, 87, 91, 96, 98). Her statement seems like an insertion. Mrs. White rejects Mrs. Dieter's statement and therefore Mrs. Dieter tries to justify her thinking (l. 101, 112). At the end she refers to the transcript when she talks about Ajdin seeing six coins at once (l. 114). Mrs. Dieter briefly **narrates** the situation at the end. The other time she does not refer to the episode.

In this episode Mrs. Otto shortly **paraphrases** that Ajdin counted the six coins when he won them (l. 115, 117). She refers to the transcript.

The moderator gives a stimulus to think about Ajdin's first pattern if it is a pattern (l. 64) and how each of the participants would put the four coins Ajdin won to his first pattern (l. 77). Then she tries to understand the statements of the teachers and demands further information. In line 104 she refers to the rule of the game that says that you have to structure your won coins, but not in a specific or given way. The moderator tries to initiate that the teachers develop different interpretations of continuing the first pattern to the second pattern of Ajdin.



Discussion of the first episode of the structured talk

If we look at the readings of the teachers we can see that they react more *biased – spontaneous* (narrate, evaluate) than *open – reflected* (paraphrase, interpret).

If we look at the generation of case knowledge we can see that the teachers use their knowledge by observation and experience they have developed. For example Mrs. White refers to her remedial teaching (l. 74) as knowledge by experience. The teachers are not interpreting the given material in detail, the video episode and the belonging transcript. They do not refer to the statements of the other participants and therefore they do not generate knowledge by transfer and interrelation.

Content of the second episode of the structured talk

In the second episode the participants discuss from where Ajdin got the first pattern. Was it his own idea or did he see this pattern on the playing field? One teacher says that Mrs. White could have asked him why he structured the pattern like this. Mrs. White says that she could ask him but his answer would not help her to know from where he got his first pattern. Then they talk about the change from the first to the second pattern. The teachers tell their own different interpretations of the second pattern. They think about how to foster the mathematical abilities of Ajdin. They believe that you only have to support children with low-level competencies. They are convinced that they do not have to support him, but to foster over the usual level. In line 320 the moderator refers to the diagnostic-talk-transcript and says that Ajdin interprets his second pattern in a second way and one teacher states that Ajdin re-interprets his second pattern when he gives another calculation.

Analysis of the second episode of the structured talk

The second episode deals with the development of several cases. They talk about the origin of the first pattern of Ajdin and again about the continuation from the first to the second pattern of Ajdin. They discuss about patterns as a mathematical content and the working process of Ajdin. Furthermore they think if they have to support Ajdin even if he is not a low achiever.

First we will look at each teacher. Each of them talks about the cases in different readings again.

Mrs. White talks more than one third of the time and like in the first episode she tells what she thinks about the patterns and what she believes how Ajdin is thinking. Probably Mrs. White has the feeling that she has to justify and to defend her actions in the diagnostic talk. It seems like that because she dominates these two episodes. She uses “I” very often differently. We already described this in the analysis of the first episode. It seems that she thinks she knows what Ajdin wanted to do. She express that she can demand explanations of Ajdin, but they will not help her understanding what Ajdin thought (l. 254, 256). Most of the time in this episode Mrs. White **evaluates** the working process of Ajdin when he builds his patterns (l. 238, 240, 242). She decides that Ajdin needs no supporting, so she also evaluates the process (l. 313) and tries to finish the discussion in this episode.

Mr. Peter talks again about the first pattern of Ajdin. He seems to be convinced that he knows how Ajdin saw his pattern. For him the only view is following the sequence of natural numbers (l. 235, 290 ff.). He refers to the transcript when he evaluates the working process of Ajdin. At the end he describes that Ajdin finds two different calculations for the second pattern. Mr. Peter **evaluates** and **narrates** in this episode.

After the moderator repeats the statement of Mrs. Dieter (l. 279) she is the only one who reacts and she explains her statement (l. 280 ff.) how she looks on the second pattern of Ajdin. Her statement seems like an insertion because nobody refers to her. It seems that only Mrs. Dieter tries to answer to the stimulus of the moderator. Mrs. Dieter **narrates** in this episode.

In this episode Mrs. Otto reacts to the statement of Mrs. White and suggests her to ask Ajdin what he thinks about his patterns. She refers to the transcript when Mrs. White says “pattern”. She reflects about the term “pattern” and the interpretation of it (l. 257 ff.). Later she points out that one can also support children who show a good performance (l. 316, 318). Mrs. Otto **paraphrases** and **interprets** in this episode.

The moderator gives feedback to the statements of the teachers with “mhm”. In line 279 she points to the continuation from the first to the second pattern and takes up the statement from Mrs. Dieter (l. 273). Later she refers to the transcript and explains that Ajdin has two different interpretations of his second pattern (l. 320 ff.). Most of the time she listens to the conversation.

Discussion of the second episode of the structured talk

If we look at the readings of the teachers we can see that all the four teachers stick to their roles. They react more *biased – spontaneous* (narrate, evaluate) than *open – reflected* (paraphrase, interpret) apart from Mrs. Otto. In this second episode Mrs. White and Mr. Peter discuss a lot, but the others are also active, but not talking that much.

If we look at the generation of case knowledge we can see that the teachers use their knowledge by observation. The teachers refer more to the transcript than in the first episode, but they rarely use knowledge by transfer and interrelation.

Comparison between the first and the second episode of the structured talk

We can see that in both episodes the teachers use almost the same readings and generate almost the same case knowledge. Only the moderator reacts more restrained in the second episode. It seems that the moderator helps the teachers to refer again to the transcript. But sometimes it seems that the teachers give the moderator the role of an inspector whom they have to answer to, especially Mrs. Dieter.

4. CONCLUSIONS AND OUTLOOK

We found out that in this first structured talk the teachers react more *biased – spontaneous* (narrate, evaluate) than *open – reflected* (paraphrase, interpret) and use mainly knowledge by observation and experience and rarely knowledge by transfer and interrelation. Probably the teachers develop a more *open – reflected* view over the course of three structured talks in one year. And perhaps they get used to this kind of discussion and interpretation as a result they refer more to the statements of their colleagues to generate knowledge by transfer and interrelation.

The influence of the moderator seems to remind the teachers to focus their attention on the transcript and to initiate reflection processes about the statements of the other participants. We have to look for more evidence what impact the moderator has on the course of the structured talks and the case knowledge the teachers develop. We also can compare the influence of the cautious moderator (“Malin”, first part of this paper) and the active moderator (“MathKiD”) on the course of the structured talks.

After one structured talk we can draw no consequences and we cannot describe lasting changes in the readings and case knowledge the teachers develop. We will investigate and describe the development over the three structured talks. At the end we will look at video graphed lessons from the beginning and the end of the project MathKiD and will investigate if the structured talks had an impact on the teaching of each participant and on their professional development. Furthermore we will reflect if the participants want to continue the collegial reflections in their school without a moderator intended of the cautious moderator (first part of this paper).

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Transcripts can be ordered from the authors.

INTERNET-BASED DIALOGUE: A BASIS FOR REFLECTION IN AN IN-SERVICE MATHEMATICS TEACHER EDUCATION PROGRAM[1]

Mario Sánchez

IMFUFA-NSM, Roskilde University, Denmark

In this paper, the asynchronous interactions of two groups of mathematics teachers in an internet-based in-service course are analyzed. During the interactions, teachers are solving a mathematical modeling activity designed to stimulate the teachers' reflections on the modeling process. In one of groups these kinds of reflections occurred frequently while they were absent in other group. The analyses reveal clear differences in the communicative characteristics of the interactions in the two groups. Some of the characteristics of the first group are argued to be important factors favoring the emergence of the teachers' reflections on the modeling process.

INTRODUCTION

In this work, the asynchronous interactions of two groups of mathematics teachers in an internet-based in-service course are analyzed. The teachers are involved in an internet-based mathematics education in-service program for teachers from different Latin American countries. The acronym for this program is PROME-CICATA, and this is an educational program sponsored by the Instituto Politécnico Nacional of México, one of the largest public universities in Mexico. I am interested in finding ways of encouraging "rich" interactions and reflections among the teachers enrolled in the PROME mathematics education program. That is why I am trying to determine when an interaction can be regarded as "rich" or not, and what characterise communication in such rich interactions.

FRAMEWORK

The concept of communication is central for this work and particularly the computer-mediated communication (CMC). There are very clear differences between the everyday communication (or face-to-face) and the CMC. Although in both types of communication some kind of information (such as thoughts and feelings) is exchanged among individuals, the CMC does not require people staying in the same place or at the same moment of time. Communication may be atemporal to some extent and free of geographic barriers. Everyday communication is primarily verbal, but the CMC fosters written communication, which can be recorded, stored and accessed by people during conversation. This creates a record of ideas and comments that can serve as a reference or *collective memory* (de Vries, Lund & Baker, 2002) for the communication process. The expression and representation of ideas, and particularly mathematical ones, can be enhanced in CMC by the use of technological tools such as software and video. The ideas can become entities with physical properties (such as a spreadsheet file in which somebody expresses a hypothesis based on graphical and arithmetical information represented in the file) which can be stored, handled and distributed.

The characteristics of the CMC influence the nature and dynamics of the interactions that I am analyzing in this study. The data analysis is based on the Inquiry co-operation model (IC-Model) of Alrø & Skovsmose (2002). This model was developed based on the observation of students, collectively solving mathematical open-ended activities. The model, strongly rooted in the critical mathematics education approach, argues that in order to have a fruitful interaction, it must be based on mutual respect, on the willingness to make public our ideas and subject them to scrutiny, as well as in a real interest to listen and analyse our interlocutor's ideas. The IC-Model is constituted by a set of communicative characteristics. According to this theoretical approach, an interaction as the previously described should have several of these communicative characteristics. In fact when these characteristics are present in an interaction, it is regarded as a special kind of interaction called *dialogue*, which possesses the potential to serve as a basis for critical learning and reflection. The communicative characteristics that define a dialogue are g*etting in contact*, *locating*, *identifying*, *advocating*, *thinking aloud*, *reformulating*, *challenging* and *evaluating*; and they could be succinctly defined as follows:

Getting in contact basically refers to the act of paying attention to the ideas expressed by our partners in an interaction. The act of *locating* takes place when you discover an idea or a way of doing that you did not know or were not aware of before. It is a process of examining possibilities and trying things out. *Identifying* is a clarifier act in the sense that appears when you explore or try to explain an idea or perspective with the intention of making it clear to all the members of the interaction (including yourself). *Advocating* appears when you present your ideas or positions and you justify them with arguments. An advocating an also implies a willingness to revisit and discuss your own ideas or positions. To *think aloud* simply means to express in public your thoughts, ideas and feelings during the interaction process. *Reformulating* means repeating some idea but with different words or in other terms, usually to try to make it clear to your interlocutors. When we question a perspective or when we try to push it toward another direction to explore new possibilities, it is said that this is a *challenging* act. An *evaluative* act appears when we examine, criticize or correct an idea or proposal from others or ourselves.

In the communicative approach of Alrø & Skovsmose (2002), the concepts of dialogue and reflection are linked. First, reflection is defined as follows: “Reflection means considering at a conscious level one's thoughts, feelings and actions” (p. 184), but the dialogical interactions are also conceived as a basis for reflection: “We find that reflections are part of a dialogue. In particular we find elements of reflection in dialogic acts like locating, thinking aloud, identifying, advocating, etc. This means that we do not follow the Piagetian line, seeing reflections as carried out by an individual. We consider reflections referring to 'shared considerations' and we see dialogue as including processes of reflection"

In the context of research on mathematics teacher education, reflection plays a key role. In her recent review, Judith T. Sowder says that several studies identify reflection as a crucial element in furthering teachers' professional development (see Sowder, 2007, p. 198).

METHODOLOGY

In this section I refer to different aspects of the production and collection process of data, namely, the mathematical activity applied, the selected population, and the collection and presentation of data.

The selected population and the research goal.

The data that I will present were taken from one of the courses of the PROME program. The course was taught between March and April 2008. The course was an introduction to the teaching and learning of mathematical modeling. The teachers who participated in this course are in-service teachers working in different educational levels, from elementary to university level. This course was part of their academic obligations in order to get a master's degree in mathematics education.

I present here the analysis of the asynchronous interactions produced in two groups of teachers while working collectively with a mathematical modeling task. I use the term 'asynchronous interactions' to specify that the sort of communication that takes place into this interaction is asynchronous. An asynchronous communication is the one that is carried out mainly by means of an exchange of written messages between two or more people (very often located in different geographical positions), but the answers or reactions that the participants get are not immediate, for example, you can raise a question or an observation and get the feedback or reactions to it several minutes or hours after. The asynchronous discussions usually last several days, allowing the participants to have more time to formulate their opinions and to reflect on comments and opinions expressed by the other participants. It is even possible to consult external sources in order to enrich and clarify a discussion in an asynchronous communication. The email messages and the discussion forums are some examples of asynchronous communication.

The activity lasted six days and although both groups of teachers solved the mathematical activity, only in one group emerged some meta-reflections about the modeling process, which were expected to be produced through the activity and the interaction. In other words, I will show an interaction that is "rich" in terms of the reflections produced and another that it is not rich, and, through the application of IC-

Model, I will try to identify the differences in the communicative characteristics that are present in each of those interactions. That is the purpose of the research.

The mathematical activity

The mathematical activity was taken from Lesh & Caylor (2007), but it was slightly modified to fit the purposes of the course. The context of the activity is a paper airplane contest in which four planes were involved, and where each of these planes were threw by three different pilots five times each. The activity includes two tables (see tables 1 and 2) containing numerical values generated during one of the tests. Table 1 shows the landing points for each launch, represented by ordered pairs (x, y); Table 2 shows data such as distance from target, length of throw and air time for those launches. In this test the three pilots flew the four paper planes. Each time the pilot was placed at the point (0, -80) on the floor, and their aim was to launch the planes so that the plane come as close as possible to the point (0, 0), which was marked with an X.

A non-explicit purpose of this activity was that teachers will experience a portion of a mathematical modeling process, enabling them to see that in an mathematical activity as such, it is possible to have several possible and valid answers (or models), depending on the assumptions and considerations in which the model is based. To support the emergence of multiple approaches and answers to the activity, I decided to replace the original request "[to explain] how they could use this data and data from future contests to measure and make judgments about the accuracy of the paper airplanes", for a more general question, namely: "Which one is the best airplane?". Any model that answered the previous question should be based on the definition or concept that the modeler holds about what does it means to be "the best airplane". This is where I expected to have a variety of definitions/concepts, and as a consequence, a variety of possible answers to the question.

	Plane 1			Plane 2			Plane 3			Plane 4		
	Flight	X	Y	Flight	X	Y	Flight	X	Y	Flight	X	Y
Pilot 1	1	45	-78	1	-37	-30	1	-42	-4	1	15	66
	2	7	-78	2	-48	-9	2	45	5	2	32	-46
	3	55	-42	3	28	26	3	35	-7	3	5	32
	4	-14	-46	4	12	-35	4	-13	5	4	14	-21
	5	21	-29	5	-19	26	5	4	-11	5	27	-5
Pilot 2	1	-12	26	1	-11	13	1	-4	12	1	-13	22
	2	-40	-20	2	-14	-36	2	24	9	2	-3	-21
	3	-38	22	3	31	13	3	-9	34	3	-11	-51
	4	-61	15	4	-43	10	4	-39	29	4	-12	56
	5	-48	61	5	-14	50	5	40	21	5	5	-69
Pilot 3	1	42	71	1	-9	6	1	24	41	1	53	-62
	2	61	38	2	-27	-13	2	-3	52	2	52	54
	3	43	27	3	17	-17	3	-31	54	3	48	-26
	4	18	50	4	14	47	4	5	64	4	45	25
	5	15	6	5	-36	41	5	36	63	5	25	40

Table 1: Where did the plane land?

Flight		Plane 1			Plane 2			Plane 3			Plane 4		
		Distance from target (inches)	Lenght of throw (inches)	Air time (seconds)	Distance from target (inches)	Lenght of throw (inches)	Air time (seconds)	Distance from target (inches)	Lenght of throw (inches)	Air time (seconds)	Distance from target (inches)	Lenght of throw (inches)	Air time (seconds)
Pilot 1	1	90	45	0.66	47.6	62.2	0.52	42.2	86.8	0.59	67.7	146.8	2.22
	2	78.3	7.2	0.58	48.8	85.7	0.91	45.3	96.2	0.7	56	47.7	1.42
	3	69.2	66.9	0.76	38.2	109.6	1.05	35.7	81	0.55	32.4	112.1	1.91
	4	48.1	36.8	0.51	37	46.6	0.51	13.9	86	0.61	25.2	60.6	1.51
	5	35.8	55.2	0.65	32.2	107.7	0.99	11.7	69.1	0.54	27.5	79.7	1.58
Pilot 2	1	28.6	106.7	0.94	17	93.6	0.95	12.6	92.1	0.73	25.6	102.8	1.88
	2	44.7	72.1	0.81	38.6	46.2	0.54	25.6	92.2	0.7	21.2	59.1	1.4
	3	43.9	108.8	0.93	33.6	98	0.92	35.2	114.4	0.77	52.2	31	1.35
	4	62.8	112.9	0.9	44.1	99.7	0.99	48.6	115.8	0.8	57.3	136.5	2.35
	5	77.6	148.9	1.28	51.9	130.8	1.1	45.2	108.6	0.78	69.2	12.1	0.97
Pilot 3	1	82.5	156.7	1.34	10.8	86.5	0.88	47.5	123.4	0.8	81.6	56	1.36
	2	71.9	132.8	1.14	30	72.2	0.73	52.1	132	0.88	75	143.7	2.39
	3	50.8	115.3	1.01	24	65.3	0.55	62.3	137.5	0.92	54.6	72.2	1.46
	4	53.1	131.2	1.21	49	127.8	1.08	64.2	144.1	0.89	51.5	114.2	2.18
	5	16.2	87.3	0.85	54.6	126.2	1.1	72.6	147.5	0.94	47.2	122.6	2.09

Table 2: Distance, time and flight sequence data for each pilot and airplane.

The activity was uploaded as a *pdf* file on the web-based educational space where all participants of the course could access it. Teachers were organized into groups of three or four members and each of those groups were assigned to a discussion forum where the activity was collectively solved.

Data collecting and data presentation

As I mentioned before, one of the characteristics of the computer mediated communication is that it can be easily recorded, stored and shared. This feature represents a significant advantage for educational research, because the need of making transcriptions disappears. In my work for instance, I am studying some of the written asynchronous discussions produced in an internet-based educational program. Those discussions are permanently recorded and accessible on the internet-based workspace, ready to be analyzed. These asynchronous discussions may be composed of dozens of utterances. Due to the space available, it will not be possible to present the complete interactions, but only those sections that I consider most significant and illustrative. I will use bracketed ellipsis [...] to denote the omission of certain segments of text; this edition was made for the sake of brevity and to increase the readability of the data. The data that I will present has been translated from Spanish into English; moreover, the original names of the teachers have been replaced to protect their identity.

To start the analysis of an asynchronous discussion, I order all its utterances in a chronological way. From this arrangement, I try to locate those sections in which two or more participants are involved in a discussion of a particular topic. Each of these sections is broken down into individual utterances, trying to 'label' them with some of the communicative characteristics that define the communication IC-Model, according to the content of the utterance and its role within the whole discussion. Let me consider utterance (1) as an example (see 'Results' section below): This is not an *evaluative* or *challenging* act, nor is *getting into contact* with someone else because

Juan is not criticizing, questioning or being referred to the ideas of another person. He is not *reformulating* because this is the first time that he presents these ideas. Juan says “I think the most important is the proximity to the target”, but he did not present any argument to be able to classify the act as an *advocating* one. The utterance could be classified as a *thinking aloud* act, but because Juan is raising different ways of tackling the problem, I have classified it as a *locating* act. A similar analysis was done with every utterance. In some cases it is difficult to carry out the categorization since the differences between some communicative acts of IC-Model are not entirely clear for some utterances.

RESULTS

Data analysis – Group A

The working group A was composed of two teachers from Argentina (Juan and Susana) and one mexican teacher (Horacio). The interaction begins with some *thinking aloud* acts where the teachers begin to make public some of their initial ideas on how to address the problem. For instance, Susana suggests that they should find a way to use the three variables contained in Table 2 (distance, length and time). Juan answered to Susana in (1):

1 *Topic: Re: The first message*

From: Juan

Date: Thursday, the 3rd of April 2008, 11:40

Colleagues. One possible option is to work with some type of weighted mean for the 3 considered variables (length of throw, distance from target and air time). I think the most important is the proximity to the target. Another option is to think on the deviation from the target (because definitely it is a measure of the dispersion) what do you think?

In (1) Juan is *locating*, I mean, he is examining different ways of facing the problem and trying things out. He is doing a specific suggestion on how to relate the three selected variables. He proposes to use a weighted mean where “proximity to the target” is the most important variable.

2 *Topic: Re: The first message*

From: Susana

Date: Thursday, the 3rd of April 2008, 13:05

Flight partners: I was planning to ask you if you have thought in a linear regression, but I read your proposal of the weighted mean. We just have to decide about the importance assigned to each variable. Since the target is point (0,0) I would give 40% to distance from target, and 30% for the other two, if you agree. […] Susana

3 *Topic: Re: The first message*

From: Juan

Date: Thursday, the 3rd of April 2008, 19:06

Fellows. I have been outlining a sketch of the things worked so far and I expressed it on this first draft that I am attaching. [...] Best wishes. Juan

In (2), Susana mentions the possibility of using a linear regression, but this possibility was not further explored because she simply leaves this alternative and without any question she adheres herself to the proposal of the weighted mean. Without a clear argumentation, Susana proposed the weight for each element of the weighted mean. In turn, Juan in (3) contributes to *not locate* Susana's idea of linear regression. In his utterance he completely ignores the timid suggestion of Susana and he only "hear" the proposal of the weights. In a file attached to his utterance (3), Juan *identifies* or clarifies in mathematical terms his perspective on the weighted mean. In this file he defines the concept of "performance" that is used to determine which one is the best airplane. The plane that gets the higher performance will be the winner. This concept is defined as follows: $Performance = 0.4x + 0.3y + 0.3z$

Where:

x = the arithmetic mean of the distances from target

y = the arithmetic mean of the lengths of throw

z = the arithmetic mean of the air times

Juan never questioned the weights suggested by Susana. He never asks which were the assumptions that Susana considered in order to establish those values, he just includes the values in his own proposal. In general, the interaction between Susana and Juan could be described as uncritical. They experienced a "smooth" interaction where they did not question nor evaluate the proposals from the other. An example of this is in the performance formula. Neither Susana nor Juan noted that this model favoured the airplanes having a landing fare away from the target. On the other hand, Juan's attitude was not the most appropriate to establish a *dialogue*, apparently Juan was more interested in delivering the solution of the task on time, that in paying attention to the proposals of his colleagues. For example, although the asynchronous discussion forum lasted until the 6^{th} of April, Juan showed in (5) a strong rejection attitude towards other proposals to his colleague Horacio (see (4)):

4 *Topic: Re: The first message*
From: Horacio
Date: Friday, the 4th of April 2008, 11:10
Susana, Juan. I am sorry but my time is very limited. I will try to communicate with you later on. Best regards. Horacio

5 *Topic: Re: The first message*
From: Juan
Date: Friday, the 4th of April 2008, 11:26
Horacio. We are against the clock, this activity started on tuesday and there is 1 day left...I think you will have to accommodate yourself to the things that Susana and I were working on...there is no time to make any modification... Do you agree? What do you think?

Thus, even though group A was able to successfully solve the mathematical modelling task (i.e. to establish a model to select the best airplane), the interaction inside the team was characterized by a poor exchange of perspectives and ideas on how to address the mathematical task.

Data analysis – Group C

The group C had three members, but almost all the interaction took place between an Argentinean teacher (Nora) and a Mexican one (Maria). Since the beginning of the interaction, Norma and Maria were *locating* different ways of tackling the problem, but always *maintaining the contact* between them, namely, listening to the proposals of the other, taking them into consideration and evaluating them. At one point, based on Maria's suggestion about excluding the pilots of the analysis, Norma proposed in (6) a new way to find the best paper airplane:

6 *Topic: Some issues*
From: Norma
Date: Saturday, the 5th of April 2008, 06:17

[...] We could choose the ten shots that are closer to the origin, and then see which of those planes did it in more time and with the biggest length, what you think? [...]

