MATHEMATICAL COGNITIVE PROCESSES BETWEEN THE POLES OF MATHEMATICAL TECHNICAL TERMINOLOGY AND THE VERBAL EXPRESSIONS OF PUPILS

Rose Vogel and Melanie Huth

Goethe University, Frankfurt am Main, Germany

Verbal expressions by students in mathematical conversational situations provide insight into the individual mathematical imagination and express what patterns and contexts children recognize in mathematical problems. Children just starting school utilize means of expression of their mathematical ideas that go from everyday speech descriptions to detailed action sequences. They already use technical facets, even though their repertoire of mathematical language of instruction has to be considered initially as tentative. In our article, by dint of methods of qualitative analysis, we want to present initial descriptions in terms of the identified capability of mathematical expression of pupils just starting school, based on a conversational situation about a combinatorial problem.

Keywords: mathematical cognitive process, mathematical language, mathematics in elementary school, combinatorics, mathematical concepts

INTRODUCTION

The mathematical cognitive process is initiated extrinsically and/or intrinsically by tangible problems or questions the young learner encounters in various contexts. This process is of a discursive nature. Furthermore the mathematical problems are expressed in manifold linguistic forms. In the process of understanding, individual prior knowledge, mathematical concepts and strategies are activated by the learner. According to the learner's estimation the activated strategies promise the most probable possibility for a solution.

Within the framework of our research, we wish to focus on linguistic activities within the mathematical cognitive process that significantly mold this very process: mathematical content is conveyed by dint of language; mathematics is talked and written about. This approach is needed to broaden the perception of language from purely verbal expressions to other activities such as gesture, body language and facial expression, as well as bringing mathematical facets into written form and presenting them. In addition, it is important to take into account what cognitive grasp, from their perspective, the respective protagonists have in terms of handling mathematical problems. It is also interesting which patterns of action are consciously or unconsciously activated in terms of the situation. Individual interpretations, concepts and models of the mathematical content, which is always restricted to context, and also social aspects (communication patterns, specific language of instruction, structure of the interactive negotiation process, teaching and learning patterns and cultural conditions) and the personal image of mathematics are especially pivotal and demand detailed consideration.

> "The emerging of mathematical knowledge is fundamentally taking place in the context of social construction an individual interpretation processes. […] it is constructed by means of social activities and individual interpretations." (Steinbring 2005, p. 7)

In the present paper we would like to give an outline of the provisional state of knowledge resulting from our activity in the field of 'The learning of mathematics and language'. Translating the mathematical content of a problem into technical terms is related to the mathematical language of young learners, in particular their mathematical concepts and individual conceptions, which are reconstructed based on verbal activities. We expect that the detailed consideration of the children's verbal expression will afford us with insights into what they view as the problem's mathematics. This identification of mathematical and individual concepts is to be deepened in the future inter alia by the interactional view of mathematical negotiation processes mentioned above. In doing so, we wish to focus on 'mathematical language' in the broadest sense of the term, that is, constituting all forms of expression accompanying the mathematical cognitive process. In our opinion learners of mathematics, especially young learners, approximate more and more to technical mathematics-orientated language in their process of learning mathematics. This "speaking mathematically" (Pimm 1989) is more than just learning vocabulary and using these words in the right linguistic form. Linked to that is the notion of utilizing this knowledge to design processes of teaching and learning