7 *Topic: Re: Some issues*
From: Maria
Date: Saturday, the 5th of April 2008, 21:44

[...] I propose to choose the other way around, let's say that the best planes are the ones who entered into a circle with center (0.0) and a fixed radio, and then to take the ones who did it in less time […] you said more time... but are we judging the fastest or the longest stay in the air[?]... both cases are possible to judge [...] in a model it should be fixed the aspects to take into account and the rest are discarded because it is a model. I think that the idea of the radio is more close to the kind of things that are considered in the accuracy competitions as in archery. Maria

8 *Topic: Re: Some issues*
From: Maria
Date: Saturday, the 5th of April 2008, 22:32

Colleagues: I am writing you because I think that a good size for the radio could be 20 because it is one fourth of the distance from the point of departure to the target point. With this we only have six throws with three planes, I mean, the fourth plane does not participate, it does not surpass the first filter, then we can evaluate the next point.... and if we estimate the maximum speed [...] It would be like the thing that I am sending you ...What do you say? [...] I will wait for your criticism

In (7) Maria is *challenging* Norma's proposal by suggesting replacing the ten shots criterion with the radio criterion. I think this intervention is particularly valuable because explicitly brings into the discussion the need to establish the criteria, assumptions or variables to consider for building a mathematical model. Her next sentence sums up this point: "[I]n a model it should be fixed the aspects to take into account and the rest are discarded because it is a model". This is the kind of meta-reflection that I was looking to produce through the activity.

Maria's utterance (8) includes a spreadsheet file that illustrates with more detail the ideas presented in (8) and (8). She concludes that the winner is the plane number 3. As a reaction, Norma in (9) *evaluates* the proposal of Maria, and qualifies as arbitrary the choice of a radio with longitude 20. Norma agrees with Maria about using the proximity to the target as a first filter for selecting the best plane, but she suggested to use the mean of the distances from target instead of the radio proposed in (7) and (8).

9 *Topic: Re: Some issues*

From: Norma

Date: Sunday, the 6th of April 2008, 12:19

Girls, Maria: The radio that you mention is a bit arbitrary, why do not we take advantage of the fact that we already have the mean of the distances from target, and then to select the planes that were above that mean??? [...]

10 *Topic: Re: Some issues*

From: Norma

Date: Sunday, the 6th of April 2008, 13:03

Well, here you have what I made according to the previous observation about the radio. But I would also mention that I love your conclusions, Maria.

If you agree, let's vote; choose one of the three options, or choose all of them because for me all of them are ok. I mean, they are all equally valuable and correct. There are as many answers as aspects and ways of evaluating we have agreed previously.

In (10) Norma attached a file showing her new calculations, in which the winner is the plane number 4. Despite she is *advocating* a different model and getting a different winner, Norma recognizes the validity of the model suggested by her colleague Maria, in fact I think that this recognition is the basis for issuing the comment made by Norma in (10), a comment linked to another reflection implicitly sought for the modeling activity: the recognition that there may be different valid answers or mathematical models to answer the same question. It may be noted that the discussion has reached an interesting point: the participants in the discussion have been able to *locate* different ways (or models) that can serve as a mean to answer the original question which one is the best airplane? Moreover, apparently they have recognized as valid each of those models, then ... what model to choose?

This discussion continued even addressing issues of *responsibility* (see Alrø & Skovsmose, 2002, p. 217). At one point Maria asked, "[I]f the owner of the plane 3 shows up, with what criteria would we justify that we do not chose the early drafts in which he would win and instead we took the other one[?]". No doubt, this was a rich interaction in terms of the reflections achieved by the teachers.

CONCLUSIONS

The analysis of the interactions through the IC-Model shows that there are some differences in the communicative characteristics that are present in the interactions of groups A and C. For example, the interaction within the A team can be described as uncritical because there is a lack of communicative acts such as *challenging* or *evaluating*; additionally they did not seize the opportunities to find additional ways to address the problem (see for example the utterances sequences (2)-(3) and (4)-(5)).

In the team C, participants were able to *locate* several ways to tackle the problem. There was a general interest in hearing (or *keep the contact*) and evaluate the proposals of the other, and they were able to recognize the existence of multiple perspectives to solve the problem.

I argue that members of team C team were able to establish a *dialogue* that fostered the emergence and recognition of multiple perspectives to solve the problem. I think that the existence of this dialogue encouraged the emergence of meta-reflections about the modeling process.

It is necessary to continue working in a more explicit characterization of the concept of reflection. It is also necessary to discuss how the characteristics that are specific to the internet-based communication affect the emergence of reflections. Methodologically speaking it is necessary to find appropriate tools to detect or to point out when a reflection takes place in an online setting, but particularly in an asynchronous interaction.

NOTES

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THE USE OF ALGEBRAIC LANGUAGE IN MATHEMATICAL MODELLING AND PROVING IN THE PERSPECTIVE OF HABERMAS' THEORY OF RATIONALITY

Paolo Boero and Francesca Morselli
Dipartimento di Matematica, Università di Genova

In this paper we consider the use of algebraic language in modelling and proving. We will show how a specific model of rational behaviour derived from Habermas' elaboration allows to describe and interpret several kinds of students' difficulties and mistakes in a comprehensive way, provides the teacher with useful indications for the teaching of algebraic language and suggests further research developments.

Key-words: Habermas, rationality, algebraic language, modelling, proving

INTRODUCTION

Habermas' work has attracted the interest of many scholars in the domain of Sciences of Education (see the review of the translation into English of *Truth and Justification* by Tere Sorde Marti, 2004). We think that at least one of his constructs, that of "rational behaviour", is of specific interest for mathematics education, if we want to analyse complex mathematical activities (like conjecturing, proving, modelling) in a comprehensive way and to deal with them not only as school subjects and sets of tasks, but also as ways of experiencing mathematics as one of the components of western rationality. In a long term research perspective, we think that Habermas' construct is a promising analytic instrument in mathematics education if we want to connect the individual and the social by taking into account the epistemic requirements of "mathematical truth" in a given cultural context and the ways of discovering, ascertaining and communicating it by means of suitable linguistic tools. Indeed, according to Habermas' definition (see Habermas, 2003, Ch. 2), a rational behaviour in a discursive practice can be characterized according to three inter-related criteria of rationality: *epistemic* rationality (inherent in the conscious control of the validity of statements and inferences that link statements together within a shared system of knowledge, or theory); *teleological* rationality (inherent in the conscious choice and use of tools and strategies to achieve the goal of the activity); *communicative* rationality (inherent in the conscious choice and use of communication means within a given community, in order to achieve the aim of communication).

In our previous research we have dealt with an adaptation of Habermas' construct of rational behaviour in the case of conjecturing and proving (see Boero, 2006; Morselli, 2007; Morselli & Boero, 2009 - to appear). In this paper we focus our interest on the use of algebraic language in proving and modelling. Algebraic language will be intended in its ordinary meaning of that system of signs and transformation rules, which is taught in secondary school as a tool to generalize arithmetic properties, to develop analytic geometry and to model non-mathematical situations (in physics, economics, etc.). In particular, for what concerns modelling (according to Norman' broad definition: see Norman, 1993, and Dapueto & Parenti, 1999, for a specific elaboration in the case of mathematics) algebraic language can play two kinds of roles: a tool for proving through modelling within mathematics (e.g. when proving theorems of elementary number theory) - *internal modelling*; or a tool for dealing with extra-mathematical situations (in particular to express relations between variables in physics or economy, and/or to solve applied mathematical problems) - *external modelling*.

Our interest for considering the use of algebraic language in the perspective of Habermas' definition of rational behaviour depends on the fact that our previous research (Boero, 2006; Morselli, 2007) suggests that some of the students' main difficulties in conjecturing and proving depend on specific aspects (already pointed out in literature) of the use of algebraic language, which make it a complex and demanding matter for students. In particular, we refer to: the need of checking the validity of algebraic formalizations and transformations; the correct and purposeful interpretation of algebraic expressions in a given context of use; the goal-oriented character of the choice of formalisms and of the direction of transformations; the restrictions that come from the needs of following taught communication rules, which may contradict private rules of use or interfere with them.

In this paper, we will try to show how framing the use of algebraic language in the perspective of Habermas' theory of rationality: first, provides the researcher with an efficient tool to describe and interpret in a comprehensive way some of the main difficulties met by students at any school level when using algebraic language; second, provides the teacher with some useful indications for the teaching of algebraic language; third, suggests new research developments, in particular those concerning the interplay between epistemic rationality and teleological rationality in the use of algebraic language, and those related to the role of verbal language as a crucial tool for a rational behaviour in the use of algebraic language, thus potentially adding new arguments to the elaboration presented in Boero, Douek & Ferrari (2008) and concerning the specific functions of verbal language in mathematical activities.

ADAPTATION OF HABERMAS' CONSTRUCT OF RATIONAL BEHAVIOUR TO THE CASE OF THE USE OF ALGEBRAIC LANGUAGE

The aim of this section is to match Habermas' construct of rational behaviour to the specificity of the use of algebraic language in modelling and proving.

Epistemic rationality

It consists in:

- *modelling requirements*, concerning coherency between the algebraic model and the modelled situation: control of the correctness of algebraic formalizations (be they *internal* to mathematics - like in the case of the algebraic treatment of arithmetic or geometrical problems; or *external* - like in the case of the algebraic modelling of physical situations) and interpretation of algebraic expressions;

- *systemic requirements* in the use of algebraic language and methods. In particular, these requirements concern the manipulation rules (syntactic rules of transformation) of the system of signs usually called algebraic language, as well as the correct application of methods to solve equations and inequalities.

Teleological rationality

It consists in the conscious choice and finalization of algebraic formalizations, transformations and interpretations that are useful to the aims of the activity. It includes also the correct, conscious management of the writer-interpreter dynamics (Boero, 2001): the author may write an algebraic expression under an intention and, after, interpret it in a different goal-oriented way, by discovering new possibilities in the written expression.

Communicative rationality

In the case of algebraic language we need to consider not only the communication with others (explanation of the solving processes, justification of the performed choices, etc.) but also the communication with oneself (in order to activate the writer-interpreter dynamics). Communicative rationality requires the user to follow not only community norms concerning standard notations, but also criteria for easy reading and manipulation of algebraic expressions.

Some comments

The previous requirements define a model of “rational behaviour” in the use of algebraic language in modelling and proving.

We are aware of the existence of several analytical tools to deal with the teaching and learning of algebraic language. In the case of most of them, the researcher adopts a specific point of view, performs in-depth analyses according to it, but usually does not take into consideration the connections between the different aspects of the use of algebraic language and suggests only partial indications for its teaching. In our opinion, Arcavi's work on Symbol sense (Arcavi, 1994; 2005) offers the most comprehensive perspective for the use of algebraic language. With different wordings, it includes concerns for teleological rationality and some aspects of epistemic rationality. Comparing our approach with Arcavi's elaboration, we may say that we add the communicative dimension of rationality. We will see how it will allow us to account for: the possible tension between private rules of communication in the intra-personal dialogue, and standard rules; and the interplay between verbal language and algebraic language. Moreover we will see how our distinctions between the epistemic dimension and the teleological dimension, and between the modelling and the systemic requirements of epistemic rationality allow to deal with the tensions and the difficulties that can derive from their coordination.

In order to justify a new analytic tool in Mathematics Education it is necessary to show how it can be useful in describing and interpreting students' behaviour, and/or in orienting and supporting teachers' educational choices, and/or in suggesting new research developments. The aim of the following Sections is to provide evidences for all the three mentioned aspects of the use of the adapted Habermas' model.

DESCRIPTION AND INTERPRETATION OF STUDENTS' BEHAVIORS

The following examples are derived from a wide corpus of students' individual written productions and transcripts of *a posteriori* interviews, collected for other research purposes in the last fifteen years by the Genoa research team in Mathematics Education. In particular, we will consider four categories of students:

(a) 9th grade students who are approaching the use of algebraic language in proving;

(b) 11th grade students who are learning to use the algebraic language in modelling;

(c) students who are attending university courses to become primary school teachers;

(d) students who are attending the third year of the university course in Mathematics.

A common feature for all the considered cases is that the individual tasks require not only the solution, but also the explanation of the strategies followed to solve the problem. Each individual task was followed by *a posteriori* interviews. However, while in the cases (c) and (d) the explanation of the strategies is inherent in the didactical contract already established with the teacher for the whole course, in the cases (a) and (b) such explanation is only an occasional request.

EXAMPLE 1: 9th grade class

The class (22 students) was following the traditional teaching of algebraic language in Italy: transformation of progressively more complex algebraic expressions aimed at « simplification ». In order to prepare students to the task proposed by the researcher, two examples of "proof with letters" had been presented by the teacher; one of them included the algebraic representation of even and odd numbers.

> THE TASK: "Prove with letters that the sum of two consecutive odd numbers is divisible by 4".

Here we report some recurrent solutions (in parentheses the number of students who performed such a solution; note that "**d**ispari" means "odd" in Italian)

- E1 (4 students): d+d=2d

In this case, we can observe how the *systemic requirements of epistemic rationality* are satisfied (algebraic transformation works well), while the *modelling requirements* fail to be satisfied (the same letter is used for two different numbers).

- E2 (8 students): d+d+2=2d+2

In this case, both the *systemic and the modelling requirements of epistemic rationality* are satisfied, but the requirements inherent in *teleological rationality* are not satisfied: students do not realize that the chosen representation does not allow to move towards the goal to achieve (because the letter *d* does not represent in a transparent way the fact that *d* is an odd number) and do not change it.

- E3 (5 students): d=2n+1+dc=2n+1+2n+1+2=4n+4 (or similar sequences)

We can infer from the context (and also from some a-posteriori comments by the students) that "dc" means "**d**ispari **c**onsecutivi" (consecutive odd numbers).

In this case *epistemic rationality* fails in the first and in the second equality, but *teleological rationality* works well: the flow of thought is intentionally aimed at the solution of the problem; algebraic transformations are used as a calculation device to produce the conclusion (divisibility by 4).

EXAMPLE 2: University entrance, primary school teachers' preparation The following task had been preceded by the same task of the Example 1, performed under the guide of the teacher. 58 students performed the activity.

> THE TASK: Prove in general that the product of two consecutive even numbers is divisible by 8

Very frequently (about 55% of cases) students performed a long chain of transformations, with no outcome, like in the following example:

- E4: $2n(2n+2)=4n^2+4n=4(n^2+n)=4n(n+1)=4n^2+4n=n(4n+4)$

In this case, we see how both requirements of *epistemic rationality* are satisfied: *modelling requirements* (concerning the algebraic modelling of odd numbers and even numbers); and *systemic requirements* (correct algebraic transformations). The difficulty is inherent in the lack of an interpretation of formulas leaded by the goal to achieve, thus in *teleological rationality*. The student gets lost, even if the interpretation of the fourth expression would have provided the divisibility of n(n+1) by 2 because one of the two consecutive numbers n and n+1 must be even. We can also observe how (in spite of the didactic contract) in general no substantial verbal comment precedes or follows the sequence of transformations (sometimes we find only a few words: "I use formulas"; "I see nothing").

In the following case, both *modelling* and *systemic requirements* are not satisfied: the same letter is used for two consecutive even numbers (note that "**p**ari"means "even" in Italian) and the algebraic transformation is affected by a mistake.

- E5: $p*p=2p^2$, divisible by 8 because p is divisible by 2 and thus p^2 is divisible by 4.

The student seems to works under the pressure of the aim to achieve: having foreseen that the multiplication by 2 may be a tool to solve the problem, she tries to justify it by considering the juxtaposition of two copies of p that generates "2". Indeed in the interview the student said that she had made the reasoning "p is divisible by 2 and thus p^2 is divisible by 4" before completing the expression. In this case we can see how *teleological rationality* prevailed on *epistemic rationality* and hindered it.

We have also found cases like the following one:

- E6: $p*(p+2)=p^2+2p=8k$ because $p^2+2p=8$ if p=2

Also in this case, from the *a posteriori* interview we infer that probably the lacks in *epistemic rationality* depend on the dominance of *teleological rationality* without sufficient epistemic control:

> I have seen that in the case p=2 things worked well, so I have thought that putting a multiple 8k of 8 in the general formula would have arranged the situation.

EXAMPLE 3: The bomb problem

> TASK: A helicopter is standing upon a target. A bomb is left to fall. Twenty seconds after, the sound of the explosion reaches the helicopter. What is the relative height of the helicopter over the ground?

The problem was proposed to groups of third year mathematics students in seven consecutive years, and to two groups of 11^{th} grade students (high school, scientific - oriented curriculum). According to the school levels, some reminds were provided (or not) about the fact that the falling of the bomb happens according to the laws of the uniformly accelerated motion, while the sound moves at the constant speed of 340 m/s. However no formula was suggested.

The problem is a typical applied mathematical problem, whose solution needs an *external modelling* process. In terms of *teleological rationality*, the goal to achieve should result in the choice of an appropriate algebraic model of the situation, in solving the second degree equation derived from the algebraic model, and in choosing the good solution (the positive one).

The first difficulty students meet is inherent in the time coordination of the two movements: it is necessary to enter somewhere in the model the information that the whole time for the bomb to reach the ground and for the sound of the explosion to reach the helicopter is 20 seconds. The second difficulty is inherent in the space coordination of the two movements: the space covered by the falling bomb is the same covered by the sound when it moves from the ground to the helicopter.

Let us consider some students' behaviours.

Most students are able to write the two formulas:

- E7: $s=0{,}5\ gt^2$, $s=340\ t$

They are standard formulas learnt in Italian high school in grades 10^{th} or 11^{th}, in physics courses. About 25% of the high school students and 20% of the university students stick to those formulas without moving further. From their comments we infer that in some cases the use of the same letters for space and time in the two algebraic expressions generates a conflict that they are not able to overcome. We can see how general expressions that are correct for each of the two movements (if considered separately) result in a bad model for the whole phenomenon. *Teleological rationality* should have driven formalization under the control of *epistemic rationality*; such control should have put into evidence the lack of the *modelling requirements* of *epistemic rationality*, thus suggesting a change in the formalization.

In the reality for those students such an interplay between *epistemic rationality* and *teleological rationality* did not work.

In other cases (about 10% of both samples) the coordination of the two times was lacking, and the idea of coordinating the spaces (together with the formalization of both movements with the same letters) brought to the equation:

- E8: $0{,}5\ gt^2 = 340\ t$

with two solutions t=0, t=68 that some students were unable to interpret and use (because 68 is out of the range given by the text of the problem). But other students found the height of the helicopter by multiplying 340x68; the fact that the result is out of the reach of a helicopter did not provoke any critical reaction or re-thinking, probably because it is normal that school problems are unrealistic!

One part of the students who introduced the third equation $t_b + t_s = 20$ added it to the first two equations without changing the name of the variable (t).

Less than 60% of students of both samples wrote a good model for the whole phenomenon:

$t_b+t_s= 20$

$h=0{,}5gt_b^2=340t_s$

and moved to a second degree equation by substituting $t_s=20-t_b$ or $t_b=20-t_s$ in the equation: $0{,}5gt_b^2=340t_s$

Many mistakes occurred during the solution of the equation (mainly due to the management of big numbers). Once two solutions were got (one positive and the other negative), in most cases the choice of the positive solution was declared but not motivated. *A posteriori* comments reveal that the fact that a negative solution is unacceptable (given that the other solution is positive!) was assumed as an evidence, without any physical motivation.

In terms of *epistemic rationality*, three kinds of difficulties arose; they were inherent: first, in the control that the chosen algebraic model was a good model for the physical situation; second, in the control of the solving process of an equation with unusual complexity of calculations (big numbers); third (once the valid equation - a second degree equation - was written and solved), in the motivation of the chosen solution.

In terms of *communicative rationality*, we can observe how (in spite of the request of explaining the steps of reasoning) very few students of both samples were able to justify the crucial steps of the solving process. How is it possible to interpret this kind of difficulty? In some cases the steps were derived from a gradual adaptation of the equations to the need of getting a "realistic" solution. In other cases the equations were written as if the idea of coordination of the spaces and times of the phenomenon was supported by an intuition, but no wording followed. *A posteriori* interviews revealed that most students who had been unable to justify their choices were sure about their method only afterwards, when checking the positive solution and finding that it was “realistic”, thus putting into evidence a lack in *teleological rationality* (lack of consciousness about the performed modelling choices). However a number of solutions was quite realistic, even if got through a bad system. Many authors of the correct solution were not able to explain (during the comparison of solving processes) why the other solutions were mistaken. This suggests that lacks in *communicative rationality* (as concerns verbal justification of the validity of the performed modelisation) can reveal lacks in *teleological rationality* (motivation of choices with reference to the aim to achieve) and even in *epistemic rationality* (control of the validity of the steps of reasoning). This conclusion can be reinforced if we consider the fact that almost all students who were able to provide a verbal justification for their modelisation were also able to explain why the other solutions were not acceptable (even if results were realistic).

DISCUSSION

As remarked in the second section, the usefulness of a new analytical tool in mathematics education must be proved through the *actual* and the *potential research advances* and the *educational implications* that it allows to get.

Research advances

In the frame of our adaptation of Habermas' construct, the distinction between *epistemic rationality* and *teleological rationality* allows to describe, analyse and interpret some difficulties (already pointed out in Arcavi's work), which depend on the students' prevailing concern for rote algebraic transformations performed according to *systemic requirements* of *epistemic rationality* against the needs inherent in *teleological rationality* (see E4). Moreover, the distinction in our model between *modelling requirements* and *systemic requirements* of *epistemic rationality* offers the opportunity of studying the interplay between the *modelling requirements* and the requirements of *teleological rationality* (see E7); we have also seen that formalization and/or interpretations may be correct but not goal-oriented (like in E2 and E4), or incorrect but goal-oriented (like in E5, E6 and E8).

Together with the other dimensions of rationality, *communicative rationality* allows to describe and interpret possible conflicts between the private and the standard rules of use of algebraic language, and the ways student try to integrate them in a goal-oriented activity (see E3).

At present, we are engaged in establishing how the requirements of the three components of rationality intervene in the phases of production and interpretation of algebraic expressions.

Further research work should be addressed to establish what mechanisms (meta-cognitive and meta-mathematical reflections based on the use of verbal language? See Morselli, 2007) can ensure the control of *epistemic rationality* and the intentional, full development of *teleological rationality* in a well integrated way. With reference to this problem, taking into account *communicative rationality* (in its intra-personal dimension, possibly revealed through suitable explanation tasks and/or interviews) suggests a research development concerning the role of verbal language (in its mathematical register: see Boero, Douek & Ferrari, 2008, p.265) in the complex relationships between *epistemic*, *teleological* and *communicative rationality*. In particular, previous analyses (see E3, E4 and Example 3) suggest not only that the request (related to *communicative rationality*) of justifying the performed choices can reveal important lacks in *teleological rationality*, but also that the development of a kind of personal "verbal space of actions" can be relevant for a successful development of the activity (even if algebraic written traces are not satisfactory from the *systemic-epistemic rationality* point of view, like in the case E3). The respective role of the space of verbal actions and of the space of algebraic manipulations should be investigated on the *teleological rationality* axe. Here Duval's elaboration about the productive interplay between different registers in mathematical activities might be borrowed to better understand and frame what students do (see Duval, 1995). Also the results by Mac Gregor & Price (1999) could help highlighting the relations, as emerged from our data, between the production of verbal justifications and the effective use of algebraic language to achieve the goal of the activity.

Educational implications

We think that the analyses performed in the previous section can provide teachers as well as teachers' educators with a set of indications on how to perform educational choices and classroom actions to teach algebraic language as an important tool for modelling and proving. Some of those indications are not new in mathematics education; we think that the novelty brought by Habermas' perspective consists in the coherent and systematic character of the whole set of indications.

First of all, the performed analyses suggest to balance (at the students' eyes, according to the didactical contract in the classroom) the relative importance (in relationship with the goal to achieve) of:

- production and interpretation of algebraic expressions, vs algebraic transformations;

- flexible, goal-oriented direction of algebraic transformations, vs rote algebraic transformations aimed at "simplification" of algebraic expressions.

These indications are in contrast with the present situation in Italy and in many other countries: teachers' classroom work is mainly focused on algebraic transformations aimed at "simplification" of algebraic expressions, and most simplifications are performed by elimination of parentheses, thus suggesting a mono-directional way of performing algebraic transformations. At the students' eyes, the importance of the formalization and interpretation processes is highly underestimated. The fact that algebraic expressions are given as objects to "simplify" (and not as objects to build, to transform according to the aim to achieve, and to interpret during and after the transformation process in order to understand if the chosen path is effective and correct or not) has bad consequences on students' *epistemic rationality* and *teleological rationality*. As we have seen, many mistakes occur in the phase of formalization (against the *modelling requirements*), and even when the produced expressions are correct, frequently students are not able to use intentionally them to achieve the goal of the activity (against the *teleological rationality requirements)*.

A promising indication coming from our analyses concerns the need of a constant meta-mathematical reflection (performed through the use of verbal language) on the nature of the actions to perform and on the solving process during its evolution. At present, the only reflective activity in school concerns checking the correct application of the rules of syntactic transformation of algebraic expressions (thus only one component of rational behaviour - namely, the *systemic requirements* of *epistemic rationality* - is partly engaged).

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OBJECTS AS PARTICIPANTS IN CLASSROOM INTERACTION

Marei Fetzer

IDM, J. W. Goethe-University, Frankfurt/M., Germany

In this article an object-integrating approach to interaction in the mathematics classroom is proposed. Accordingly, not only human beings, but also non-human objects are considered as participants in the course of action. Symbolic interactionism and Actor-Network-Theory both serve as a theoretical basis for the development of the object-integrating approach to classroom interaction outlined in this article.

Keywords: objects, classroom interaction, Symbolic interactionism, ANT, analysis

INTRODUCTION

Research on teaching and learning processes in the mathematics classroom focuses on different aspects. Mathematical language, or communication in a broader sense, are possible points of interest. In this article I take an interactionistic perspective on processes of teaching and learning. I investigate classroom interaction as it is developed by its participants. My current interest is on the role of objects in such interactional processes. How do they affect the proceeding of interactional learning processes in primary education? My concern is the development of an object-integrating approach to interaction in the mathematics classroom.

OBJECTS AND CLASSROOM INTERACTION

The ‘discovery’ of the mere existence of objects in the mathematics classroom is rather innocuous. Besides, the observation that objects have an influence on interaction in mathematics primary education is not new either. Moreover, systematic implementation of objects such as books, paper and pencil, blackboards, calculators, cubes or dice in teaching and learning activities is a commonly shared practice. It gains wide acceptance amongst researchers as well as amongst primary teachers. Undoubtedly, objects play a role in the course of mathematical teaching and learning. But how can one describe the objects’ role in the course of classroom interaction theoretically? Interactionistic perspectives on primary mathematics education traditionally focus on students and teachers (see e.g. Mehan, 1979; Cobb & Bauersfeld, 1995). These persons are the actors developing the interactional process. However, no special attention is paid to non-human objects, and no interactionistic thought is given to them. Thus, there remains uncertainty concerning things and their role within the interactional development. Subsequently I am going to outline a theoretical approach to interaction in which objects have “agency” (Latour 2005, p. 63) as well. Proposing this object-integrating approach to classroom interaction, I draw on the framework of symbolic interactionism (Blumer, 1986) and on Latour’s Actor-Network-Theory (ANT) (Latour, 2005). Referring to ANT I go beyond the more common idea of interpreting objects as tools or instruments in human’s hands.

Nor do I concentrate on mediated thinking or objectification (Radford 2006). Instead, I accept objects as participants in classroom interaction. Thus, Latour's theory serves as an impetus for a radical change in studying mathematical learning processes. While the suggested object-integrating approach is not yet a fully developed theory, I suggest it as a thought–provoking impulse.

Symbolic Interactionism

Blumer (1986) gives an outline of the nature of symbolic interactionism, calling in three premises. The first premise is that "human beings act toward things on the basis of the meanings the things have for them." (ibid., p. 2). Here, Blumer's use of the term 'thing' differs fundamentally from the understanding of 'things' throughout the rest of this article. It is as broad and overarching as possible. Blumer defines: "Such things include everything that the human being may note in his world – physical objects […], other human beings […], institutions […], guiding ideals […], activities of others […] and such situations as an individual encounters in his daily life." (ibid., p. 2). In contrast, I apply the everyday-term 'thing' with regard to ANT in a much closer form. I use it as a colloquial and sensitizing version of the term 'object', taken as short for non-human physical object.

The second premise refers to the source of meaning. Meaning is not intrinsic to the thing. Nor is it a psychical accretion like a sensation, memory, or feeling brought into play in connection with perceiving the thing. Instead, "symbolic interactionism sees meaning as arising in the process of interaction between people. The meaning of a thing for a person grows out of the ways in which other persons act toward the person with regard to the thing." (ibid., p. 4). Thus, the meaning of things is formed in the context of social interaction. It is seen as a social product.

The meaning of a thing is derived by the person from the interactional process. But meaning is not an already established application to a thing. It is nothing that has to be arisen from the thing itself. In contrast, the use of meaning by the actor occurs through a process of interpretation. And this leads to the third fundamental premise put forward by Blumer: "The meanings are handled in, and modified through, an interpretative process." (ibid., p.5). Thus, interpretation becomes a matter of handling meanings. It is considered as a formative process in which meanings are used and revised as instruments for the guidance and formation of action.

Analysing interaction in the mathematics classroom on the basis of the framework of symbolic interactionism is a matter of interpretation. It is an interpretative effort to reconstruct, as in the case of my research work, *processes* of meaning making. How is meaning formed and negotiated in the process of interaction? How do actors collectively create mathematical meaning? In order to investigate the process of meaning making, every single *action* is interpreted extensively in the sequence of emergence. The analyst tries to generate as many alternative interpretations as possible. Thus, he or she opens up the range of potential ways of understanding and construing the action. In order to get hold of the process of *inter*-acting, actions are considered to be related to each other. They are interpreted as turns to previous actions. Analysing turn by turn the process of meaning making can be reconstructed.

Actor-Network-Theory (ANT)

Latour (2005) poses the question who and what participates in the course of action. He criticises the established definition of action: If action is limited a priori to "what 'intentional', 'meaningful' humans do" (ibid., p. 71), objects have no chance to come into play. Instead, he recommends a broader understanding of action and agency. He defines that "*any thing* that does modify a state of affairs by making a difference is an actor" (ibid., p. 71). In doing so, he equips objects just as well as humans with agency. All actors, human or not, are "*participants* in the course of action" (ibid., p. 71). Thus Latour extends and modifies the list of actors assembled as participants fundamentally. He gives several reasons why ANT accepts objects "as full-blown actor entities" (ibid., p. 69). One is that the social world will "retain a sort of provisional, unstable, and chaotic aspect" if it was made of local face-to-face interaction. However, such temporary and fugacious interactions can become far-reaching and durable. Latour calls the "steely quality" (ibid., p. 68) of things to account for this durability and extension. What is new is, that objects are highlighted as actors that might "authorize, allow, afford, encourage, permit, suggest, influence, block, render possible, forbid, and so on" (ibid., p. 72). Latour does not give privilege; human as well as non-human participants in the course of interaction have agency. Latour refrains from imposing "some spurious asymmetry among human intentional action and a material world of cause relations" (ibid., p. 76). He denies loading things into social ties. Objects do not serve as a "backdrop for human action" (ibid., p. 72). Neither do they determine the interactional process; they are not the causes of action. But he does not propose some sort of equality either (ibid., p. 63; p. 76). Instead, he emphasises the varieties and differences in modes of action (ibid., p. 74ff.).

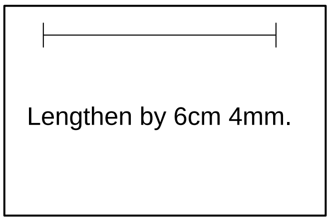
Doing research on mathematical education from an interactionistic perspective, the merge of ANT and symbolic interactionism might be a fruitful effort. Latour considers objects as actors contributing to the process of interaction in different modes of action. They participate in the process of meaning making, even though they have different options open. Concerning methodology, Latour preaches to "follow the actors" (Latour, 2005, p. 156) and "describe" (ibid., p. 144; p. 149). Blumer emphasizes that non-human objects as well as human activities have no intrinsic meaning. They do not carry an established meaning that has to be revealed. Meanings are formed in the process of interaction. Meaning making, according to Blumer, is a matter of interpretation. Symbolic interactionism serves as a point of reference for interpretative research trying to reconstruct the process of meaning making. Merging symbolic interactionism and Latour's approach might help to bring the consuetudinary excluded objects into the course of interaction. It might contribute to the development of an object-integrating theory of learning in mathematical classroom interaction. Latour states with regard to interaction, that “the number and type of ‘actions’ and the span of their ‘inter’ relations has been vastly underestimated. Stretch any given inter-action and, sure enough, it becomes an actor-network” (2005, p. 202). But how do you investigate interactional processes if you consider objects as full-blown actors? How do you deal with the modified list of participants and with the increased modes of action? In the following paragraph, I propose an object-integrating approach on classroom interaction.

OBJECT-INTEGRATING APPROACH TO CLASSROOM INTERACTION

Empirically grounded development of an object-integrating theory of learning in mathematical classroom interaction includes the development of analytic tools, analysis of numerous scenes, and the comparison of interpretations to various scenes. Below, methodological thoughts are discussed as a basis for analysis of object-related classroom interaction and accordingly as a contribution to the development of an object-integrating theory of learning. To exemplify the methodological points of interest, a short episode taken from a third year German primary class is introduced (first published in Fetzer, 2007).

Example

In this scene the task is to lengthen a graphically given straight segment by 6cm 4mm (compare fig.). First the children work on the problem on their own. They are asked to put written notes on their problem solving process. Afterwards some children present their approaches on the blackboard. Sonja is the first to explain her proceeding. The teacher requests those students that “can’t follow anymore” to “ask what’s going on”. Sonja selects Sabina as next speaker. She says: “Somehow I don’t get it.” This last utterance will be the focus of investigation.



Person	Aktivität	Activity
Sonja	*Steht an der Tafel, schaut zur Lehrerin*	*Stands at the blackboard, looks towards the teacher*
Teacher	Die andern- da sind viele gewesen da kann ich mir vorstellen die kommn jetzt schon nicht mehr mit- da müsst ihr auch mal fragen was da los iss- aber wenn die nich meinn sie müssten fragen erklär weiter-	The others- there have been many I can imagine who can’t follow anymore- you have to ask what’s going on then- but if they don’t bother asking keep on explaining-
Sonja	*Schaut in die Klasse* Sabina-	*Looks towards the class* Sabina
Sabina	Ich kapier des irgendwie net-	Somehow I don’t get it-

First, I will give a ‘traditional’ analysis of the scene focussing on the verbal activities of the human participants. This brief analysis may serve as a basis for the subsequent theoretical and methodological discussion.

An extensional analysis of Sabina’s utterance in the last line of the transcript opens up a wide range of possible ways of understanding. Here only a small selection is given. By stating “somehow I don’t get it”, Sabina perhaps intends to express that she could not follow Sonja’s explanation. On the one hand this could be a statement referring to herself and her own learning process. On the other hand her utterance could be understood as a statement concerning Sonja’s performance. In the context of the latter interpretation, Sabina would indicate that Sonja’s explanation was not comprehensible. Alternatively one might understand her utterance as an expression of her troubles in solving the given task. If so, her difficulties would not relate to Sonja’s explanation, but to the task itself. Eventually her utterance might be interpreted as a contribution to the classroom interaction in order to demonstrate alertness. In this case, the mathematical substance of her contribution could be minimal.

Who could Sabina possibly refer to? The turn-by-turn analysis basically reveals two alternatives: Sabina’s utterance could be understood either as a turn on Sonja, or alternatively as a turn on the teacher. Following the first interpretation, Sonja addresses Sabina and picks her as the next speaker. Sabina gets active and paraphrases the teacher by translating “can’t follow anymore” into “somehow I don’t get it”. In the context of this interpretation, Sabina would invest hardly any mathematical effort. According to the second understanding, Sonja might just as well get active as a turn on the teacher’s invitation “You have to ask what’s going on”. Again her utterance might be understood as a paraphrasing of the teacher’s “can’t follow anymore” (see above). Following this interpretation, not much mathematical content can be attested to her utterance. An alternative understanding would suggest that Sabina indeed could not follow Sonja’s explanation. She then actually belongs to those who were addressed by the teacher and were invited to get active. Again, Sabina takes the turn offered by the teacher. In the context of this latter understanding the mathematical content attributed to her utterance would be (slightly) increased.

On actors

According to an object-integrating approach to classroom interaction, not only humans but also objects have agency. This modified understanding of who and what acts in mathematical interaction entails a modified way of transcribing as demonstrated below.

Actor	Aktivität	Activity
Board	5+6=11 4+7=11 *[Tafelanschrieb bleibt während der gesamten Szene unverändert und sichtbar]*	5+6=11 4+7=11 *[Notes on the blackboard remain untouched and visible throughout the whole scene]*
Sonja	*Steht an der Tafel, schaut zur Lehrerin*	*Stands at the blackboard, looks towards the teacher*

Teacher	Die andern- da sind viele gewesen da kann ich mir vorstellen die kommn jetzt schon nicht mehr mit- da müsst ihr auch mal fragen was da los iss- aber wenn die nich meinn sie müssten fragen erklär weiter-	The others- there have been many I can imagine who can't follow anymore- you have to ask what's going on then- but if they don't bother asking keep on explaining-
Sonja	*Schaut in die Klasse* Sabina-	*Looks towards the class* Sabina-
Sabina	Ich kapier des irgendwie net-	Somehow I don't get it-

The first column indicates the interacting participants. It is captioned with 'actor' because the term 'person' solely refers to human beings and excludes other participants. The second and third columns give the actions in English and in German, differentiating verbal (regular font) and non-verbal actions (*italic font*). In contrast to 'conventional' transcripts, activities of objects are included as well. In the illustrating scene, for example, the notes on the blackboard are highlighted in grey.

Who and what participates in the given scene? Sonja, the blackboard, the teacher, and Sabina are actors in the scene. Besides, the children have their own written work at hand. Accordingly, Sonja's and/or Sabina's written approach might just as well enter into account. Working with an object-integrating approach to learning processes casts a different light on the selection of participants. The identification of the actors becomes more difficult for two reasons. Firstly, the fact that objects enter into account does not as a matter of course show in the restricted lines of a transcript. The reason is the time-spreading quality of things. Some-*thing* lying on the table like Sabina's written work or written on the blackboard as in the given example might not be mentioned in the specific scene selected for analysis. Nevertheless, board and written work might become participants within the course of action. Secondly, indicating participants in object-related classroom interaction is not a matter of fact, but a matter of interpretation. Some objects may be appraised as participants in one interpretation, but remain unconnected to the course of interaction in another interpretation. Regarding the interpretation on Sabina's utterance given above neither the board nor Sabina's work get connected to the interaction. However, understandings that take the blackboard as well as Sabina's work as actors can be reconstructed, if an object-integrating approach is applied. As a consequence, the selection of the actors of a given scene can always be no more than a *pre*-selection. Supplementary nominations of participants are likely to become necessary within the process of analysing. Accordingly, the pre-selection of participants should accept a wide range of possible actors. Concerning the example, Sabina's work should be at hand for analysis.

The selection of actors is one crucial point in implementing an object-integrating approach to classroom interaction. Another striking aspect is the matter of sequence and time span of participating. Who and what assembles as participants in the course of action might change very quickly. Especially non-human objects may enter into account one moment and recede into the background an instant later. They appear associable with one another only momentarily. Analysis of interactional processes focuses on visible actions and the process of interweaving. Consequently, children, teachers or things can become the 'object' of analysis just as long as they leave an observable trace. If no trace is produced, no information is offered to the observer. If humans as well as things remain 'silent', they are no actors anymore. They remain unaccountable (Latour, 2005, p. 79). The written work on the table is not an actor. But Sabina and her notes might weave together and both become active participants in the interactional process as soon as Sabina picks up her sheet or has a glance on her notes. With Sabina and her written work entering the course of interaction, they may be captured by analysis. Interaction analysis based on the framework of symbolic interactionism takes a micro perspective and proceeds sequentially. Thus, intermittent existence and rapidly changing assembling of participants may be captured appropriately. But in the context of an object-integrating approach, durability and lasting time spans have to be considered as well. The blackboard might show Sonja's notes for quite a while. Consequently it is a potential actor for a certain length of time. This abiding participating could be indicated in the transcript, for example, by implementing an additional column.

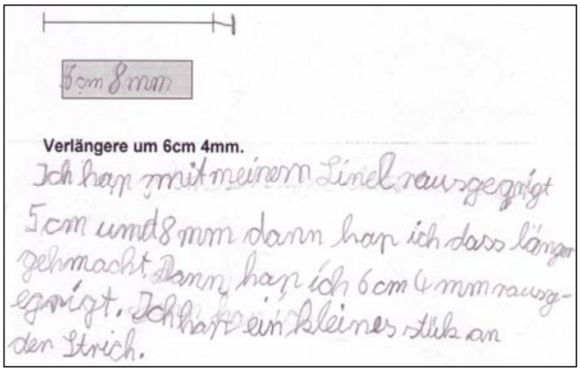
On modes of action

Investigating processes of teaching and learning in mathematics education actions are analysed in their order of emergence (see above). The analyst generates as many sensible interpretations to the given action as possible in order to expand the range of potential understandings. Reconstructing the process of meaning making in the context of ANT widens the spectrum and modes of actions under investigation. Both, human and non-human actions have to be analysed. However, analysing non-human's actions on the first glance appears to be a bold venture. How can an object's agency be interpreted? In order to investigate the object's agency one may firstly explore the object itself 'nakedly'. What does this object tell the analyst, what does it remind him of? What might it express, suggest, allow, forbid, enable, etc.? This mode of analysis compares to a methodical dodge often applied in analysing human action: the variation of the interactional context. The action is taken out of the given context and conveyed into another. This is an established proceeding in interaction analysis in the theoretical framework of symbolic interactionism. What is new is to implement the variation of the context to objects and their activities. This analytic move raises the analyst's awareness and sharpens his or her analytic senses when it comes to interpret the object's actions. This is possible as soon as objects get visibly connected to other participants in the course of action. Once they become associated with one another, their action might be captured by analysis. With Sabina glancing on her notes, the written work becomes a participant in the interaction. It is no longer a sheet of paper on the table, but a tangible link between now and earlier. It is a concrete backing of argumentation or a means of distraction. As an actor, the written work in front of Sabina might demonstrate alertness, or it might assign her to be the current speaker. The assumption that objects have agency, too, widens the range of observable actions. Consequently, the analysis of the sequentially emerging action must be implemented to human as well as to non-human actors' activities.

Interesting enough an object may well be there unaltered or untouched for a couple of minutes or half an hour. In the selected example this applies for both, the blackboard and for the written work(s). Their 'steely quality' persists, although objects just momentarily enter into account, and become active only from time to time. In the context of the traditional analysis of interaction we are used to focus on actions as momentary affairs producing visible or otherwise perceivable traces only here and now. Objects prompt the analyst to open the perspective. The potentially long lasting effect of an object's activity on classroom interaction has to be considered. The blackboard *is* there. Any participant might refer to the notes any time within the interaction. Thus the notes on the board become participants.

These theoretical thoughts have an impact on the method of analysis in the context of an object-integrating approach to classroom interaction. To illustrate the effects on the analysing procedure, the investigation presented above is adopted and supplemented accordingly. Subsequently, the blackboard and Sabina's work are explored.

On the blackboard there are two number problems. Both are additions, both sums are eleven. Due to a lack of space, again, only a selection of possible interpretations is given. The two lines seem to refer to an arithmetic problem. They might for instance be related to each other by the mathematical strategy of inverse changing of summands. Assuming that Sonja's notes are related to the given task on measuring and calculating lengths, the two sums might be read as operations with numerical values omitting the units (cm and mm). In this case, the two sums could be interpreted as short versions of 5cm+6cm=11cm and 4mm+7mm=11mm. From a mathematician's point of view, this interpretation would give the written sums the touch of side notes. Taking a (weak) student's perspective, these two lines could be seen as the extract or the fundament of the problem: Plain numerical values, assorted by different values. One rather complex calculation with units is reduced to two simple arithmetic problems that can be managed easily. Anyhow, the blackboard displays an arithmetic problem. The geometric element of the graphically given straight segment does not show anymore.



Below the task (Lengthen by 6cm 4mm) Sabina's work says: "I found out with my ruler 5cm and 8mm then I have lengthened that Then I found out 6cm 4mm. I had a little bit to the line." (See fig.). Her work shows a rather geometric approach based on the idea of adding *up to* 6cm 4mm (instead of *lengthen by*). The little figure on the right hand side can be interpreted as the answer to the given task; it is the missing bit to the requested length. The written text proves this interpretation valuable. The ruler is assigned to be the clue to the solving process. First, it serves to find out the length of the given line. Afterwards, it shows the gap between given and requested length.

On turns

In order to reconstruct the process of meaning making in mathematical classroom interaction according to symbolic interactionism, actions are understood as turns on previous actions. As soon as objects are accepted as actors in the ongoing course of interaction, not only the concept of 'action' has to be adopted (see above). The concept of 'turn' as originally introduced by Sacks (1996) has to be re-thought as well. In his book "Lectures on Conversation" he works on the subject of turn-taking and introduces the adjacency relationship if utterances are related to each other as turns (Vol. II, part 1, p. 41ff.). This utterance-based understanding of 'turn' does not meet the demands of interactions. It is not only verbal, but rather all sorts of activities that might be related to each other as turns. The teacher's utterance might be interpreted as a turn on Sonja's look at her. Sabina's "Somehow I don't get it" might be a turn on the written notes on the blackboard or her working sheet. As a consequence, in the context of an object-integrating approach to classroom interaction, I use the term 'turn' in a broader sense: Actions are interpreted as turns, if they are closely related to previous actions. The underlying concept of 'action' is closely linked to ANT. It includes different modes of actions carried out either by human beings or by objects. If the concept of action and turn is extended in this way, analysis on the basis of the framework of symbolic interactionism will serve as an appropriate method to reconstruct object-related classroom interaction. Objects and things will be integrated into the course of interaction again. To me, re-thinking the concept of turn is the decisive approach in investigating object-orientated classroom interaction. It is the adopted understanding of turn that helps to trace object's activities. On the level of turns objects leave observable marks and become visibly connected to one another. Human as well as non-human actors get involved as soon as it comes to think about possible relations between actions as turns.

Analysis on the basis of the adopted concept of turn may work as presented below. Again I refer to the example "Somehow I don't get it." In addition to the interpretations suggested above, I now propose an interpretation taking Sabina's action as a turn on her own written work. Sonja presented her arithmetic proceeding to the task, based on the idea of adding two specific lengths. She did it in a convincing way, and Sabina could follow well. Consequently, she remains silent when the teacher asks those, who got in trouble, to become active. However, looking onto her written work causes confusion. Two different approaches, yet both convincing, show neither conformance nor consensus. The ideas of lengthen *up to* on the one hand and lengthen *by* on the other hand seem incommensurate. The geometrical and the arithmetic approach simply won't merge. According to this interpretation, the utterance “Somehow I don’t get it” appears to be a mathematically spoken reasonable statement. The last line of the transcript can be interpreted as a mathematically substantial statement. Its mathematical relevance is closely connected to the two objects, blackboard and written work.

ANALYSING OBJECT-RELATED CLASSROOM INTERACTION

Based on the presented outline of an approach to object-integrating interaction in the mathematic classroom, I will eventually point out some key points concerning the related method of analysis.

The identification of the actors in the scene to be investigated is an interpretative act. Thus, assembling of the list of participants is a pre-selection. In order to leave space for a wide spectrum of alternative interpretations, the list of (potential) participants should not be prematurely limited.

In order to maximize the range of possible interpretations to an observable action, the analytic dodge of variation of the context might be called on. This applies both for human as well as non-human actions.

Actions are related to each other as turns. On the one hand, actions are interpreted as turns on previous human-actors’ actions. On the other hand, actions are explicitly related to non-human actions that may be perceived in distinct ways. How could a certain action be interpreted if it was a turn on an object-participant’s action? Performing such an object-integrating turn-by-turn analysis prevents from accidental neglect or premature exclusion of objects as actors. However, the list of participants might need reassembling or supplementation in the context of this analytic move.

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THE EXISTENCE OF MATHEMATICAL OBJECTS IN THE CLASSROOM DISCOURSE

Vicenç Font, Juan D. Godino, Núria Planas, Jorge I. Acevedo

U. de Barcelona, U. de Granada, U. Autònoma de Barcelona, U. de Barcelona

In this paper we are interested in the understanding of how the classroom discourse helps to develop the students' comprehension of the non ostensive mathematical objects as objects that have "existence". First, we examine the role of the objectual metaphor in the understanding of the mathematical entities as "objects with existence", as well as in some of the conflicts that the use of this type of metaphor can provoke in the students' interpretations. Second, we examine the mathematics discourse from the perspective of the ostensives representing non ostensives that do not exist.

INTRODUCTION

In this report we present some findings from our current research on the role of objectual metaphors in the interpretation of the existence of non ostensive mathematical objects within the classroom discourse. We illustrate these findings with a reinterpretation of data from Acevedo (2008). In particular we analyze certain remarks of different teachers that have in common the use of metaphors in their teaching practices. In that study, the fourth author presented an analysis of some teachers' discourses while teaching the graphic representation of functions in Spanish high schools. The focus was on the teachers' discourses and practices when interacting with the students in certain lessons. The main data was gathered by means of video and audio tapes, together with written tests, students' work and filed notes.

We organize the report from theory to example in order to deal with language and communication issues in mathematics classrooms from a semiotic point of view. We begin by briefly reviewing part of the literature on metaphors and presenting the notions of image schema and conceptual metaphor, which are drawn on the theories of the embodied cognition. When introducing some findings, we show how the use of metaphorical expressions of the objectual metaphors in the teachers' discourses leads the students to understand the mathematical entities like "objects with existence". Finally, we show how the mathematics discourse on ostensives representing non ostensives that do not exist and on the identification of mathematical objects with some of its representations, leads the students to separately interpret the mathematical objects and its ostensive representations.

IMAGE SCHEMAS AND METAPHORICAL PROJECTIONS

In recent years, several authors (see, for instance, Bolite, Acevedo & Font, 2006; Lakoff & Núñez, 2000; Núñez, Edwards & Matos, 1999; Pimm, 1981, 1987;

Presmeg, 1997) have pointed to the role of metaphors in the teaching and learning of mathematics, and some of them have reflected on the embodied cognition theory. Sriraman and English (2005), in their survey of theoretical frameworks that have been used in mathematics education research, talk about the importance of the embodied cognition theory. On the other hand, the discursive emergence of mathematical objects is interpreted as a research focus within that theory. Sfard (2000, p. 322) has stressed some of the metaphorical questions concerning the existence of the mathematical objects:

> To begin with, let me make clear that the statement on the existence of some special beings (that we call mathematical objects) implicit in all these questions is essentially metaphorical.

We argue that the use of objectual metaphors in the mathematics classroom discourse leads to talk about the existence of mathematical objects. Our notion of objectual metaphor is highly related to the notions of image schema and metaphorical projections (Johnson, 1987; Lakoff & Johnson, 1980). The image schemas are basic schemas, in the middle of the images and the propositional schemas that help to construct the abstract reasoning by means of metaphorical projections. These schema are constituted by multiple corporal experiences experimented by the subject. Some of these experiences share characteristics that are incorporated within the image schema. Both the experiences and the shared characteristics are a consequence of situations that have been physically and repeatedly lived.

Lakoff and Núñez (2000) claim that the cognitive structure for the advanced mathematical thinking shares the conceptual structure of the non mathematical daily life thinking. The metaphorical projection is the main cognitive mechanism that permits to structure the abstract mathematical entities by means of corporal experiences. We interpret the metaphor as the comprehension of an object, thing or domain in terms of another one. The metaphors create a conceptual relationship between an initial or source domain and a final or target domain, while properties from the first to the second domain are projected. In relation to the mathematics, Lakoff and Núñez distinguish two types of conceptual metaphors:

- Grounding metaphors: they relate a target domain within the mathematics to a source domain outside them.
- Linking metaphors: they maintain the source and the target domains within the mathematics and exchange properties among different mathematical fields.

Within the group of grounding metaphors, there is the ontological type, where we find the objectual metaphor. The objectual metaphor is a conceptual metaphor that has its origins in our experiences with physical objects and permits the interpretation of events, activities, emotions, ideas... as if they were real entities with properties. This type of metaphor is combined with other ontological classical metaphors such as that of the “container” and that of the “part-and-whole”. The combination of these types leads to the interpretation of ideas, concepts... as entities that are part of other entities and are conformed by them. This interpretation is clear in the axioms of existence and link, as they are mentioned in a classical Spanish textbook on Geometry (Puig Adam, 1965, p. 4):

> Ax. 1.1. We recognize the existence of infinite entities called <points> whose set will be called <space>.
>
> Ax. 1.2. The points of the space are considered grouped in partial sets of infinite points called <planes> and those from each plane in other partial sets of infinite points called <lines>.

METAPHORICAL EXPRESSIONS OF OBJECTUAL METAPHOR

We consider it necessary to make a distinction between the metaphorical expressions and the conceptual metaphors, as highly interrelated but different ideas. This distinction permits to establish generalizations that, otherwise, would remain invisible. The metaphorical expressions may be grouped into conceptual metaphors, and seen as isolated, they can be thought of as individual cases of particular conceptual metaphors.

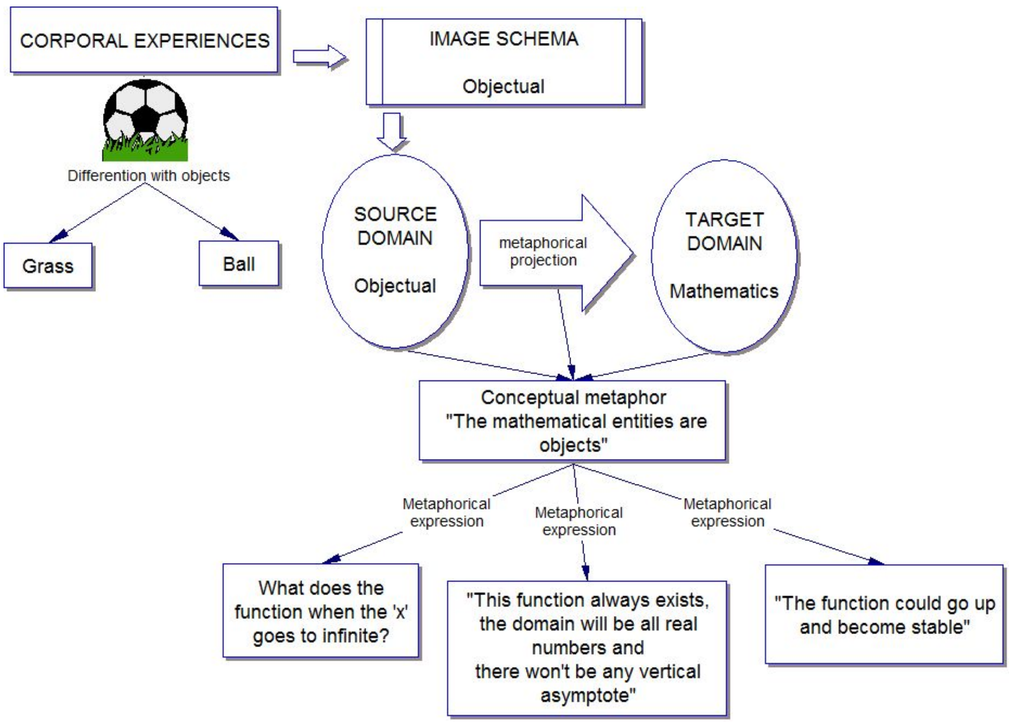


Figure 1. A representation of the objectual metaphor

The conceptual metaphor “The mathematical entities are physical objects” is a grounding ontological metaphor. Figure 1 (Acevedo, 2008, p. 138) illustrates the metaphorical projection with the different metaphorical expressions that appear when using this conceptual metaphor in a mathematics classroom where the graphical representation of functions is being taught to students in high school. Figure 1 shows our experiences in the world of things and the interpretation of the physical objects as separated from this world context; these experiences generate the “objectual” image schema that become the source domain that is projected into the world of the mathematical objects. Table 1 refers to the source and target domains that intervene in the interpretation of this metaphor.

"The mathematical entities are physical objects"	
Source domain: Image schema	Target domain: Mathematical entities
Physical object	Mathematical object
Properties of the physical object	Properties of the mathematical object

Table 1. Domains of the metaphorical projection

THE OBJETUAL METAPHOR IN THE TEACHERS' DISCOURSE

The objectual metaphor is always present in the teachers' discourse because here the mathematical entities are presented as "objects with properties" that can be physically represented (on the board, with manipulatives, with gestures, etc.). In Acevedo (2008), metaphorical expressions of the objectual metaphor occur when the mathematics teacher refers to the graphic of a function as an object with physical properties. When he talks about the application of mathematical operations in order to obtain the first derivative of a function, the teacher uses verbal expressions and gestures that suggest the possibility of manipulating mathematical objects as if they were things with a physical entity (Acevedo, 2008, p. 137):

> Teacher1: The derivative of the numerator, no! You multiply by the denominator as it is, minus the numerator multiplied by the derivative of the denominator. Ok. Now you divide it by the denominator... square, it is. (...) This is the first derivative. Now, what's next? To operate, to manipulate... What's left?

The use of the objectual metaphor facilitates the transition from the ostensive representation of the object –written on the board, drawn with the computer, etc.– to an ideal and non ostensive object. Hence, the use of this type of metaphor leads to talk in terms of the "existence" of the mathematical objects. This use may lead the students to interpret that the mathematical objects exist within the mathematical discourse (internal existence) and, sometimes, may lead them to interpret that they exist like chairs and trees do (external existence, physical or real). In Acevedo (2008, pp. 136-137), we first find a classroom discussion on the domain of the logarithm function and later a discussion on the domain of the square root function, during the instruction of the graphical representation of functions. Here the "existence" is considered within the language game of the mathematical discourse, in comparison to the former teacher's comments on the existence of the first derivative of a function:

Teacher2: The domain goes from zero to infinite because logarithms of *negative numbers do not exist*, logarithm *of minus one does not exist*. Shall we say with the zero included?

Teacher2: Not the negative… because *the square root of a negative number does not exist.* We could also say the real numbers without the negatives, or even easier, all the positive numbers, we can write it like this, with an interval, from the zero to the infinite, now the zero is included.

If the teacher is not careful enough with the way of using (or not using) the verb "exist" in his discourse, the students in this class may not remain within an "internal existence" position. Instead, they may change the "language game" (Wittgenstein, 1953) and assume the "external existence" of the mathematical objects. In the following paragraph, a third different teacher explains the graphical representation of functions to the students in the class and explicitly mentions the idea of existence, although he does so in a rather controversial way (Acevedo, 2008, p. 137):

Teacher3: Then...this function always exists, the domain will be all real numbers and there won't be any vertical asymptote.

We observe a deviation in the "expected" use of the word "exists" within the language game of the mathematics discourse. It would be reasonable to affirm that the images of the values in the domain exist or are defined. When attributing the existence to the whole function instead of talking about its images, the teacher is making a use of the word "exists" that can lead to the understanding of the function as a "real" object with properties, like a chair or a person. Moreover, by doing so, the teacher can promote the movement from the mathematical internal existence of the object to a physical external existence.

DIFFERENTIATON BETWEEN OSTENSIVES AND NON OSTENSIVES

We draw on the theoretical distinction between ostensive and non ostensive objects as established by the onto-semiotic approach to mathematics education (Godino, Batanero & Font, 2007, p. 131):

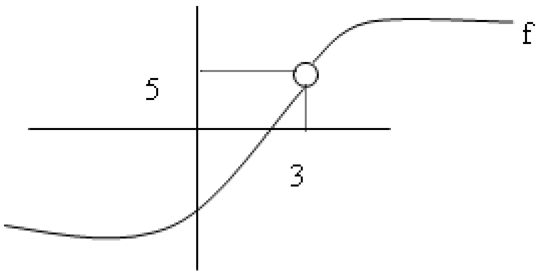
> Ostensive–non-ostensive Mathematical objects (both at personal or institutional levels) are, in general, non-perceptible. However, they are used in public practices through their associated ostensives (notations, symbols, graphs, etc.). The distinction between ostensive and non-ostensive is relative to the language game in which they take part. Ostensive objects can also be thought, imagined by a subject or be implicit in the mathematical discourse (for example, the multiplication sign in algebraic notation).

In the mathematics discourse, it is possible to talk about ostensives representing non ostensives that do not exist. For example, we can say that f'(a) does not exist because the graphic of f(x) has a pointed form in $x = a$. This gives us another example of the semiotic and discursive complexity of the classroom discourse when referring to the existence of mathematical objects. In Acevedo (2008, p. 320) we find the following remark made by a teacher in his classroom discourse:

> Teacher4: As you can see, the one-sided limits are not the same and then the limit does not exist… or the limit is infinite, I mean it is more or less infinite.

In García (2008, appendix 2, p. 8), we find a teacher who uses a discourse with ostensives (*f*(3)) that represent non ostensives that do not exist. He does not say that they do not exist but literally says that "we cannot have them". The instances from García's research were obtained in a similar methodological setting –in regular high school classrooms focused on functions and graphs–, to that constructed for the study that was developed by Acevedo.

> Teacher 5: […] Let's imagine this function:



> What is the domain of *f*? [He answers on the board $\Re - \{3\}$]. And *f*(3)? Don't make the mistake of saying five, because it is not in the domain and we cannot have an image. We are not worried about *f*(3), but about going as closer as possible to three, before and after the three. Attention, where are the images? Now I don't have a formula.
>
> Students: Near the five.
>
> Teacher 5: And now if I get closer to three on the right, where are its images?
>
> Students: Over the five.
>
> Teacher 5: Yes we can say limit of *f*(*x*) when *x* goes to three.
>
> Students: But *f*(3) does not exist.
>
> Student: But the asymptote does not touch it either.
>
> Teacher 5: It is curious but $\lim_{x \to 3} f(x) = 5$ [on the board]. It is not defined in three but its limit does exist. That limit exists without having the analytical expression and without having *f*(3).

In order to talk about the existence of certain non ostensives, we have to use a discourse with ostensives constituted in accordance to the "grammar" that regulate the construction of the well-established formulas. This type of discourse is frequently used by many students, as the following remark shows (Acevedo, 2008, p. 368):

Student: Then you do the same here, well you first put the zero here because it is… you look for it, it is the number that you have obtained and the derivative is zero. Then in minus one and in one, you also have to write a zero, but as you have vertical asymptotes you can say that the derivative does not exist, neither does it exists the function. Then you do it with minus one and zero and you get a negative, with the same procedure, and then with the zero and the one you get a positive. As it is positive, it means that you have a minimum here because you have this drawing and it is a minimum.

The use of ostensives that represent non ostensives that do not exist may create confusions in the students' thinking, although it also can turn into philosophical implicit reflections for them. This is the case with a student (Acevedo, 2008, p. 213) that makes a distinction between "to be" and "to exist". He misunderstands the vertical asymptote and makes a mistake:

Teacher5: Could you explain a bit more about the vertical asymptote?

Student: I understand that the vertical asymptote is the value that does not exist in the function.

The existence of well-established ostensives that represent non ostensives that do not exist facilitates the consideration of the non ostensive object as something different from the ostensive that represents it. Duval's work (2008) has pointed to the importance of the different representations and transformations between representations in the students' understanding of the mathematical object as something different from its representation.

Many textbooks of mathematics, implicitly or explicitly make the students observe that an object has many different representations and it is needed to distinguish the object from its representation. In a popular Catalan textbook (Barceló et al., 2002, p. 89), for instance, the following is written:

> In all the activities made, you have been able to observe the different ways of expressing a function: as a statement, as a table of values, as a formula and as a graphic. You always have to remember these four forms of representation and know how to go from one to another.

However, these textbooks frequently tend to identify the mathematical object with one of its representations. In the same Catalan textbook (Barceló et al., 2002, p. 90), it is said "Given the function f(x) = 1/x …" The explanation is that the representation is identified with the object or differentiated from it depending on the purpose. Peirce (1978, §2.273) mentions this idea in his work:

> To stand for, that is, to be in such a relation to another that for certain purposes it is treated by some mind as if it were that other. Thus a spokesman, deputy, attorney, agent, vicar, diagram, symptom, counter, description, concept, premise, testimony, all represent something else, in their several ways, to minds who consider them in that way.

In the mathematical practices, we constantly identify the object with its representations and, on the other hand, we make a distinction between the object itself and some of its representations. The rules of this language game, where the objectual metaphor is crucial, may be difficult to learn for some students. When we deal with physical objects, we can differentiate the sign from the object (for instance, the word "watch" and the physical object "watch"). The objectual metaphor as it is used in the mathematics discourse permits to transfer this differentiation to the mathematical objects and, therefore, we also differentiate the "representation" from the "mathematical object". Moreover, the type of discourse that we produce within the mathematics classroom, leads us to infer the "existence" of the object as something independent from its representation. This situation let us conclude about the existence of a mathematical object that can be represented by means of different "representations".

FINAL REMARKS

In this report we have argued that the objectual metaphor plays a central role in the pedagogical process in the classroom, where teachers (and, consequently, the students) talk about mathematical objects and physical entities. We have shown how the use of metaphorical expressions of objectual metaphors in the mathematics classroom discourse leads the students to interpret the mathematical entities like "objects with existence". On the other hand, the mathematics discourse about ostensives representing non ostensives that do not exist and about the identification (differentiation) of the mathematical object with one of its representations leads the students to interpret the mathematical objects as being different from its ostensive representations. As a consequence, the classroom discourse helps to develop the students' comprehension of the non ostensive mathematical objects as objects that have "existence".

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MATHEMATICAL ACTIVITY IN A MULTI-SEMIOTIC ENVIRONMENT

Candia Morgan and Jehad Alshwaikh

Institute of Education, London

Abstract: Different semiotic systems provide different sets of resources for the construction of mathematical meanings. In this paper, we argue that a multi-semiotic environment not only affords rich potential for developing mathematical concepts but may also affect more fundamentally the goals of student activity. We present a multimodal analysis of an episode from a teaching experiment with software that allows students to construct animated models using equations. In the course of this short episode, the students made use of drawing and gesture as well as mathematical and everyday speech in ways that transformed the purpose of their activity from drawing a static pattern to constructing an animation, changing the mathematical problem from using velocities to determine the direction of motion to considering how to stop a moving object.

INTRODUCTION

The study of mathematical language and other sign systems has developed in recent years with increasing recognition of the importance of a variety of specialised mathematical systems, including graphical and diagrammatic forms as well as linguistic and symbolic (Alshwaikh, 2008; O'Halloran, 2005), and of interaction between the various systems (Duval, 2006) in the development of mathematical discourse. Moreover, where mathematical communication takes place in face-to-face contexts, body language and gesture also play a part (see, for example, Bjuland, Cestari, & Borgersen, 2007; Radford & Bardini, 2007). The development of new modes of representation through the medium of new technologies has generated further interest in this area by opening up possibilities for dynamic forms and for interactions between systems (such as graphs and algebraic equations) in ways that were previously inaccessible.

From a social semiotic perspective (see Morgan, 2006), each semiotic system provides a different range of meaning potentials (Kress & van Leeuwen, 2001). For example, as O'Halloran argues, visual modes such as graphs allow representation of 'graduations of different phenomena' rather than the limited categorical distinctions available through language or algebraic symbolism, while dynamic modes additionally allow the representation of temporal and spatial variation (2005, p.132). Such different potentials have been exploited in the design of interactive learning environments (for example, Yerushalmy, 2005) and research from various theoretical perspectives has focused on the kinds of mathematical meanings constructed by students working with such novel representations, especially in the contexts of use of dynamic geometry (for example, Falcade, Laborde, & Mariotti, 2007).

In this paper we report a teaching experiment, involving a multi-semiotic interactive learning environment, MoPiX, produced as part of the ReMath project [i]. This environment and the associated pedagogical plan were designed to provide multiple linked representations to support students' development of concepts of velocity and acceleration [ii] by allowing them to experience and connect formal symbolic definitions and dynamic animations. We report elsewhere how the semiotic resources provided by this environment appear to support students' development of ways of operating with velocity and acceleration compatible with their formal definitions and with Newtonian laws of motion (Morgan & Alshwaikh, 2008, 2009). Here, however, we discuss the influence of the multi-modal environment on the process of problem solving, presenting an example of an episode in which interaction with the various available semiotic systems transformed the goals of the activity.

A MULTI-SEMIOTIC ENVIRONMENT

The interactive learning environment of MoPiX allows users to construct animated models and investigate their behaviour. It is conceived as a constructionist toolkit (Strohecker & Slaughter, 2000), providing fundamental elements (in this case objects, represented by shapes such as squares or circles, and equations) with which students can build models, form and investigate hypotheses by activating their constructions and observing their behaviour. The environment of MoPiX is essentially multi-semiotic, linking symbolic representations (equations) using a variation of standard mathematical notation, with animated models and graphs. In addition, the planned pedagogy of the teaching experiment, the social environment of the classroom and the nature of the technology (individual tablet PCs) were intended to encourage use of a range of modes of communication, including talk, gesture, various paper-and-pencil representations and the electronic sharing of constructions through the ReMath portal [iii]. The variety of semiotic systems provides a range of meaning potentials and hence rich opportunities for users to construct meanings for the mathematical objects and concepts represented.

$x(\text{object\_1},t)=x(\text{object\_1},t-1)+Vx(\text{object\_1},t)$

x-coordinate of the circle (object_1) is augmented by the value of Vx as time (t) increases

$Vx(\text{object\_1},0)=3$

variable Vx, assigned an initial value of 3 (when time=0), may be considered the velocity of the circle

$Vx(\text{object\_1},t)=Vx(\text{object\_1},t-1)+Ax(\text{object\_1},t)$

Vx (velocity) is augmented by the value of Ax as time (t) increases

$Ax(\text{object\_1},t)=-0.1$

variable Ax, in this case assigned a value of -0.1, may be considered the acceleration of the circle

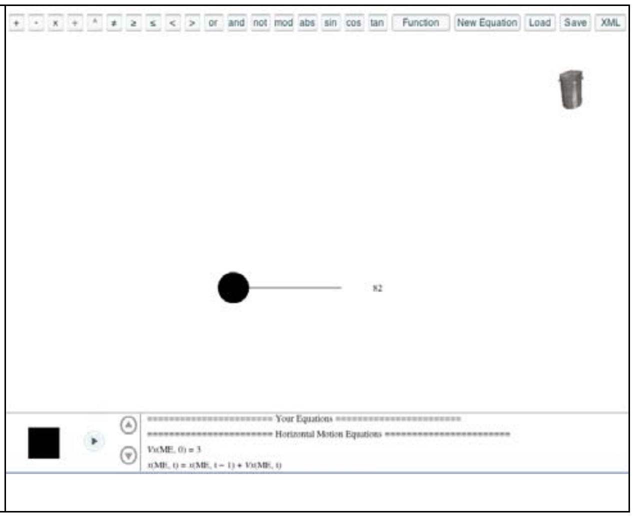


Figure 1: A set of equations defining horizontal motion

A MoPiX object is caused to move by applying a set of parametric equations defining how its position will change over time. For example, the set of equations shown in Figure 1 would cause object_1 (the circle in the screen shot) to move in the horizontal direction with an initial velocity of 3 and constant acceleration -0.1 [iv]. Horizontal and vertical components of motion are defined separately. The notation thus draws attention to vector concepts of velocity and acceleration, while the form of the equations embodies the definitions of velocity as change in position and acceleration as change in velocity. Equations may be taken from a library of basic equations, edited or authored directly and applied to objects. Once equations have been added to one or more objects, the model may be played and each object in the model will move according to its own set of equations. (It is also possible to apply equations defining interactions between two or more objects.) Visual feedback from the animated model allows students to test their hypotheses about the functioning of the equations they have used. They may then continue their investigations: editing the sets of equations and adding new objects to their model.

THE TEACHING EXPERIMENT

A pedagogic plan was devised, in collaboration with teachers in a London tertiary college, with the educational goal of developing understanding of ideas of velocity, acceleration and force. A group of seven students (aged 17-18 years) volunteered to participate in the study, which took place during 10 weekly one-and-a-half hour sessions outside the normal curriculum. The participants were all enrolled in an Advanced level mathematics course. They had not previously studied the mathematics of motion (though some had studied physics) and, though all were familiar with the formal definitions of velocity and acceleration as rates of change, a pre-course paper-and-pencil questionnaire revealed that they nevertheless relied on informal non-Newtonian intuitions in order to describe and explain motion. Participation in the study was presented to the students as extra preparation for the Applied Mathematics (Mechanics) module that they were to start the following term.

The intended pedagogy was founded on constructionist principles, providing students with access to the means of manipulating the elements of the MoPiX microworld while posing challenges that would encourage them to experiment, shaping their own goals and hypotheses. The episode we consider in this paper is taken from the second session. During the first half of this session, the students had been given a worksheet with a sequence of tasks introducing them to the equations needed to create straight line motion, to the idea that the direction of motion is determined by a combination of velocities in the horizontal and vertical directions and to the equations for drawing a trace of the motion of an object. Having done the set tasks, they experimented in a playful way with these and a range of other equations taken from the MoPiX equation library, creating multi-coloured objects moving in various ways, not only in straight lines. They then had their attention drawn to the next task on the worksheet: '*As a group, plan a design formed by several lines.*' In designing this challenge, it was anticipated that students would make use of the combination of horizontal and vertical motions to make objects move in different directions drawing straight lines with different gradients, thus developing their appreciation of relationships between components of motion in two dimensions.

DATA ANALYSIS

During the teaching experiment we gathered data in the form of video and audio records of pairs of students, together with any incidental paper-and-pencil work. In addition we administered paper-and-pencil pre- and post-questionnaires. Our broad research aim was to investigate how students would make use of the semiotic resources offered by MoPiX and the broader classroom environment in the course of their work on tasks related to motion. We were particularly interested to see what contribution the various resources might make to students ways speaking about and operating with ideas of velocity and acceleration.

Extracts of video were identified as ‘of interest’ and were transcribed. In accordance with our research focus on multiple semiotic resources, extracts chosen for transcription included, in particular, those where several modes of communication were being used together. We consider the form of transcription to be part of the analytic process as a preparation for the multi-semiotic analysis needed to address our research questions. The use made of each mode of communication was thus recorded in a separate column of a spreadsheet, allowing both horizontal (a snapshot of all simultaneous semiotic activity at each ‘moment’) and vertical (an overview of semiotic activity within a particular mode through the whole period of the extract) examination of the data. The transcript was divided into ‘moments’ of communication that were considered to have some meaningful coherence; this division was a pragmatic consideration with no explicit theoretical basis.

Our approach to analysis involved both the application of a priori categories and the iterative definition and refinement of categories derived from the data. In the episode discussed below, we discuss the interaction between mode of communication (an a priori categorisation) and the goal of the students’ activity (a strand of analysis arising from our exploration of the data). The episode is a five-minute extract from about half way through the second session, focusing on two male students, Baz and Vin as they start to work on the design task.

CM if two of you think about a pattern maybe with some parallel lines and perpendicular lines and a number of lines to make some sort of a pattern on the screen. Yeah? And design that in advance and then one of you does some of the lines, the other does the other set of lines and then you combine the two to make the whole pattern. Yeah? So you might want to do some pencil paper work first. think about your design

Vin Do you have a pen?

Baz Just use the computer

Vin Yeah.. in Paint [*this refers to the Paint drawing programme on the PC*]

Baz, Vin [*laughing together*] yeeaah.. Paint

Vin Bring it over

[… *about a minute trying to find the Paint programme on an unfamiliar PC*]

Baz Here we go. All right so we can do the horizontal lines and vertical lines.

Vin Can't we do the diagonal ones

Baz We can do squiggly lines, but

Vin Like in our thing, if it has a formula, then it's not going to be random is it

Baz Yeah exactly

Vin Do a log [*i.e. logarithmic function*] actually you can't do log because it'll get kind of mad because it'll go on for ever

Baz You can have different colours right [*both laugh*] so make it like a firework so it goes like that and then you could have vertical ones like that and diagonal ones and another horizontal, I mean vertical one going even further up

Vin like a sparkler

Baz yeah but we need it to start from here and then these start after this one and then .. I don't know how that'll work

We originally identified the extract for detailed transcription and analysis because it seemed interesting for two reasons. In the first place, the students chose to make use of the *Paint* programme on their PCs, thus providing us with an opportunity to consider how they were making use of the various modes of communication available to them. Secondly, the mathematical nature of the problem they were working on and the focus of their MoPiX programming task changed through the course of the episode.

Strand 1: Mode.

This strand of analysis was identified as a fundamental component of our social semiotic theoretical framework and of importance in addressing our research questions. It was initially defined by a priori categories. Each moment was first coded according to the mode or modes in use. The initial categories used were:

- spoken language (subdivided into everyday/ mathematics/ MoPiX registers)
- written language (natural language/ conventional mathematics notation/ MoPiX notation)
- drawing (outcome of MoPiX animation/ aid to problem solution)
- gesture (pointing/ mimicking MoPiX motion/ other)
- MoPiX equations (library/ authored/ complete models)

During the coding process, however, it became clear that this categorisation was not sufficient by itself to capture the ways in which the meanings produced during the extract were realised using the available semiotic resources. In particular, the functional relationship between the various modes used in any moment appeared significant. For example, Baz, creating the initial design, used simultaneous words and drawing (see Table 1). The initial causal connection 'so' made by Baz between the possibility of using *colours* and the decision to make the design ‘like a firework’ draws attention to the significance of the semiotic potential of the available technology. Both the *Paint* programme the students had chosen to use instead of paper-and-pencil and MoPiX itself afford easy application of a range of colours. It seems that the availability of colour as a resource suggests representational possibilities that might not have been chosen when working with traditional tools.

	spoken language	*drawing (in Paint)*
Baz	You can have different colours right [both laugh] so make it like a firework	
	so it goes like that	draws vertical bottom to middle twice
	and then you could have vertical ones like that	horizontal middle to left; horizontal middle to right twice
	and diagonal ones	3 diagonals: middle to NW; middle to NE; middle to SW
	and another horizontal, I mean vertical one going even further up	vertical middle to top

Table 1: Interaction of speech and drawing

There is a direct congruence between Baz’s words (*spoken -mathematics*) and his *drawing*; as he speaks the word ‘vertical’, he draws vertical lines (although he initially confuses vertical and horizontal). In addition, however, the motion of drawing (*gesture*) mimics the imagined motion of the firework (*spoken -everyday*) thus combining use of the static meaning potential of the descriptive language - vertical, horizontal, diagonal - and the completed drawing (displaying the outcome of the intended MoPiX animation) with the dynamic meaning potential of gesture.

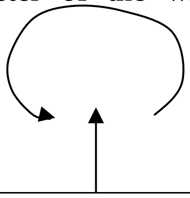
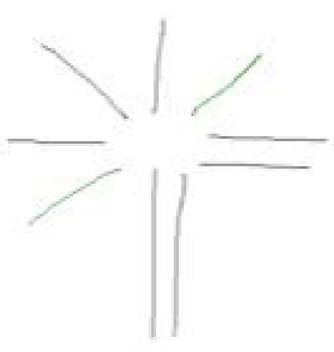
	spoken language	*gesture*	*drawing*
Vin	like a sparkler		
Baz	yeah but we need it to start from here and then these start after this one and then I don't know how that’ll work	slide-pointing bottom to middle, then slide-pointing anticlockwise circle around the perimeter of the whole shape 	

Table 2: Interaction of speech, gesture and drawing

In the next moment (see Table 2), Vin echoes Baz's original everyday discourse identification of the design as a firework, now specifying it more concretely as a sparkler, then Baz uses *gesture* to interact with the now complete *drawing*, simultaneously verbalising the process needed to construct the design with moving objects (*spoken -MoPiX*). In this case, the students use the drawing mode as readers, producing new meanings for the drawing through their use of spoken language and gesture. The spoken language naming of the design as firework/ sparkler here provides a holistic (everyday) image of the outcome of the design, while Baz's simultaneous use of language and gesture affords a dynamic representation of the development of the animated design over time.

Strand 2 Goal of the design activity: static versus dynamic outcome

In order to capture the complexity of the relationships between modes in use in any moment, the coding was developed to take account of the changing nature of the design activity. This strand of analysis was developed after initial examination of the whole extract, emerging as a theme from the data. It was observed that the ways in which the participants talked about their pattern included attending both to the properties of the lines drawn as traces of the MoPiX animation (a static outcome) and to the properties of the motion itself (a dynamic outcome). At the beginning of the chosen extract, the task is introduced by the teacher/researcher, using what we have now characterised as a static representation of the goal of the task:

> think about a pattern maybe with some parallel lines and perpendicular lines and a number of lines to make some sort of a pattern on the screen.

This static goal is taken up initially as the students discuss the types of lines they might make using MoPiX (horizontal, vertical, squiggly, defined by a formula). By the end of the episode, however, the focus of the activity is related to the motion of objects needed to construct the pattern. This focus was not the anticipated task of coordinating horizontal and vertical components of motion in order to draw lines with particular gradients. Rather, the students identified an important new goal that influenced the progress of their work through the remainder of the session: to find a way of stopping a moving object. This proved a substantial problem for them as its solution demanded a more analytic use of MoPiX equations than they had developed up to that point, in particular the use of equations specifying values of velocity or acceleration at a given time.

The question thus arises as to why this change from static to dynamic goal may have occurred. We coded references in any mode to the pattern or to components of the pattern as static or dynamic, identifying for each reference the mode and the indicators used to apply the code. Through this process of coding it became apparent that significant moments in the students' developing image of their pattern occurred as they moved between different modes of representation (see Table 3). In particular, the naming of the pattern as a 'firework' (apparently influenced by the articulated recognition of the possibility of using colour in their design), and interaction using gesture with the drawing of their design introduced new semiotic resources with meaning potentials that highlighted dynamic aspects of the design.

(i)	The original MoPiX programming challenge focuses on the direction of lines: “parallel”, “perpendicular”.	written and spoken language - mathematics	static
(ii)	Vin discusses the need for mathematical formulae to define MoPiX motion.	spoken language - mathematics; MoPiX programming	static
(iii)	Vin introduces of the idea of using a formula involving ‘log’ and the dynamic idea that it will ‘go on forever’, perhaps invoked by a concept image of a logarithmic graph (note O’Halloran’s (2005) identification of the dynamic meaning potential of mathematical graphs).	spoken language - mathematics; imagined graph?	static - dynamic
(iv)	The use of Paint or perhaps the use of MoPiX enables the suggestion to use different colours.		
(v)	This suggestion then seems to trigger the naming of the design as a “firework”.	spoken language - everyday; imagined dynamic object	dynamic
(vi)	The firework idea is realised in Paint.		
(vii)	Interaction with this drawing through gesture introduces a temporal aspect.	drawing; gesture	dynamic
(viii)	This temporal aspect is taken up immediately by Baz's verbal description of the motion "we need to start from here and then these start after this one"	drawing; gesture; spoken language - MoPiX	dynamic
(ix)	The MoPiX programming challenge then becomes the problem of how to make motion stop.	MoPiX programming	dynamic

Table 3: Change from static to dynamic

CONCLUSIONS AND DISCUSSION

The analysis we have offered here has focused on the multiple modes of communication used by these two students. Not only does each mode have its own set of meaning potentials but the different modes also interact, providing further potential. The complex interaction of use of language, drawing, gesture and MoPiX programming thus contributes to the construction of new meanings in the communication between the two students. The new semiotic resources provided by

MoPiX play relatively little explicit part in the episode we have considered. Nevertheless, we would argue that they play an influential role in shaping the students' activity, not only because the overt goal of the task involved use of MoPiX but also because the students were influenced by their recent use of MoPiX and their awareness of its potential. Moreover, the technological environment and the students' familiarity with its capabilities enabled them to choose to use Paint and its colour resources rather than traditional monochrome paper-and-pencil.

The resources afforded by gesture have been identified as significant in the move from a static to a dynamic goal. We consider here not only the pointing gestures accompanying the deictic spoken language seen in Table 2 but also the bodily movement implicit in the act of drawing in Table 1. This draws attention to the duality of the *drawing* mode: it is both a product - the outcome or picture - and the process by which the outcome is produced. In different moments it thus has both static and dynamic meaning potential and may play an important part in shifting focus between the two types of meaning.

However, the change from a static to a dynamic focus for the students' problem solving activity was not solely a product of the multi-semiotic environment. The nature of the pedagogic discourse of the classroom also played an important role. In particular, the students had enough agency within the classroom to enable them to make decisions about their own activity. In the first place, they were able to decide to ignore the teacher/researcher's suggestion to use paper-and-pencil, choosing to use Paint instead. Further, they were able to follow their own interests in designing their firework, thus enabling the change in the focus of their attention. Indeed, at a later stage in the same lesson, the teacher/researcher worked with this pair to help them solve the MoPiX programming problem of making a moving object stop, using techniques whose introduction had been planned for a later lesson.

Our analysis of this episode illustrates the very complex space of communication and learning and, we hope, contributes to Kress's call for development of theory of learning from a social semiotic perspective (Kress, 2008). The focus of students' attention and the direction of their learning are shaped by the multi-modal resources available and the interactions between them. However, this takes place within a learning environment that affords and/or constrains students' agency and their ability to change the direction of their activity in ways that will be considered legitimate.

NOTES

[i] ReMath (Representing Mathematics with Digital Technologies) funded by the European Commission FP6, project no. IST4-26751.

[ii] MoPiX also has potential to be used in many other areas of mathematical modelling.

[iii] MoPiX version 1 is available at http://remath.cti.gr; version 2.0 is under development at http://modelling4all.nsms.ox.ac.uk/

[iv] Units are non-standard and not identified explicitly in the notation.

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ENGAGING EVERYDAY LANGUAGE TO ENHANCE COMPREHENSION OF FRACTION MULTIPLICATION

Andreas O. Kyriakides

The Open University, United Kingdom

Dedicated to the memory of the Cypriot teacher Georgia Kyriakidou

Using as analytic frames the Pirie-Kieren model and theoretical constructs on the role language and communication could play in the process of learning, I attempt to sketch the pathway of understanding of a sixth-grade student (Avgusta) while she is attempting to make sense of fraction multiplication. The viewing of mathematical understanding as a dynamic process proved supportive in enabling me to identify the role language could play both at any level and in the growth between levels of Avgusta's understanding. Occasioning learners to fold back to everyday language in order to collect the spontaneous interpretation of the word "of" and combine it with the scientific notation of multiplication could awaken learners' awareness that the interpretation of multiplication involves finding or taking a part of a part of a whole.

INTRODUCTION

The story to be recounted here evolves in a public elementary school in Cyprus, where I work as a full-time teacher. It is part of a two-year research studying the complexities of learning to compute fractions as revealed from the use of a novel peda-cultural tool. Though in Cypriot culture school mathematics textbooks introduce the concept of fraction with images of partitioned rectangles and circles, they make little or no use of diagrams when they show students the way to compute.

During the first year of the study I was the teacher of a fifth grade class (10 boys & 12 girls) and had to address all subjects' objectives set by the curriculum. Once a week, I took the role of a teacher-researcher and taught students how to learn fractions through manipulating diagrams. To be consistent and learn from my experiences I revisited my group of students a year later and conducted individual interviews in order to collect some retrospective evidence about the nature of their learning. It is the purpose of this paper to zoom in on one of those interviews and describe how one girl, Avgusta, could derive meaning in multiplication of fractions. Worthy of consideration is that in sixth grade my ex-students had been exposed to a different teacher's instructional mode which gave no emphasis on diagrams as a learning tool.

This study is of interest because it refers to an educational culture unused to use diagrams to compute fractions and more used to show and tell than to getting learners to make sense by using the diagrams as mediating tools. Its contribution lies in offering Avgusta's learning as grist for the learning and development of other pupils, beyond the local boundaries of the particular school.

THEORETICAL BACKGROUND

The role of language in learning and particularly the social role of other people in the development and use of language was explicitly stressed by Vygotsky when he emphasized the importance of getting students talking about their thinking in order to help them make sense of, or construct, mathematical meaning. Vygotsky also observed that there are differences between what pupils can achieve working alone and what they can achieve when assisted by someone more experienced, such as a teacher. He captured this in a phrase which in English is usually rendered by "zone of proximal development" (Vygotsky, 1978). This term suggests that the teacher wants to support awareness that is imminent but not yet available to learners and not do those things which learners can do, since this will only raise dependency. Bruner (as cited in Wood et al., 1976) while presenting Vygotsky's ideas in English, made use of the metaphor "scaffolding" to refer to the assistance that a teacher some time may offer, which can be gradually withdrawn as students are able to function independently. The critical part of scaffolding is its removal or fading because when the support has not been removed, pupils may become dependent upon the teacher or any employed pedagogical tool (Love & Mason, 1992).

Zack (2006) appears in synch with Vygotsky's and Bruner's observations when she claims that because "students use sophisticated reasoning but may not see the power in the reasoning they are doing", it might be useful if teachers could "revisit what students have said, and connect their talk with the ways in which a mathematician would express those ideas" (p. 211). Linking everyday and scientific ways of knowing in order to support learners' imminent awareness is, according to Zack (1999), a much more challenging task than most researchers have appreciated.

The Pirie-Kieren theory and its associated model [Figure 1] is a well-established and recognized tool for listening and looking at growing understanding as it is happening. Growth in understanding is seen as a dynamical and active process involving a continual movement between different layers or ways of thinking, with no implication of a linear ladder-like system. These layers, which are intentionally represented in the form of eight nested circles so that the accent is put on the embedded nature of understanding, are named Primitive Knowing, Image Making, Image Having, Property Noticing, Formalising, Observing, Structuring and Inventising. A critical feature of this theory is the act of returning to an inner layer, or re-visiting and re-working existing understandings and ideas for a mathematical concept. This act is called "folding back" (Pirie & Kieren, 1989). A slightly differentiated form but equally important to folding back is that of "collecting". Its major difference from folding back is that, in collecting, the inner level activity does not involve a modification (or thickening) of the individual's earlier understandings. Instead, learners' efforts are concentrated on finding and recalling what they know they need to solve a task. They are consciously aware that this knowledge exists but their understanding is not sufficient for the automatic recall of it (Pirie & Martin, 2000).

METHOD AND METHODOLOGY

Avgusta, 12 years old when the interview was conducted, was one of the twenty two students participating in the study. I have chosen to present here selected pieces of her responses to a scenario on multiplication [Table 1], as well as explanations of these responses. By choosing particular moments and voicing them through a temporal sequence, I aim to convey not only a succession of Avgusta's learning experiences but also how she experienced this succession. What counts is not only the content and structure of the practice itself but also the ways in which it is talked about, perceived and assimilated by the learner.

When the principal of the school entered the classroom and asked the children what they were doing, they replied that they were learning how to multiply fractions. Then the principal asked who could come up to the board and show to her how to find the product 2/3 x 1/2 without performing any calculations but using only the area models. Orestes wrote the following on the board but the principal did not seem satisfied. If Orestes asked for your help, what would you say to him?

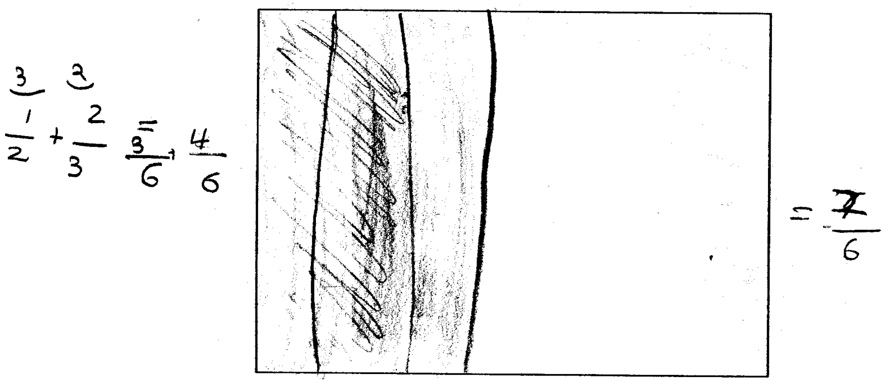


Table 1: Interview scenario

Using as analytic frames the Pirie-Kieren model for the growth of one's understanding, theoretical constructs on the role language and communication could play in the process of learning, as well as personal reflections on pedagogy, I shall attempt to map the growth of Avgusta's understanding. Throughout the analysis, my specific goal is to explore her thinking "in-change" and how this is accomplished and shared. In other words, how shifts in Avgusta's thinking occur and in what ways such shifts in thinking supported her understanding of the meaning of multiplication.

Taking the position with Doerr and Tripp (1999), I argue that shifts in thinking could be described in terms of an initial interpretation of the task situation and a later interpretation that stands in opposition to the initial interpretation. It is sensible to assume that somewhere between the two interpretations there will be evidence of what precipitated the change in Avgusta's thinking. For this reason, attention will be cautiously focused on the sequence of events between initial and later interpretations, as well as on identifying those characteristics that illuminate the growing understanding of Avgusta throughout the interview.

INTERVIEW FINDINGS

The conversation I had with Avgusta about the multiplication scenario [Table 1] is the focus of this section. The quoted transcript has been intentionally split into three parts each of which has a distinct subheading. This division is absolutely artificial and it does not imply any linearity in the girl's growth of understanding. Rather, it is meant simply to organize structurally the data and facilitate the development of discussion later on.

Avgusta's tenacious-but-futile struggle to recall and apply a half-remembered algorithm in order to shed meaning to the procedure of multiplying fractions

What really strikes me here is Avgusta's "trapped" awareness of the falsehood of her actions.

507 Interviewer: Would you like to write down what Orestes [Table 1 - scenario on multiplication] should have done?

508 Avgusta: Yes.

[Avgusta is drawing the first and second figure of sheet 5. See Table 2 below, read left to right, up to down direction].

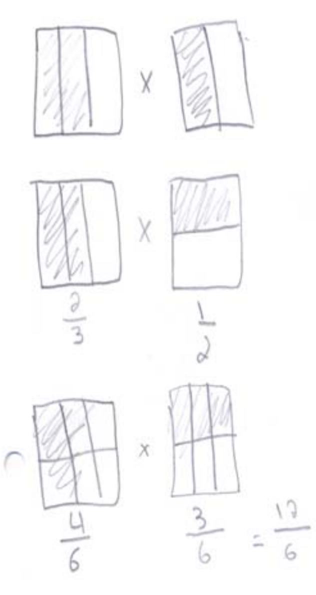
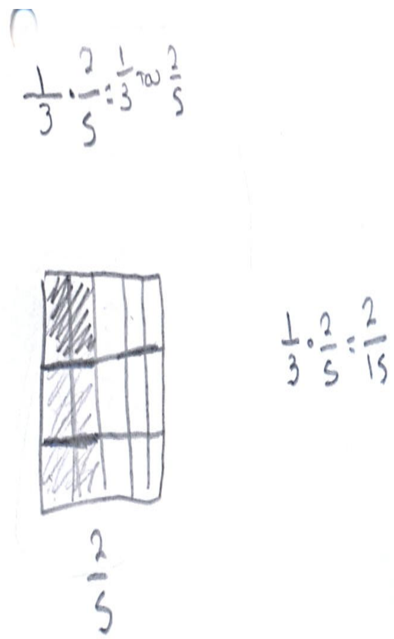
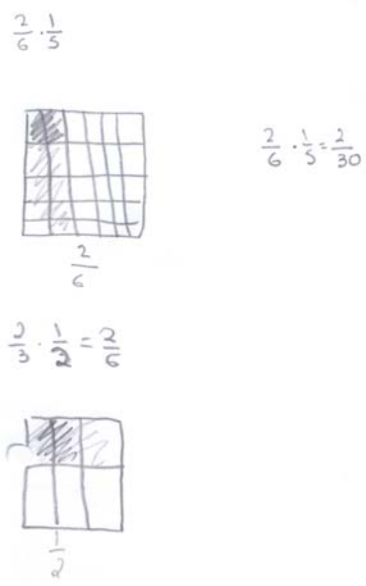
Sheet 5	Sheet 6	Sheet 7
		

Table 2: Avgusta's handwritten notes

509	Interviewer:	What are you doing now?
510	Avgusta:	The two thirds. He takes the two. Then… times one half. He takes the one and then we reverse them. No, I did it wrong.
511	Interviewer:	Why?
512	Avgusta:	I should have done it like that, a line.
513	Interviewer:	How about doing it below to see what you mean?

[Avgusta is drawing the third and fourth figure of sheet 5 – Table 2]

514	Avgusta:	Like this.
515	Interviewer:	Yes?
516	Avgusta:	We will reverse them and…we will reverse them.
517	Interviewer:	Why?
518	Avgusta:	To find…to find the same number of small boxes…to do them common fractions.
519	Interviewer:	Okay, you could do whatever you think Avgusta and we will see.

[Avgusta is drawing the fifth and sixth fig of sheet 5 – Table 2]

520	Avgusta:	We will reverse them.
521	Interviewer:	Okay.
522	Avgusta:	The two thirds…we will bring the one half…one minute…this one and then we will do times….We will reverse the one half and…
523	Interviewer:	And what do we have now?
524	Avgusta:	The small squares are now the same.
525	Interviewer:	Yes?
526	Avgusta:	But we have…
527	Interviewer:	What do you have there?
528	Avgusta:	Four sixths and here three sixths.
529	Interviewer:	Yes.
530	Avgusta:	And it becomes twelve sixths [She writes it at the bottom of sheet 5 – Table 2]
531	Interviewer:	So is this your answer?
532	Avgusta:	I think it is wrong.
533	Interviewer:	Why do you think so?
534	Avgusta:	[pause]
535	Interviewer:	Would you like to tell me why do you think it is wrong?
536	Avgusta:	But I don't know sir.

An invocative intervention aimed to occasion the link between everyday language and multiplication notation

The point that merits attention here is that Avgusta's folding back to everyday language could open the door for her to notice fractional symbols from a lens, which in turn could affect her way of thinking.

569 Interviewer: Okay. Now I would like to ask you something else. What does "times" mean? For instance, when we say one half times one hundred, what does that mean? You may write it down if you want.

[Avgusta is writing on the top of sheet 6 – Table 2]

570 Avgusta: We will multiply one half times one hundred.

571 Interviewer: Yes. Could you not say "we multiply"? How about our everyday language? Will you say one half times? Or, do we use any other word?

572 Avgusta: The word of?

573 Interviewer: How about saying it to see what you mean?

574 Avgusta: One half of one hundred.

575 Interviewer: That is? What does it mean? One half of one hundred is what?

576 Avgusta: Fifty.

577 Interviewer: Could you tell me Avgusta what does one half mean?

578 Avgusta: They are two and we are taking the one.

579 Interviewer: Nice. If I had one fourth, what does that mean?

580 Avgusta: There are four and I take one of them.

Educating awareness through encountering conflicting results and detecting the origin of the conflict

After Avgusta had been exposed to the foregoing intervention, she worked on the examples 1/3 x 2/5 [Table 2 – sheet 6] and 2/6 x 1/5 [Table 2 – sheet 7]. Lines 720-759 are indicative of what had been exchanged between me and Avgusta later on. Of great importance here is the gradual refinement of the girl's awareness of what it means to multiply two fractions, and the restructuring of ill-defined algorithmic knowledge.

720 Interviewer: Which way from the two, do you think, could help a child to understand what multiplication means? If you show him that you should multiply the… But, first, Avgusta do you know how we could multiply two fractions?

721 Avgusta: Yes, don't we do them common fractions?

722 Interviewer: Could you show me the example two thirds of one half, with the way of area models?

[Avgusta is drawing the second figure of sheet 7 – Table 2]

723 Avgusta: We will do the one half, we will take the one and then we will divide it in three…vertical ones and we will take the two.

724 Interviewer: Would you like to shade again what are you going to take?

725 Avgusta: These here [She shades again the two left small squares of the top row of the second figure of sheet 7 – Table 2].

726 Interviewer: Could you now tell me which your result is?

727 Avgusta: Two sixths.

728 Interviewer: Right. Earlier Avgusta we had this example again, it was on sheet 5 [Table 2]…and you found what?

729 Avgusta: Twelve sixths.

730 Interviewer: You found twelve sixths and now you found two sixths. Which of the two is the correct one? Earlier you said that when we multiply we do the fractions common ones, didn't you?

731 Avgusta: Yes.

732 Interviewer: Here [He points to sheet 5 – Table 2] you did common fractions, didn't you?

733 Avgusta: Yes.

734 Interviewer: You did two thirds, four sixths, and one half, three sixths. And what did you do then?

735 Avgusta: I did it times.

736 Interviewer: Could you explain a bit more?

737 Avgusta: I did four sixths times three sixths.

738 Interviewer: And how much did you find?

739 Avgusta: Twelve sixths.

740 Interviewer: How did you find twelve?

741 Avgusta: Four times three.

742 Interviewer: And how about six?

743 Avgusta: Because the denominators are…

744 Interviewer: But here [He points to sheet 7 – Table 2] how much did you find?

745 Avgusta: Two sixths.

746 Interviewer: Which of the two is the correct one?

747 Avgusta: This one, the two sixths.

748 Interviewer: Could you tell me why?

749 Avgusta: [pause]

750 Interviewer: You saw it here Avgusta, didn't you? Whereas there [He points to sheet 5 – Table 2]?

751 Avgusta: I didn't see it.

752 Interviewer: What should you have done here [He refers to sheet 5 – Table 2], do you think?

753 Avgusta: The same with this one [She points to sheet 7 – Table 2].

754 Interviewer: So, how do we multiply Avgusta? Do you see here [He points to sheet 5 – Table 2]? There was something wrong. When we multiply two fractions, we multiply the numerators…

755 Avgusta: And the denominators.

DISCUSSION

Avgusta's main difficulty seems to be a dependence on a half remembered algorithm. The way she manipulates the rectangles she drew [Table 2 – sheet 5], her rapid but purposeful shift from solely vertical to both vertical and horizontal type of partitioning [lines 507-518], as well as the multiplying of the numerators of the newly formed common fractions [lines 527-530], all could suggest that her understanding of multiplication is compartmentally drawn upon a vague memory of the standard change-into-common-denominators rule.

The ability to produce a partition of a partition in the service of finding the product of 2/3 x 1/2 might not be straightforward to Avgusta because it entails the composition of the operator “2/3 of” and the operator “1/2 of”. This idea is complex because it is removed from the whole number knowledge that learners could employ when first introduced to a single operator, such as “1/2 of”.

In lines 532-536 Avgusta is observed to express concerns about the correctness of her actions but is failing to exemplify the origin of this uncertainty, at least in the short term. This could indicate that after using diagrams, Avgusta pauses and reflects by considering what it is that the results tell her. It is possible that while checking against her intuitions that the results seem to be reasonable and roughly what she expects, the girl encountered an internal conflict which, in turn, generated doubt. Avgusta’s assertion that she knows that something went wrong [line 532] but does not know what [line 536], catches my attention and opens the possibility that I could provide for her some cognitive “scaffolding” (Wood et al., 1976) to support, and perhaps transform that state. There was a sense of her having, and being aware that she has the necessary understandings but that these are just not immediately accessible.

One of my enduring questions, thus, while interviewing Avgusta [lines 569-580] was in regard to the role I could play in pulling to the forefront of her mind the “Primitive Knowing” (Pirie & Kieren, 1989) that was going to be the basis for locating the source of perplexity. My intention was to encourage the girl to keep in touch with her personal way of knowing mathematics and sustain a back and forth movement, not unidirectional, between that understanding and the conventions of the culture. It is for this reason I occasioned [lines 569-580] Avgusta to “fold back” (Pirie & Kieren, 1989) to everyday language, “collect” (Pirie & Martin, 2000) the spontaneous interpretation of the word “of” and combine it with the scientific notation of multiplication. This invocative intervention resulted in the student returning to an inner, more localized layer of understanding, which, in turn, seems to have given rise to a succession of “Image Making” activities (Martin, 2008). The handwritten notes on sheets 6 and 7 [Table 2] are indicative of the replacement of faded images of multiplication by meaningful diagrammatic illustrations linking recursive area partitioning with the respective symbolic notation.

It is of great importance to stress here that it is the response of Avgusta to the particular intervention that determined the actual nature of it, namely, to occasion folding back to existing understanding, searching for, finding and then remembering this understanding (Martin, 2008). If the girl did not assign herself the everyday meaning of the word “of” to “x” or “times” [lines 569-576], it is ambiguous whether Avgusta would awaken her awareness that the interpretation of multiplication involves finding or taking a part of a part of a whole. Standard multiplication symbols appear, hence, not mere marks on paper for her but become manageable and confidence-inspiring so as to be used in further manipulation.

After successfully re-collecting the image she needed and through experiencing a series of Image Making activities [Table 2, sheets 5-7], the last of which was centered on the same example she worked on at the very beginning, Avgusta noticed a conflict between the two images she had constructed for the product of 2/3 x 1/2. This discerned contradiction [lines 728-747] between 12/6 [Table 2 – sheet 5] and 2/6 [Table 2 – sheet 7] is likely what occasioned Avgusta to reject her initial way of using diagrams and revise her existing Formalizing level of understanding by re-structuring the procedure of multiplying two fractions [lines 748-755]. Figure 1 is an attempt to illustrate by means of the Pirie-Kieren onion model (Pirie & Kieren, 1989) the pathway of Avgusta's growth of understanding. Based on my observations, this is seen to grow in a non-linear way: from the Primitive Knowing layer to the Image Making and Image Having layers. Then, evidence exists of folding back to the Primitive Knowing in order to collect an earlier understanding to use it anew at the Image Making layer. Avgusta seems to reach the Formalizing layer having first gone through the Image Having and Property Noticing layers.

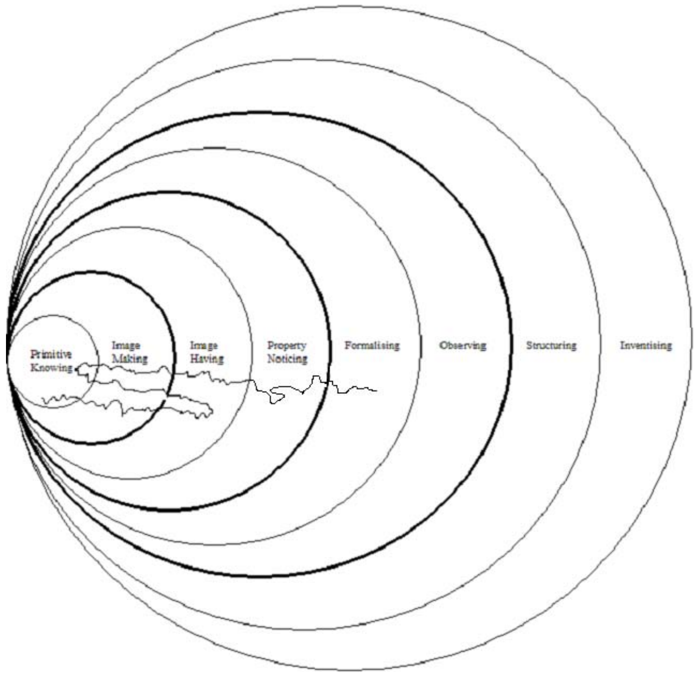


Figure 1: Avgusta's growth of understanding

The case of Avgusta comes to question the generalization of the assumption that once the meaning of a mathematical concept has been discussed, explained, formally articulated in class and students have at one time proven fluent with the corresponding algorithm, then the learning of this concept has been accomplished and a degree of readiness has been achieved for more sophisticated ones (Rasmussen et al., 2004). The fact that Avgusta struggled with the idea of fraction multiplication that had been taught to it while in fifth grade, neither speaks of a teacher's nor of a learner's failure per se. Rather, it points to the need for teachers to occasion students to re-encounter ideas that they already have, in a different light or in relation to unfamiliar circumstances.

The viewing of mathematical understanding as a dynamic process proved in the current study supportive in enabling me as a teacher-researcher to identify the roles language and thought could play both at any level and in the growth between levels of Avgusta's understanding. If, as in the case of Avgusta, the student needs to activate a link between everyday language and mathematical notation, then in order to allow that student to progress in making sense, occasioning –not imposing- an awareness as to what to collect could be of assistance.

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TENSIONS BETWEEN AN EVERYDAY SOLUTION AND A SCHOOL SOLUTION TO A MEASURING PROBLEM

Frode Rønning

Sør-Trøndelag University College
N-7004 Trondheim
Norway

This paper reports on an empirical study from a mathematics lesson in a Norwegian 4th grade classroom. The pupils are making batter for waffles, and the mathematical challenges are mainly connected to measuring. The paper will focus on the process of determining the correct amount of milk for the batter and furthermore on the tension that can be observed in the discursive practice as a result of the pupils' and the teacher's conflicting goals.

THE CLASSROOM SITUATION

This study is done in a group of 20 4th grade pupils in a Norwegian primary school in a mathematics lesson. During the lesson the pupils come in groups of five to the kitchen area in the back of the classroom where they make batter for waffles that are going to be prepared later the same day and eaten by themselves and the rest of the 4th graders at the school. Each group is supposed to make an equal amount of batter based on a recipe that is written on a poster. Before starting the actual work with the batter each group had a discussion where the task was to find out how much of each ingredient they would need in order to make three times as much as indicated on the recipe. The teacher expressed to me that her main mathematical focus with the waffle making was the discussion about the three folding. I will not report on this discussion but I will go into the part of the working process where the pupils are actually going to measure out 15 dl of milk. The milk comes in boxes marked "1/4 liter", and the pupils have measuring beakers available that can take 1 litre. The beakers are transparent, with a scale reading "1 dl, 2 dl, …. 9 dl, 1 lit" from bottom to top. Each group has to determine the number of boxes needed to get the correct amount of milk.

THEORETICAL BACKGROUND

The notion of a complex mediated act goes back to Vygotsky (e.g. 1978) and has led to the idea of sociocultural artefacts that mediate between stimulus and response. Such artefacts can take many forms and they shape the action in essential ways (Wertsch, 1991). In mathematics the tools are often signs and symbols that represent an abstract mathematical concept, and the signs and symbols also often refer to a context or a specific object. A sign typically has two functions, a semiotic function – something that stands for something else – and an epistemologic function as the sign contains knowledge about that what it stands for (Steinbring, 2005).

One of the pioneers of semiotics is the American mathematician and philosopher Charles Sanders Peirce (1839-1914). He defines the terms involved in his triadic model of semiosis in the following way.

> A *sign* is a thing which serves to convey knowledge of some other thing, which it is said to *stand for* or *represent*. This thing is called the *object* of the sign; the idea in the mind that the sign excites, which is a mental sign of the same object, is called an *interpretant* of the sign. (Peirce, 1998, p. 13, emphasis in original)

Peirce describes three kinds of signs (or representamens), *icons*, *indices* and *symbols* referring to three ways the representamen is related to its object. An icon stands for its object by likeness, an index stands for its object by some real connection with it or because it makes one think about the object, whereas a symbol is only connected to the object it represents by habit or by convention (Peirce, 1998, pp. 13-17, 272-275).

Presmeg (2005) turns the triadic model of semiosis into a nested model. This nestedness is based on the idea that the totality of the triad (representamen, object and interpretant) becomes reified (Sfard, 1991) as a new object to which one can assign a representamen and an interpretant. This gives a nested chaining of signs which can serve as a model to describe processes leading to more general or more abstract situations.

An important justification for mathematics in school is often the alleged usefulness of mathematics in other subjects and in situations outside of the school. It has been questioned whether it is possible to use a school subject such as mathematics outside of its own domain, and in this context it has been found fruitful to investigate the boundaries between the in-school and out-of school practices (Evans, 1999).

On areas where an overlap between in-school and out-of-school practices occurs it could be expected that there is some tension between the motives and goals lying in the school mathematics and the specific out-of school activity. To analyse this tension I will use the framework from activity theory. Leont'ev writes that *activity* is energised by a *motive*, and that "[t]here can be no activity without a motive" (Leont'tev, 1979, p. 59). Further he talks about the components of the activity as *actions* – processes that are subordinated to certain *goals*. On the third level there are the *operations* – the means by which the action is carried out. It is possible to carry out the same action by means of various operations, which means that the chosen operation "is defined not by the goal itself, but by the objective circumstances under which it is carried out" (Leont'ev, p. 63). Hence, the choice of operation may depend on the specific *conditions* in the given situation. It is henceforth possible to envisage one particular action but different operations that may be chosen depending on whether one is situated within a school practice or within an out-of school practice. According to Leont'ev the activity is driven by a motive, and the actions are directed towards certain goals. An important point is that each activity answers to a specific need of the active agent. "It moves towards the object of this need, and it terminates when it satisfies it" (Leont'ev, p. 59).

METHOD

I have been collaborating with all the teachers in grades 1-4 at this particular school for two years. This collaboration has involved working with the teachers in workshop activities, discussing in small groups and observing in classroom situations. When observing in the classrooms I have videotaped the activities going on. On some occasions parts of the videotapes have been shown and discussed with the teachers afterwards. Prior to the episode reported on here the teachers and I had been working with aspects of multiplication and division in a sequence of several workshops. We had agreed that on two given days in February I was going to videotape a session from each of the four grades 1-4. Each teacher, or group of teachers, was free to design the activities in accordance with the normal progression in the class. The only constraint was that it should have something to do with multiplication and division, or preliminary work leading up to these concepts. I did not partake in designing the lessons.

In the grade four class, which is the focus of this paper, the mathematics lesson was scheduled for two hours. I stayed in the kitchen area all the time, and with a hand held video camera I tried to capture as much as possible of the activity going on. During the lesson I was mostly passive but as can be seen from the excerpts of the dialogue I sometimes posed questions to the pupils.

THE HANDLING OF THE MEASURING PROBLEM IN EACH GROUP

Group 1

One measuring beaker is filled with flour, and Ellie is mixing flour and eggs. Lucy (the teacher) asks what they think is a good idea to do to avoid lumps, and they agree to start adding milk. James and Jessica fetch one box of milk each, and they agree that altogether they need 15 dl. Jessica looks at the box on which is written “1/4 liter”.

1.1	Jessica:	This is one four litre
1.2	James:	One four litre
1.3	Jessica:	Yes, so we take one of these first. One whole of these
1.4	Lucy:	How are you thinking now?
1.5	James:	Have no idea
1.6	Jessica:	Yes, it should be five
1.7	James:	Yes, fifteen so now you must. We just say that this is one and a half
1.8	Jessica:	It is one comma[1] five. No, we are supposed to take … like this
1.9	Lucy:	Emily, what do you think?
1.10	James:	Now it will be two comma eight, now it is two comma eight if we take
1.11	Ellie:	You are supposed to measure in the other decilitre measure

Jessica starts by looking at the text “1/4 liter” on the box but she and James do not have a clear sense of what this means and how it relates to the 15 dl that they know they are supposed to have. In utterance 1.10 James states that the two boxes they have will be “two comma eight” which indicates that one box would be “one comma four”. It is not clear which unit this relates to, and it is also not clear what is the meaning of the words (two comma eight) that are spoken out. The teacher perceives what the pupils are saying as not correct and asks them what they are thinking. When they do not give a satisfactory answer she turns to Emily (#1.9) but she does not react to the question. Ellie comes to rescue by pointing to the existence of one more measuring beaker (#1.11). The existence of the second measuring beaker makes the meaning of “two comma eight” or “1/4 liter” redundant. After this Jessica and James are no longer interested in how much there is in one box, and the conversation that follows is about practical solutions, for example how to avoid lumps. The teacher also seems to be mainly interested in the practical solutions at this point.

After having put in the first litre of milk Jessica and James start to measure out another 5 dl. Jessica pours in one box, looks at the scale and says “three decilitres”. She does not seem to make any connection between the sign on the scale (level of milk being close to 3 dl) and the sign 1/4 liter on the box. Then she gets another box and gives it to Emily who asks “How much is it we need?” Jessica answers: “We had ten before and then we need fifteen.” Up to now I have not contributed to the discussion at all but at this point I ask a question which seems to shift the focus somewhat for the rest of the lesson.

1.12	Frode:	How many decilitres are there in one of these? (Jessica looks at the box)
1.13	Lucy:	How many decilitres are there in one box?
1.14	Jessica:	It is one comma four litres. (Emily pours in the content of the box. Jessica looks at the scale.)

I suggest that they keep track of how many boxes they have used. They figure this out by counting the empty boxes but make no connection to the number of decilitres. I do not push this any further but Lucy repeats the question about how many decilitres there are in one box, and James answers:

1.15	James:	One comma four
1.16	Lucy:	One comma four?
1.17	James:	One comma four litres.
1.18	Jessica:	Yes, but she asked about decilitres.
1.19	Lucy:	Is it more than one litre?
1.20	Ellie:	No, it isn’t. It is less. This isn’t even half a litre.

As in the beginning of the episode 1/4 is read as “one comma four”, this time with the emphasis “litres”. Jessica realises that the question was about decilitres, and on Lucy’s expressed doubt whether it could be more than one litre (#1.19), Ellie gives a practical estimate, stating that it is indeed less than half a litre (#1.20). After this I end the conversation on this topic suggesting that it might be better that they work on the batter.

The pupils in Group 1 make notice of the sign 1/4 liter but they never develop a meaning of it. They also have no real need to find out what the sign means because they solve the practical task using the measuring beaker. The pupils answer the question about how many boxes they have used but they do not make any connection between the number of boxes and the number of decilitres.

Group 2

Also this group starts by looking at the milk box and the pupils pay attention to the text 1/4 liter.

2.1	Chloe:	One (looking at the box)
2.2	Chris:	slash four, what does that mean?
2.3	Chloe:	Four and a half
2.4	Chris:	Four and a half
2.5	Chloe:	And we need fifteen.

The teacher asks the same question as to the previous group about how much is in one box.

2.6	Chris:	Four and a half
2.7	Lucy:	Four and a half?
2.8	Chris:	Decilitres. No, litres.
2.9	Lucy:	Is it four and a half litres in here?
2.10	Chris:	No, decilitres.

The answer is first given in terms of the number words only (four and a half), and when Lucy wants them to be more precise they hesitate a little between decilitres and litres but stick to litres (#2.8). To this Lucy expresses astonishment (#2.9), and Chris changes to decilitres. Lucy is still not satisfied, and she takes Chris and Matthew to the board at the other end of the room. Lucy writes $\frac{1}{4}$ on the board. She also draws a circle that she partitions into four equal sectors, and she fills one of the sectors. This evokes the concept “one fourth” in the children. Lucy links this to “one fourth of a litre” and asks how many of these go into one litre. This evolves into a discussion that moves between various issues; how many decilitres in one litre, how many boxes in one litre, how many decilitres in total, and how many boxes in total.

Group 3

Joseph and Thomas find the crate with the milkboxes and Joseph starts by asking how much one box is. Thomas says that it is a quarter of a litre. At first Thomas will not engage in Joseph’s thinking when he wants to find out how many boxes they need. Joseph asks Lucy if he may use the measuring beaker. Lucy encourages him to try without it and after a brief discussion he accepts this.

3.1	Joseph:	Ohh. A quarter of a litre, that is … a quarter … ten decilitres is one litre. We have to have three of these then, then it will be. Five of these I think … no not five. How much should we, Thomas, if we take three

		of these, no four, then it is one litre and we want fifteen decilitres, and that is, and ten decilitres that is one litre. But how many more than four do we have to take then?
3.2	Thomas:	Then we have to take four, and then we have to take … two
3.3	Joseph:	Then we have two, and ten decilitres here. And then it is fifteen.
3.4	Thomas:	Yes.
3.5	Joseph:	Lucy, is this correct?

In turn 3.5 Joseph asks the teacher for reassurance of the solution, and then she makes him explain his reasoning. Joseph explains that four boxes equal one litre, and that two more boxes are two quarters which is equal to a half. Joseph and Thomas now state that they have one and a half litre which is the same as fifteen decilitres.

Group 4

Group 4 starts in the same way as Group 1 by pouring milk into the beaker. When they cannot find 15 on the beaker they decide that they have to split, and they choose to measure 9 dl first and 6 dl afterwards. They do not pay any attention to the number of boxes they use or to what is written on the boxes. When fetching the sixth box Katie says “it could be that it will be enough”. Grace looks at the scale saying “no, it is … it is exactly enough”. Katie replies “yes, exactly. Good.” Lucy asks how many boxes they have used. Katie counts them and answers “six”. Again Lucy asks the pupils to figure out how many boxes they need without using the measuring beaker. The following dialogue takes place.

4.1	Grace:	Put in three milkboxes … no six
4.2	Lucy:	Yes, but why?
4.3	Grace:	(…)
4.4	Lucy:	Yes, because you know that now
4.5	Grace:	Yes.
4.6	Lucy:	Yes, but if you hadn’t known
4.7	Adam:	Then we could have imagined having one like this (pointing to the measuring beaker)
4.8	Grace:	Then I could have walked home to get one

Lucy pushes them further and Katie asks how much is in one box. They come up with some suggestions, and I suggest that maybe something is written on it. They look at the box.

4.9	Hollie:	There, one comma five.
4.10	Katie:	No, one comma ….
4.11	Grace:	Comma, this is a slash. One slash four litres.
4.12	Lucy:	What does that mean?
4.13	Hollie:	Haven’t a clue.

Adam suggests “one fourth”, Lucy completes this to “one fourth of a litre” and goes on to ask how many they would need to get one litre. The pupils suggest that they need four fourths, and Lucy asks how many boxes that will be. They agree that this will be four, and Lucy points to the original problem to explain why they need two more to get the correct amount of milk.

4.14	Lucy:	Why do you need two more then?
4.15	Grace:	To get six, no
4.16	Adam:	To get three times as much
4.17	Grace:	To get fifteen – fifteen decilitres
4.18	Lucy:	Mmmm
4.19	Adam:	Can we put in the flour now?

Lucy is pushing the issue further and wants to know how many decilitres there are in four boxes which she states to be equal to one litre. In the dialogue that follows answers like “four fourths”, “four decilitres”, and “four litres” can be heard. At the end Lucy holds up one box at a time and they count one fourth, two fourths, three fourths and four fourths. Lucy states that four fourths is one whole. The pupils add “litre” and Katie says “plus two more is one half”.

DISCUSSION OF THE EPISODES

The semiotic issues

Central to the task is the sign or symbol 1/4 liter printed on the milk boxes. The pupils read the sign in various ways (one comma four, one slash four, four and a half) but many of them do not have a clear meaning linked to it. Groups 1 and 4 solve the measuring task completely by using a measuring beaker holding 1 litre. For these groups it is irrelevant to know the meaning of 1/4 liter to solve the task. They relate to the fact that they need 15 dl of milk and by using the measuring beaker as a mediating tool (Vygotsky, 1978) they are able to get the correct quantity. When the teacher asks these two groups to figure out how many boxes they would need without using the measuring beaker they are facing a difficult problem. I interpret the teacher here to be working with 1/4 liter as the representamen and the amount of milk in the box as the object. The teacher’s interpretant is that this is a fourth of a litre and that four boxes are needed to get one litre. The pupils are working within another triad where the representamen is the scale on the measuring beaker, an indexical sign pointing to the quantity of milk in the beaker as the object. The interpretant is the concept “fifteen decilitres” or “one and a half litre”, which they know that they need. I see the problem as having to do with creating a link between these two semiotic triads. As it is the symbolic sign 1/4 liter is not seen as a representamen for the semiotic triad involving the measuring beaker. Since the pupils do not have a clear meaning of what 1/4 liter means, the sign might just be an index connected to the box. In Group 3 the situation is quite different. The pupils make the connection between the sign 1/4 liter and the amount of milk, and as a result they are able to identify 4 + 2 boxes with one and a half litre.

In Group 2 the teacher physically moves from the kitchen part of the classroom to the opposite end where the blackboard is. She writes $\frac{1}{4}$ on the blackboard and also draws a circle partitioned in four sectors, filling one of them. Here the interpretant 'one fourth' is evoked in the pupils, and the teacher and the pupils seem to be working within the same semiotic triad, situated in a school practice. However, the sign $\frac{1}{4}$ is not seen as a representamen for the triad in which 1/4 liter is the sign, and therefore the link to the actual measuring problem is also missing in this case.

The sign $\frac{1}{4}$ is a symbol, clearly embedded in the school practice. The scale on the measuring beaker is an index, firmly based in the everyday practice. The sign 1/4 liter could be seen as a symbol representing the amount of milk in one box but for some of the pupils it might seem as if it is an index by its connection to the box, or a symbol with no interpretant. Based on this I identify three semiotic triads; the first where the scale is the sign, the second where 1/4 liter is the sign, and the third where $\frac{1}{4}$ is the sign. The everyday solution to the measuring problem is to pour milk into the measuring beaker until the indexical sign (the scale) points to 15 dl (seen as 1 litre + 5 dl, or 9 dl + 6 dl). The school solution could for example be to establish the relation $6 \cdot 1/4 = 1{,}5$ (litres) or $6 \cdot 2{,}5 = 15$ (decilitres). I have showed various attempts to create connections between these two practices. Based on Presmeg's (2005) model I suggest that a nested chaining of the semiotic triads described above could establish a connection between the practices, and I have showed that lack of connection can be explained by lack of connection between the semiotic triads.

The discursive practice

Seen as a task from school mathematics the measuring problem could be formulated as follows. "Each milk box holds ¼ litre of milk. How many boxes are needed to get 15 decilitres of milk?" All four groups were able to find a solution to the practical problem of getting the right amount of milk, so indirectly they also know how many boxes of milk they need. Therefore they have all found the solution to the question in the imaginary school task, albeit not in a school like manner. I perceive the main *motive* for this lesson to be to produce batter for the waffles, and this determines the direction of the *activity* in the lesson. The activity consists of a number of different *actions* that can be linked to specific *goals.* Some of these actions can be carried out in a number of different ways, using different *operations.* The choice of operations depends on the *conditions* that are there at any given time (Leont'ev, 1979). My main objective in this section is to analyse the teacher's and the pupils' goals and actions in the lesson. My interpretation is that there is some tension between the teacher's and the pupils' goals, and that this tension is due to the fact that the lesson is operating on the border between a school practice and an everyday practice.

In Group 1 it seems that both teacher and pupils share the same goals in the beginning. The pupils (Jessica and James) have the idea to try to figure out how many boxes of milk they will need (#1.1-1.11). The teacher sees that their idea will not work and she tries to guide them or bring in Emily to help (#1.4 and 1.9) but when Ellie (#1.11) points to the fact that there is one more measuring beaker the teacher just lets them go on with the measuring without going any further into their interpretation of 1/4 liter. The measuring beaker is the only tool they rely on to get the correct amount of milk. When I pose the question about how many decilitres there are in one box (#1.12), the situation changes somewhat. This question seems to bring in new goals that guide the teacher's action and in turn influences the pupils' goals. The teacher becomes more concerned about the mathematical content of the situation (e.g. #1.13). The fact that her attention to the mathematics appears after my question leads me to characterise her new goals as 'seeing the mathematics' and 'satisfying me'. The pupils do not relate this question to the work they are doing so their new goal can be expressed as 'answering the questions' or maybe 'satisfying the teacher'. They stick to reading 1/4 as "one comma four" (#1.15), emphasising "litres" (#1.17). Ellie is aware that there is not more than one litre in one box, "[t]his isn't even half a litre" (#1.20), indicating a lack of meaning to "one comma four".

In Group 2 the process with the milk starts with the pupils reading on the box "one slash four" (#2.1-2.2) which they suggest means "four and a half" (#2.3), but they are not quite sure whether it is litres or decilitres (#2.8). With this group the teacher to a much larger extent goes into the role of the mathematics teacher, and she literally crosses the boundaries between practices by walking over to the blackboard at the other end of the room. In a funnelling pattern of interaction (Bauersfeld, 1988, p. 36) the teacher leads the group to a conclusion about how many boxes are needed.

Group 4 solves the whole measuring problem using the measuring beaker, thereby reaching their goal. It is only on the teacher's request that the number of boxes being used is brought into the picture. The pupils give an answer, because that is what is expected of them as pupils, but without enthusiasm. They have reached their goal, and they have no need to use any more energy on this. Each activity, here the measuring of the milk, answers to a specific need of the active agent, here getting the correct amount of milk for the batter, and when this need is satisfied the activity stops (Leont'ev, 1979). The answers of the pupils (some examples are shown in turns 4.14 to 4.19) indicate little interest. The numbers that come up can be connected to certain incidents throughout the process but not necessarily corresponding to the questions that the teacher asks. For example in turn 4.15 when Grace answers "to get six", she applies the fact that they used six boxes, which she already knows, but this is not in line with the hypothetical situation that the teacher has constructed. Towards the end the teacher leads the pupils via the question about how many boxes they need to get one litre. Even this evokes answers that indicate that the pupils do not engage in the problem.

I have shown that by operating on the border between practices, the mediating tools from the non-mathematical practice offer alternative possibilities for solving a task. The teacher, being pulled between the two practices, is seen to struggle in order to keep the pupils' motivation to solve the task in the mathematical context when they already have solved it in the practical context.

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[1] In Norwegian the sign for the decimal point is a comma. Since this sign is central in the interpretation of the dialogues I am using, I will keep the word 'comma', and I will also for example use the notation 1,5 instead of 1.5 which would be the standard English notation. Also when I directly refer to the text on the milk box I will use the Norwegian word 'liter' instead of 'litre'.

LINGUISTIC ACCOMPLISHMENT OF THE LEARNING-TEACHING PROCESS IN PRIMARY MATHEMATICS INSTRUCTION

Marcus Schütte

Goethe-Universität Frankfurt am Main, Germany

The linguistic accomplishment of a mathematics lesson, based on an illustrative example of an everyday lesson in a Hamburg fourth grade class, was analyzed via the person instructing. The linguistic accomplishment of instruction, for the purpose of analysis and with the help of qualitative procedures of interpretative classroom research of German mathematics education (Krummheuer/Naujok 1999), was analyzed on the basis of three hierarchical levels, developed from an existing theory. The results of these analyses grant on the one hand a hypothesis of the learning opportunities for a multilingual pupil body in German classes. On the other hand the results in the sense of local theory genesis can be integrated into a theory concept, which the author designates Implicit Pedagogy.

1 Introduction

If one looks into the classrooms of German schools, one notes that the pupil body is increasingly becoming shaped by multilingualism and various cultural backgrounds; currently, almost a third of all pupils in the German educational system hold a migrant background. Despite the increasingly linguistic and cultural diversity in German schools, instruction seems to be only slightly flexible and adapted to the needs of the diverse pupil population. Students with a migrant background or students who grow up in a semi-illiterate environment perform worse, according to the findings of international and national scholastic achievement tests, in comparison to their classmates who grow up in a monolingual German environment (compare the results of PISA 2000 and 2003, as well as IGLU 2003). It appears to be indisputable, that the origin of this poor performance is in a not insignificant manner to be found in an insufficient mastery of the language of instruction. However, these differences in the mentioned studies are often gladly categorized as unchangeable via school and their cause legitimized by the socio-economic background and/or language of the family.

The goal of the article at hand is thus to demonstrate the underlying reasons for the poorer performance of students with a migrant background and/or who grew up in a semi-illiterate environment. The achieved results will then be subsequently explained with the assistance of theoretical approaches and in this manner demonstrate possible consequences or potential for change. On the basis of this, further studies may be able to develop concrete possibilities of how to fit instruction better to students affected by lingual-cultural plurality.

Linguistic accomplishment of instruction constitutes a substantial aspect of the adjustment of instruction to suit the needs of multilingual pupil bodies. In accordance with some approaches in the field of mathematics education, language and communicative competence both have a special significance for the learning of mathematic content. Above all, Maier (compare e.g. 2006, 2004 and 1986) was concerned with research in the field of language and mathematics within German-speaking countries. Maier (2006) justified, that language holds a special relevance in Mathematics instruction, as objects in Mathematics, " ... do not have a material nature and thereby are not accessible through the senses" (p.137, translated by the author). This consequently accounts for the significant focus of Maier's works on the observation of technical terminology in Mathematics instruction. In the international community there are several authors who can be mentioned, who concern themselves with the relevance of language in the learning of mathematics. In the following, it should be initially referred back to Pimm (1987) who understood Mathematics as a social activity that is structurally and closely connected with verbal communication. From this, Pimm introduces the metaphor "Mathematics is a language?" (ibid, p.XiV) as a question of whether Mathematics could be evaluated not in the sense of a natural language, but as its own style of language. He compares, for this purpose, teachers as a role model of a "native speaker" of Mathematics and other people, for whom Mathematics appears to be incomprehensible, as per a foreign language, to which they are not empowered (ibid, p.Xiii).

The empiric material of the underlying research consists of transcripts from video recordings of an everyday primary lesson. The video recordings took place over a time period of four months in three classes of the fourth grade in two Hamburg primary schools with an approximate 80% migration contingent amongst its pupils.

In section 2 of this article, the analytical findings of the analysis of interactions within a selected instructional episode will be presented. In connection, a methodologic indexing of the procedure of the underlying research will be taken as preparation of further analysis. The selected episode will be used in section 3 as an illustrative example to demonstrate how lingual accomplishment of primary mathematics instruction lends itself to be described and analyzed with the here-accepted theoretical perspective. To this, three hierarchical levels are developed from this theory, by which the linguistic accomplishment of the lesson in the selected episode will be deeply analyzed. In section 4, the possible outcomes will be described, that yield from the results of the analysis to learning opportunities for pupils in German primary school classes. Furthermore, the results will be presented for the purpose of local theory development in a theoretical concept developed by the author from the entire research.

2 An Episode from the Lesson Sequence "LCM"

In the following a short transcribed episode of an everyday primary school mathematics lesson during the introduction of a new mathematic concept will be looked at.

2.1 Prehistory and Transcript of the Lesson Episode

At the beginning of the scene "LCM" Ms. Teichmann along with 25 female and male pupils, 17 of which have a migration background, are situated in the classroom. In this lesson the introduction of a new mathematic concept should take place: the LCM- the Least Common Multiple.

It is Wednesday morning Ms. Teichmann asks initially what the abbreviation LCM stands for. Thereafter she allows the multiples to be calculated. Finally she draws two circles on the board, that she divides into four and three segments respectively, with an addition symbol between them and an equals sign. She marks for each circle one of the segments in pink. While one pupil very quietly says, “1/3 plus 1/4,” Ms. Teichmann asks the pupils which equation stands on the board. The pupils begin to guess and first give the answer, “1 plus 1,” or, “2,” and then somewhat later label the segment with 1/3 and 1/4. The teacher notes this in the drawing on the board and adjusts the fractions from 1/3 and 1/4 to 4/12 and 3/12. Several pupils offer many creative solutions for their addition, such as for example “2/7”. In closing, her generalization of the procedure follows.

241	16:30	<L:	right/ you may not- add a large piece of pizza [points to the left circle]
242		>L:	and a small one and a smaller -.one together [points to the right circle]
243		L:	that is not equal right/
244		<L:	you must practically...
245			chop them into such pieces that they are equal\
246		<L:	[makes a chopping motion with her hand]
247		>L:	..right/these pieces are equal\ [points to the left circle]
248		<L:	[points to the right circle] These pieces as well\
249			only here it is less\ right/ here there are only three-
250		>L:	and here there are four pieces. [Points to the left circle]
251		S:	ah now I understand it
252	16:57	L:	and for that reason one need this\. if you at all want to (add) fractions-
253			so that you can add together such pieces of cake together\
254			right/one can not simply
255			say three and four is seven and from above
256			we will take two and then I have two sevenths\
257	17:11		Two sevenths is something completely different
258			no that doesn’t work\

2.2 Concise Analysis of the Interaction

At the end of the episode the teacher attempts to show the pupils a generalization of the addition of fractions. She uses for this purpose the everyday example of the division of a pizza, respectively cake and makes the division of them visual through gestures. Hereby both levels of the illustration on the basis of the everyday and the generalization of the rules of fractional arithmetic meld together. This is shown in the statement by Ms. Teichmann in *<252-258>*. The reference to “LCM” seems to have been completely lost, or left as implicit. Alone the, “…and for that reason one needs this…” in <252> from Ms. Teichmann gives us the idea that there is still a reference to the “LCM”, since one needs an “LCM” in order to find the least common denominator for the addition of the two fractions. Ms. Teichmann does not further explain this connection. Also the final generalization by hand of the cake example <252-258> can barely be accounted for as a further clarification of the procedure, since Ms. Teichmann says that one may not simply add three and four together and means thereby apparently the denominators of one third and one fourth. Through the selected example, however, pupils did indeed have to add three and four in order to ascertain the solution of the task – though, on the level of the numerator. They added 3/12 and 4/12. Moreover, the addition of the numbers three and four are everyday tasks for primary pupils in basic arithmetic operation. Why one may no longer carry out this arithmetic remains unexplained. Since one cannot assume, that the pupils are competent to differentiate between numerators and denominators, one can classify the statement of the teacher as contradictory. Consequently, pupils in the end of this episode were merely able to solve an addition task, which they were already capable of solving before and whose correctness would now be put into question.

2.3 Methodology

After having summarized the analysis of the scene, I would like to offer as preparation of further analysis a few explanatory notes to the methodological situating of the underlying research. The underlying research to this article is qualitatively oriented and grounded in interpretative classroom research. More exactly: in the domain of the interactionist view of interpretative classroom research in the field of mathematics education. Through the analysis of the units of interaction in the videotaped instructional episodes, I oriented myself to a reconstructive-interpretative methodology and on a central element of the research style of Grounded Theory- the methodic approach of comparative analysis. The goal of interpretative classroom research is to pursue a local theory genesis through "understanding" of interactions of individuals in concrete instructional practice. The scope of this concept theory is related to the interpretative classroom research, however, to be decidedly restrained, since this is in many areas mostly globally and universally connoted. The theoretical results of research of such a reconstructive-interpretative procedure present hypothetical outcomes, which do not follow the claims of the development of globalizing and universalizing theoretical approaches (compare Krummheuer/Naujok 1999, p. 105). These hypotheses stay arrested to the fact, that they are directly connected to the respective context of the researched field of study and are thereby rich in empirical elements and feature inner consistency. A universality of underlying results does not lend itself to be understood as, "is always applicable," rather may be related to only a limited scope of classes, who are taught and will learn under similar conditions.

3 The analysis of the linguistic accomplishment

Here subsequently follows the analysis of the linguistic accomplishment of the instruction on the basis of the selected instructional episode on three hierarchical levels.

3.1 Technical terminology versus everyday language

Since objects of Mathematics are according to Maier (1986, p.137) of an abstract nature, the introduction of new mathematic concepts allows for particular attention to the technical language of Mathematics, as objects of Mathematics can ultimately be handled and represented only on a linguistic-symbolic level (compare ibid, p. 137). The question, which should be answered in the following sections, is how these technical terms of Mathematics are introduced into the analyzed lesson. To this, Maier (2004) refers to the fact that in the technical language of Mathematics, as well as in other technical languages, there is a problem of ambiguity within the technical language, since it interferes with the everyday language of the pupils (compare ibid, p. 153). The problem of ambiguity within the technical language of Mathematics, according to Maier (2004, p.153), carries a significant relevance in the verbal actions of teachers. Maier writes, that teacher language moves in a stress-ratio between technical linguistic "*Hypertrophy*" and accordingly "*Hypotrophy*"[i]. The goal should be, according to Maier, to have the teaching language, which moves on a scale between these two extreme points of Hypertrophy and Hypotrophy, positioned "in the middle." Thus a necessary technical language development of the pupil body can be assured and on the other side the pupils can be given the opportunity to comprehend mathematic phenomena with their own language. In which forms the usage of mathematic concepts let themselves be differentiated from the usage of everyday language concepts in instruction follows as next in the first level of hierarchisation.

<u>The analysis of the selected episode</u>

In the underlying episode the teacher attempts to give a generalization for the addition of fractions. She stresses here the relevance of LCM for the addition of fractions in line <252> in saying, *"and for that reason you need this."* In this statement she uses the place holder *"in addition"* and *"this"* instead of the technical terminology. In her entire generalization she uses a multiplicity of everyday language concepts such as, "*a piece of pizza*" <241>, "*pieces"* (of a pizza or cake) <245, 247, 248, 250>, "*chopping"* <245>, "*pieces of cake"* <253>. From the terminology she used, the following language can be found in everyday language as well as in technical language: *"to add together"* <241-242, 253>, *"not equal"* <243>, *"equal"* <245, 247>, *"less"* <249>. Only the expressions of *"fractions"* <252> und *"two sevenths"* <256, 257> suggest, on the other hand, technical linguistic terminology. With this analysis in mind, the procedure of the above-mentioned teacher would surely be described, according to Maier, more in terms of technical Hypotrophy, since the teacher through the generalization of the procedure, where the greatest level of abstraction could have been conjectured, reverted only minimally back to technical terminology.

According to the statements of Maier one could reason, that such a procedure enables pupils to describe mathematic phenomena with their own language, but also endangers the development of technical language. Since, however, these attempts to explain multiplicity are through everyday language concepts and the usage of placeholders, the general principle remains implicitly hidden (see section 2.2) and it is doubtful, that pupils are in a position to shift into their own language to describe this mathematic phenomena.

3.2 The embedding of mathematic concepts in a mathematics register

The second level of analysis of the linguistic accomplishment of instruction via the teacher by the introduction of a new mathematic concept lends itself to a reference of the statements of Pimm (1987). Pimm compares teachers as a role model of a "*native speaker*" of Mathematics (ibid, p. Xiii) and other people, for whom Mathematics appears to be incomprehensible, as per a foreign language, to which they are not empowered (ibid, p.2). In this context, Pimm (1987) is speaking of a "*mathematics register*" (p. 74). With the term register, Pimm is referring to Halliday (1975). Halliday understands a register as an assemblage of meanings that are intended for a particular function of language, that together with the words and structures are able to express these meanings. Halliday subsequently talks of the mathematics register only when a situation is concerned with meaning, that is related to the language of Mathematics, and when the language must express something for a mathematical purpose. Mathematics register in this sense can be understood as not merely consisting of terminology and that the development of this register is also not merely a process to which new words can be added (Halliday 1975, p. 65). The task of the pupils to learn mathematical concepts in their lessons contains, according to Pimm (1987), more a deeper learning of linguistic competence than is the case by Maier (e.g. 2004). In Maier's approach the focus lies on the acquisition of technical linguistic competence through a well-balanced application of technical linguistic terminology and everyday language concepts in the linguistic accomplishment of instruction via the teacher. Pimm (1987, p.76) sees the task of pupils, however, as to become proficient in a mathematics register and in this way to be able to act verbally like a native speaker of Mathematics. The second level of hierarchisation of the linguistic accomplishment of instruction falls into what extent the newly learned mathematic concepts in the researched lesson were integrated into a mathematics register or if they were to be introduced and regarded as isolated units.

The analysis of the selected episode

In the selected episode the teacher appears to attempt to explain the mathematic concept "LCM" in connection with the addition of fractions. In the beginning of this episode the teacher produced for this purpose a reference to the concept of multiples in allowing pupils to calculate them. According to the theoretical perspective of Pimm (1987) the attempt by the teacher to reconstruct the concept of "LCM" only allows itself to be incorporated, not as an isolated conceptual unit, but through its connection with other mathematic concepts in a mathematics register. According to Pimm, it should be the goal to make pupils competent native speakers of Mathematics. In the introduction by the teacher, however, there was no time point in the entire scene in which the mathematic concepts of denominator, numerator, fractions, fraction strokes, or multiples were verbally and content-wise clarified in the official classroom discourse. They remain implicit and are integrated without reflection in the already familiar calculation routines. Even the teacher herself seldom uses the concepts to be learned actively, such as is shown in the first analysis, rather reverts back predominantly to the everyday language concepts. Pupils must extract the meanings of the new concepts by themselves from the illustration on the board. Pupils are then additionally given only the possibility to calculate the multiple as an active manner in which to solely understand the meaning of the concept of a multiple. That pupils are able to extract the concepts, without a verbal contextual explanation of the concepts by the teacher seems questionable. For example, in the analysis at the beginning of the scene there were alternatives for interpretation, in which the pupils interpreted the fraction stroke as minus sign. Pupils must extract the subject with this *implicit procedural method* from their everyday background or from that which they already know from their lessons and will thus be able to take no decisive steps in the direction of becoming a native speaker of Mathematics.

3.3 The embedding of the mathematic concepts in a formal language register

The third level of analysis of linguistic accomplishment of instruction unfolds from the reference of the theoretical explanations of Bernstein (1977), Gogolin (2006), and Zevenbergen (2001). According to Gogolin (2006), pupils in German schools are submitted to the normative standard, that they are receptively and productively in command of the cultivated linguistic variations in class. This language of school-described by Gogolin as *"Bildungssprache"* [ii] (ibid, p.82 ff., according to the concept of *"Cognitive Academic Language Proficiency"*, Cummins 1979)- has on a structural level more in common with the rules of written linguistic communication. It is in large part inconsistent with the characteristics of the everyday verbal communication of many pupils.

Bernstein (1977) and Zevenbergen (2001) target, with their discussion of the language of instruction, the children from the working and middle class for differentiation. According to them, the linguistic abilities of formal language that are required in schools set a line of demarcation in everyday language, that is more in accordance to the abilities of the middle class, than to those of the working class. This formal language of instruction stands out through its precise grammatical structure and syntax as well as through its complex sentence structure. Through proficiency in this formal language, pupils develop - those in the middle class in particular - a sensibility in regards to the structure of objects and the structure of language, that helps them to solve problems in life and in school in a relevant and goal-oriented manner. Successfully receptive in *"being (a) part (of)"* and productive as in *"taking part (in)"* (Markowitz 1986, p.9, translated by the author) a linguistic discourse of instruction is something that is only possible for pupils, according to the above-mentioned authors, when they have competence in the formal language or the *Bildungssprache* of instruction. In this way it is possible for them to understand abstract concepts independent of concrete context and to be able to transfer them into written decontextualized form. In the third level of hierarchisation of the linguistic accomplishment of primary mathematics instruction there follows the question, to what extent, and how pupils are introduced during instruction to a formal *Bildungssprache*.

The analysis of selected episode

In her attempt to make a generalization, the teacher says in <241-242> "*Right/ you may not add a small piece of pizza and a small one and smaller one together*" <241 – 242>. She also uses the comparative form of the adjective "small" for this purpose, but does not go into the "Least Common Multiple" more explicitly. However, it is not self-explanatory that all pupils- most especially those who have grown up multilingual- are familiar with the comparative forms of adjectives in the German language. It is not self-explanatory that pupils will be able to differentiate between “Small Common Multiple” and “Least Common Multiple”. This interpretation is supported by analysis of previous episodes, in which pupils used the incorrect comparative form when attempting to use the term “Least Common Multiple”. Another correlation to this can be seen in the procedure at the beginning of the scene where the teacher allowed the pupils to calculate multiples. At no point in time did the teacher explain the connection between the terms “multiple” and “Least Common Multiple”. In this way it is made difficult for students to be able to recognize that the “Least Common Multiple” is really a subset of all “multiples”. It is not attempted on the part of the teacher to integrate the new concept into a related text. Hereby the question may be asked if and how the students should be empowered to understand such abstract concepts independent of concrete examples and to be able to transfer them into written form.

Summary of the analysis of the linguistic accomplishment of instruction

In the underlying research of this article there were 15 different episodes in total which were analyzed[iii]. These episodes with the help of comparative analysis were systematically compared. The comparison thereby of the three hierarchical levels of the linguistic accomplishment of instruction resulted in the following structure characteristics:

In the case of the first level, the application of technical terminology or everyday language by the teacher in instruction, allows no structural commonalities to be reconstructed. A unified procedure by the usage of mathematics register and everyday language does not seem to make a difference in the episode. The teachers use either predominantly everyday language concepts or several new and unexplained mathematic concepts. Unlike the first level, the results of the analysis of the other two levels behave in a different way. The implicitness of learning content, as a phenomenon in the introduction of a new mathematic concept, allows itself to be reconstructed as the common basic structural characteristic of the linguistic accomplishment of instruction via the teacher. The implicitness of the learning content defeats itself by the usage of different mathematics and formal linguistic registers. In this introduction of new mathematic concepts one can reconstruct through mathematics register, that the meanings of the concepts, just as the content references between the new mathematic concepts to be learned or the already known everyday language concepts is not made clear or only implicitly. The meanings or connections are not explicitly taken up in the instructional discourse and find thus no consideration in the classroom discourse. The meaning or the reference are explicitly assimilated by the teacher into the instructional discourse and thus find no consideration in the interaction of the classroom discourse. The formulated goal of Pimm (1987, Xiii; see Ch. 2.4) that students should learn to speak Mathematics like a native speaker, will be difficult for students to achieve, as the native speaker of Mathematics - the teacher - does not exemplify this active speaking themselves. A similar picture shows itself in the way the teachers commit themselves to linguistic particularities of formal linguistic register. Also here there is an implicitness that rules the teaching. The teacher only refers back to the grammatical structure implicitly, in which the mathematical concept is embedded, or to that which characterizes the meaning carrying elements. With which linguistic methods the complex and abstract mathematic concept, in the sense of the conceptual writing, is expressed to a connected text is left, as regards content or implicitness, in the end of the attempted explanations, unconnected. An integrated embedding of the mathematical concept in a *Bildungssprache* is not noticeable.

4 Implicit Pedagogy and its consequences

In the basis of the research the reconstructed procedures of the teacher in the linguistic accomplishment of the lesson alone was with mathematics teaching approaches not enough to explain, and for this reason further pedagogical, sociological and linguistic approaches were expanded into the theory genesis (compare Bourne 2003; Bernstein 1996; Walkerdine 1984). Through this opening of the theoretical framework of the underlying research, there allows for the procedure of the teacher to be conceptualized under the concept of "*Implicit Pedagogy" (compare "Implizite Pädagogik" Schütte 2009)*. This displays itself in the introduction of new mathematical concepts, in the manner, that decisive aspects of meaning negotiating of the individuals and the thereby possible constructions of enduring, non-situational bodies of knowledge for the individuals, remain concealed. One such Implicit Pedagogy is attached to the main idea, that students alone on the basis of the abilities they bring along with them can unlock meanings. Not the lesson, the qualifications of the teachers, nor their efforts can bring a deciding influence on the possible educational success of students in school, but rather, and above all else, the abilities that the children have brought with them decides this. The linguistic accomplishment of the instruction via the teacher, that follows such fundamental ideas, would not appear to make enough adjustments to the existing relationships of linguistic-cultural plurality in the classroom, since the procedure as it stood only served to reproduce existing social relationships in the educational system. The consequence of such an implicit procedure by the teacher can be, for example, that the comprehensive development of the relevance of the new concepts to be learned, on the side of the students, can be hindered. On the other hand it is a possible consequence that the students could be hindered by, or could refuse to participate in, a formal linguistic educational discourse in their lessons. Additionally, the opportunity is taken away from them to participate actively, that means productively, in the lesson, and through this accomplish the lesson. This happens for the main reason that the teacher, through her primarily implicit procedure, presents no model for her students to follow in her interactions with the formal linguistic *Bildungssprache*.

[i] The excessive use of almost "*pure technical language*" (ibid) by teachers and instructional media is viewed by Maier (2004, p.153) as technical linguistic hypertrophy. The excessive use of almost

"*pure colloquial language*" (ibid) by teachers and instructional media is characterized by Maier (2004, p. 153) as technical linguistic hypotrophy.

[ii] *Formal linguistic instructional language* (translated by the author).

[iii] This episode under consideration deals primarily with a shortened extract from the original episode, since for reasons of space limitations no analysis of the entire episode was possible. The detailed analysis of this episode can be found in Schütte (2009).

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MATHEMATICAL COGNITIVE PROCESSES BETWEEN THE POLES OF MATHEMATICAL TECHNICAL TERMINOLOGY AND THE VERBAL EXPRESSIONS OF PUPILS

Rose Vogel and Melanie Huth

Goethe University, Frankfurt am Main, Germany

Verbal expressions by students in mathematical conversational situations provide insight into the individual mathematical imagination and express what patterns and contexts children recognize in mathematical problems. Children just starting school utilize means of expression of their mathematical ideas that go from everyday speech descriptions to detailed action sequences. They already use technical facets, even though their repertoire of mathematical language of instruction has to be considered initially as tentative. In our article, by dint of methods of qualitative analysis, we want to present initial descriptions in terms of the identified capability of mathematical expression of pupils just starting school, based on a conversational situation about a combinatorial problem.

Keywords: mathematical cognitive process, mathematical language, mathematics in elementary school, combinatorics, mathematical concepts

INTRODUCTION

The mathematical cognitive process is initiated extrinsically and/or intrinsically by tangible problems or questions the young learner encounters in various contexts. This process is of a discursive nature. Furthermore the mathematical problems are expressed in manifold linguistic forms. In the process of understanding, individual prior knowledge, mathematical concepts and strategies are activated by the learner. According to the learner's estimation the activated strategies promise the most probable possibility for a solution.

Within the framework of our research, we wish to focus on linguistic activities within the mathematical cognitive process that significantly mold this very process: mathematical content is conveyed by dint of language; mathematics is talked and written about. This approach is needed to broaden the perception of language from purely verbal expressions to other activities such as gesture, body language and facial expression, as well as bringing mathematical facets into written form and presenting them. In addition, it is important to take into account what cognitive grasp, from their perspective, the respective protagonists have in terms of handling mathematical problems. It is also interesting which patterns of action are consciously or unconsciously activated in terms of the situation. Individual interpretations, concepts and models of the mathematical content, which is always restricted to context, and also social aspects (communication patterns, specific language of instruction, structure of the interactive negotiation process, teaching and learning patterns and cultural conditions) and the personal image of mathematics are especially pivotal and demand detailed consideration.

> “The emerging of mathematical knowledge is fundamentally taking place in the context of social construction an individual interpretation processes. […] it is constructed by means of social activities and individual interpretations.” (Steinbring 2005, p. 7)

In the present paper we would like to give an outline of the provisional state of knowledge resulting from our activity in the field of ‘The learning of mathematics and language’. Translating the mathematical content of a problem into technical terms is related to the mathematical language of young learners, in particular their mathematical concepts and individual conceptions, which are reconstructed based on verbal activities. We expect that the detailed consideration of the children’s verbal expression will afford us with insights into what they view as the problem’s mathematics. This identification of mathematical and individual concepts is to be deepened in the future inter alia by the interactional view of mathematical negotiation processes mentioned above. In doing so, we wish to focus on ‘mathematical language’ in the broadest sense of the term, that is, constituting all forms of expression accompanying the mathematical cognitive process. In our opinion learners of mathematics, especially young learners, approximate more and more to technical mathematics-orientated language in their process of learning mathematics. This “speaking mathematically” (Pimm 1989) is more than just learning vocabulary and using these words in the right linguistic form. Linked to that is the notion of utilizing this knowledge to design processes of teaching and learning. If we demonstrate our considerations in the following and represent our thoughts by means of an example, we focus at first verbal expressions and unique actions in the mathematical cognitive process that are examined as unique expressions for now. We will present an exemplary conversational situation of first-graders concerning a combinatorial problem. Our research perspective is guided by the super ordinate question about mathematical language and a potential mathematical language development in the process of learning mathematics on the part of young learners. In the present paper we want to focus on the following embedded questions:

What language do the here described pupils have at their disposal when handling a combinatorial problem in the conversational situation being presented?

What individual conceptions and mathematical (‘target-consistent’) concepts can be surmised behind the described children’s verbal-linguistic activities in the examination with a combinatorial problem?

What patterns of actions are activated or what conceptions about ‘to do mathematics’ in the examination with an explicit structured combinatorial problem can be reconstructed by means of verbal activities?

The data and considerations used have emerged from our research within the framework of an initial exploratory pilot study. We conducted this exploratory study with a focus on designing and interpretating situations which could be analyzed in the view of mathematical concept development and the linguistic means of expression in discourse situations. This study could contain useful information and serves as a trial of such situations. It is embedded in the context of a current developed longitudinal study to investigate early steps in mathematics learning (related to the Centre for Research on Individual Development and Adaptive Education of Children at Risk (IDeA), a centre of DIPF (German Institute for International Educational Research) and the Goethe university, Frankfurt/Main in cooperation with the Sigmund-Freud-Institute, Frankfurt/Main).

THEORETICAL FRAMEWORK – MATHEMATICAL LANGUAGE ACTIVITIES OF CHILDREN IN ELEMENTARY SCHOOL

At the beginning of their time in school, young, monolingual, linguistically inconspicuous learners have at their disposal a fundamental passive and active vocabulary. Their language acquisition in the unique grammatical sub-systems can be termed basic. Now what becomes relevant in terms of language is the growth of special communication and action patterns to be ascribed to the institution of the school, such as the acquisition of a certain language of instruction (cf. "cognitive academic language proficiency," according to Cummins 2000 after Gellert 2008, p. 140). For mathematics lessons in particular, a vocabulary and a specific language have to be acquired in which symbols are employed or terms from everyday speech adopt a different meaning (like 'equal,' 'less,' 'greater'). Negotiation processes in the social context have to be mastered linguistically within the learning process so as to understand mathematical teaching contents and be capable of participation. Verbal expressions are thus embedded in the interaction process in which they are uttered. The process of analysis documented here represents an initial approach to a form of analysis yet to be developed, which would permit one to make statements about the applied forms of language in the context of mathematical cognitive processes. Beside that, the analytical method to be developed could be interlocked with other approaches like interaction, argumentation and participation analysis (Brandt & Krummheuer 2000; Krummheuer 2007).

The approach presented here in an initial outline bears a certain resemblance in several parts to Steinbring's (2005, 2006) epistemological approach. In the epistemological triangle developed by Steinbring, the interactively constructed mathematical knowledge is of central importance. This knowledge, which is again based on pre-existing conceptual ideas, is generated by creating relations between the signs being utilized and reference context. In our approach the children used signs in the form of verbal, gestural and also written expressions to communicate their meaning or interpretation of the given mathematical content. In doing so, they needed to revert to their pre-existing conceptual ideas. Their expressions or signifier could only refer to the reference context or signified, whereas a common interpretation of this mathematical content has to be negotiated in interaction.

The question is how these mathematical pre-existing conceptual ideas and knowledge in Steinbring's approach can be described. The point of departure of our analysis is the problem's so-called mathematical content. While handling the 'mathematical content,' we try to describe the mathematical concepts or mental models (here in the meaning of Prediger 2008) that are of import for solving the problem. Mathematical concepts or mental models, according to Prediger (2008), are contrasted with the personal conceptions of the individual who is learning, which are reconstructed here by means of pupils' expressions. These individual "students' conceptions" (Prediger 2008, p. 6) which are comparable with Steinbring's pre-existing conceptual ideas (Steinbring 2006, 140), sum up the conceptions of the individuals who are learning, which could be developed up to now to handle similar mathematical problems. Any other structurally similar mathematical problem will re-activate these "individual models," which are then confirmed in the situation or may lead to irritations and potential expansions of these individual models. Mathematical experts and novices alike use individual mathematical models to be able to approach the abstract and immaterial mathematical objects and develop mental images for them: "[…] mathematical concepts are sometimes envisioned by help of 'mental pictures' […]. Visualization […] makes abstract ideas more tangible, […] almost as if they were material entities." (Sfard 1991, 6) Should a discrepancy arise between the individual model and the 'mathematical concept' relevant to the problem and prove to be too large to overcome, this may create learning opportunities that can be utilized more or less beneficially.

RELEVANT MATHEMATICAL CONCEPTS IN SOLVING COMBINATORIAL PROBLEMS

Combinatorics involves the determination of the number of elements of finite sets. The point is to select elements from a given total (basic set) and re-combine and rearrange them according to specific criteria (cf. Krauter 2005/2006). The description "combining selected elements" refers to the formation of new combinations of sets. The description "arranging selected elements" focuses on the order and thus on the formation of variations (cf. Selter & Spiegel 2004, 291). Again the determination of the number, of the sets or lists that arise this way, will be of importance. Thus, combinatorics centers around counting. Although here we are moving in the context of discrete mathematics and hence in the range of countability, this will frequently take on a theoretical character and provoke mathematical methods that go beyond the act of counting. These arithmetical "counting methods" are documented as formulas that in a compressed form describe the appropriate algorithm. In addition to the formulas, instructions are described having the function of activating inner images with the learner. These images help to translate familiar situations into the unknown mathematical problem and encourage the utilization of a suitable formula (for instance, without regard to order and without replacement).

The conversational situation that our analysis is based on is a part of an explorative study in which a total of eight first-graders were under examination. We selected this particular situation because its progress is comparable with all other videotaped and transliterated situations. Furthermore we choose such situation with a combinatorial problem, because this requires from the pupils counting and manipulation with sequences in practice. For the explorative study we developed mathematical problems of different mathematical areas, e. g. combinatorics, and then presented one problem to a student-duad in a conversational situation. The setting which is important in the following descriptions was hence set up as follows: The researcher presents a combinatorial problem to two first-graders. The pupils had the joint task of solving the combinatorial problem. In the progress of the situation, the researcher simply joins the conversation of the children in an appropriate way. As material at their disposal the children had paper, pencils and a bag full of candies.

Problem: Emma has two red cherry candies and six green apple candies in her bag. She pulls four times from her bag and gives the candies that have been pulled to her brother Tom. What candies can Tom get? Find all the options that are not identical!

The problem describes precisely how the desired subsets – consisting of four elements – are to be generated. Four pullings in a row are to take place. Replacement does not make sense, as the generated subset is to be given away. This makes it quite explicit that one element of the initial set cannot be pulled more than once. Thus, the problem describes the combinatorial figure of pulling without replacement (a total of four pullings) of k elements from n. The second criterion of order is irrelevant to the problem (cf. set concept). Thus, the act can be translated into a pulling all at once, that is, without replacement and without regard to order (cf. Kütting & Sauer 2008, p. 93).

Cardinal number concept / set concept

The point of departure for the problem is an n-element set ($n = 8$), which is comprised of two subsets with the element numbers $r = 2$ and $g = 6$. In tangible terms, the problem is about the set of eight candies that differ in color (two subsets). In this way the cardinality of set or the subset comes to the fore. There are eight candies which consist of six green apple candies and two red cherry candies. Within these subsets, there exists no possible differentiation; hence no specific sequences that would be distinguishable are imaginable. For the subsets of four candies that are to be created anew, as well, the only thing that can be said is that each subset consists of candies that might be different in taste. A specific sequence is neither necessary nor would it make sense in the chosen everyday situation. Thus, all combinations of four candies that are distinguishable from one another have to be found from a set of eight candies.

Selection concept / combinatorial concept

Initially, all possible cases of distinguishable combinations according to the given assumptions of the problem have to be considered: With $k = 4$ pullings 0, 1 or 2 red candies and correspondingly 4, 3 or 2 green candies can be pulled. The following k-element sets are possible: $\{g, g, g, g\}$; $\{g, g, g, r\}$; $\{g, g, r, r\}$. The number of possible outcomes of the experiment could be found by a lexicographical counting of the combinations, following the formula of hypergeometric distribution (cf. Kersting & Wakolbinger 2008, p. 28) or by dint of a tree diagram. With the latter method, the doubles that are generated have to be discarded.

What is important for this concept is that there be combinations of selection distinguishable from one another that are created in a specific way, namely without replacement. In addition, a selection of candies may occur consisting of only one kind, since there are only two of the other kinds in the initial set. Moreover, fictitious combinations are generated mentally, of which only one will actually occur (cf. randomness concept). For that reason the initial situation (eight candies in the bag) has to be restored after each pulling, although there must be no replacements for each four-time pulling. For the discovery of all possibilities, it is advisable to compare the combinations that have been found and written down, thus eliminating doubles. Hence, this approach provokes a certain kind of documentation, since the process of pulling has to be repeated until all the various combinations have been discovered. Furthermore, written documentations often indicate a certain order, which in this context is unimportant, though.

Randomness concept / combinatorial concept

Which of all the possible combinations will occur cannot be definitively predicted. All imaginable possibilities can be pulled, but the pulling does not lead automatically to all the different combinations. It is possible that the same combination is pulled several times. Hence, a situation has to be considered that will only possibly occur. With the facet of the randomness concept that is relevant here, it is less the probability of particular combinations than the determination of all possible events that is in the foreground. The combination of the four candies that have been pulled is random. The missing combinations have to be added by thought experiment.

TECHNICAL TERMINOLOGY – MATHEMATICAL COGNITIVE PROCESS – PUPILS' EXPRESSIONS

Mathematical cognitive processes take place between the poles of mathematical and individually formed concepts. Mathematical as well as individual concepts are expressed in signs in form of the respective language culture (mathematical technical language, mathematical language of instruction, mathematical everyday speech). In this paper, we define the mathematical technical language as a language, which is used in the conversation between mathematical experts with a focus on formalization in verbal and written contexts in support of an agreed form of communication over a particular issue. The mathematical technical language is hence the result of many discursive negotiation processes that lead to a formal presentation. The mathematical everyday speech displays a discursive, processual character and serves more for individual formation of concepts and the approach to mathematical concepts.

Using the example presented above, figures 1 and 2 (see below) illustrate mathematical and individual pre-concepts, which, at best and naturally individually formed, approach one another. Verbal orientated signs that would be used by an expert (e.g. mathematician) are listed in the category of mathematical technical language and expresses mathematical knowledge which is adequate for the given problem. This mathematical knowledge and the expression of it also emerged in discursive negotiation processes and in build a relation between signs and reference context and aim at an agreed form of communication – language culture among mathematicians (Morgan 1998). The pupils' expressions specific to the situation are listed in the right column and are conceptually oral as well.

ANALYSIS

At first glance, the language of the pupils is molded by phrases taken from everyday speech and child-like action patterns like "which should I take [using a counting-out rhyme]" as well as by terms from the text of the posed problem. In Steinbring's words you can reconstruct out of these expressions the children's given pre-existing conceptual ideas or in Prediger's words their individual concepts. These conceptions are tried to communicate by dint of signs or signifiers which should convey the children's interpretation of the meaningful mathematical content.

<table>
<tr><th rowspan="2">mathematical technical language ⇔</th><th>mathematical concepts</th><th>individual mathematical concepts</th><th rowspan="2" colspan="2">⇔ verbal expressions by students</th></tr>
<tr><th colspan="2">mathematical cognitive process</th></tr>
<tr><td rowspan="2">There is a finite set, called A, with n elements. This set is composed of two disjoint sets, called A1 and A2, with r respectively g elements. From the finite set A should be removed a set of k elements, in this case four elements.</td><td colspan="2">set concept /
cardinal number concept</td><td rowspan="2">Student 1:
Student 1:

Student 1:

Student 2:
Student 1:</td><td rowspan="2">Emma has six candies
yes two cherry candies and em six apple candies
she wants to give her brother four of them
and we must pull them
Yes, two plus two</td></tr>
<tr><td>formally abstract conceptions of sets</td><td>additive conception, tangible objects – candies</td></tr>
<tr><td rowspan="2">There are four different combinatorial cases. In view of this task the case can be described as pulling without replacement and without regard to order. You can use the formula of n over k and adhere to the lamination.</td><td colspan="2">selection concept /
combinatorial concept</td><td rowspan="2">Student 1:
Student 1:

Student 1:</td><td rowspan="2">how much possibilities
which should I take [using a counting-out rhyme with one hand in the sac]
at first we must always pull them and later then we have to lay all of them back into the sac</td></tr>
<tr><td>'inner action images':
for instance, urn model, 0-1 sequence</td><td>playful action</td></tr>
</table>

Figure 1: Mathematical cognitive process, exemplified by the set concept / cardinal number concept and the selection concept

The technical-language part described here uses phrases that are more typically mathematical: "There is a finite set, called A, [...]". It can be determined that the pupils utilize terms like "pull" or "replacement," which they probably connect with their everyday conception of pulling situations (pulling lottery tickets, for instance). In the situation that is presented, the pupils skillfully focused mathematical facets: not the taste or type of candies (cherry, apple) but the number, the color as a differentiator, and the possible combinations under the given assumptions constitute the focus of their consideration. "At first we must always pull them and later then we have to lay all of them back into the bag," is the description of the combinatorial figure of pulling without replacement and, in addition, something actually in contrast to that: the restoration of the initial situation after pulling four times. Here, the close connection of context and mathematical conception (→urn model) – intended by the text of the problem – is presumably taking hold.

In terms of the technical language, mathematical terms are used also as typical formulations like “as pulling without replacement and without regard to order” for modeling, which are applied in a way relevant to the problem. The students are still in the process of model discovery, which is displayed in such comments about possible combinations: “Ah we can’t red, red, red, red we can’t because there are only two red,” which presents an interactive verbal negotiation of this cognitive process and suggests mathematical concepts that are still developing but are already target-consistent and are moving within the domain relevant to combinatorics. The production of relations between signs and reference context here therefore generate new mathematical knowledge. While the children at the beginning of the situation seem to utilize more operational and process-oriented dynamic concepts (they pull, put down, count by dint of a counting-out rhyme), they use in the proceeding of the situation more and more also structural descriptions: “We have red, red, green […].” Sfard (1991, p. 5) said, that seeing both “[…] a process and […] an object is indispensable for a deep understanding of mathematics [...].”

mathematical technical language	mathematical concepts	individual mathematical concepts	verbal expressions by students	
⇔	mathematical cognitive process		⇔	
[in continuation of figure 1] With the different given sets taking into account, you can hence say, that r is less than k, which is less than g, which is finally less than n. Accordingly you can suggest, that k could be equal g or g plus r but never just r.	selection concept / combinatorial concept	Negotiation, what combinations are possible? ⇓ selection concept / combinatorial concept	Student 1:	[Looking at the subscription]: we have red, red, green / red, green, green, red / green, green, green, red / green, green, green, green
			Student 2:	we can also take red, red red/
			Student 1:	Ah we can’t red, red, red, red we can’t because there are only two red
			Student 2:	I would say I take three red and a green but it doesn’t work because we have only two red
The result set could be defined as gggg, gggr or ggrr. You have to use the classic propability formula and divide the number of well cases by all possible cases.	randomness concept		Student 2:	Don’t look while pulling
			Student 1:	perhaps we will pull the same
			Student 1:	[By pulling]: please, please, please
	stochastic model: classic probability	random access by dint of desire, luck, bad luck	Student 1:	please no red, no red please
			Student 1:	[By pulling]: if we get a green

Figure 2: Mathematical cognitive process, exemplified by the selection concept

The pupils’ randomness concept is molded by child-like pre-conceptions where events are attempted to be ‘wished’ to come true, which becomes implicit in expressions like “please no red, no red please.” Mathematically, randomness becomes comprehensible by dint of the formula about classic probability. Nonetheless, the students already have at their disposal the skill that is crucial for handling combinatorial problems: being capable of mental imagining the configurations of possible combinations and they can also communicate this by dint of verbal signs: "Perhaps we will pull the same." It becomes manifest that the pupils have a concept of mathematizing at their disposal. Certainly in part guided by the setting, the students activate action patterns, which focus on those facets of the problem that are relevant to combinatorics and express mathematical thought processes verbally. Individual conceptions converge with mathematical concepts.

CONCLUSIONS

With our initial attempts at analysis, preliminary insights in the mathematical utterances of first-graders can be described. Concerning our introductorily questions we can summarize the following conclusions:

1. In view of the presented analysis of this exemplary situation we suggest that there is first evidence that children, who just starting school obviously have at their disposal manifold forms of expression in terms of mathematical problems. They convey these forms of expression by dint of everyday speech as well as of first technical language, e. g. in using mathematical terms like "possibilities" or abstract from the given context in using "red, red, red" rather than the concrete objects (here: candies). Terms belonging to combinatorics are utilized in a meaningful and productive way during the process of handling the problem and suggest mathematical concepts that have been already acquired or are developing.
2. In reference to the problem's core question, language is dominant for action steps that are in need of explanation, or when considering an action result (here the combinations of candies that have been pulled). Concepts are verbalized that have to be tested or that only develop in – and through – the process of verbalization. In doing so, the individual mental concepts converge with mathematical concepts, which can be partially considered as already acquired.
3. The young learners in the presented situation utilize process-oriented and structural concepts, which indicate they are focusing on what doing mathematics means to them in the context of the specific combinatorial problem.

These initial conclusions have to be examined in further research to follow, in other mathematical areas or different problem arrangements, for instance. Moreover, it is essential to approach the analytical procedures mentioned above and, for one, to examine more closely the construction of mathematical knowledge in the focus of interaction. In our further investigations we want to deepen this analysis and adopt it to other comparable situations in which children solve problems in different mathematical areas. In this context we plan to investigate the mathematical development in the age of kindergarten children in a longitudinal study (a study inside IDEA, in front explained). This could enable us to describe over the period in which the children visit the kindergarten the development of mathematical thinking. The project of research is applied as a cooperation study with researchers of language acquisition, which should enable us to investigate in particular the coherency of mathematical development and language acquisition. Furthermore it is possible to broaden the perception of language from purely verbal expressions to other activities such as gesture or body language as well as written and presented mathematical facets and also focus on interaction processes for an implication of a social perspective.

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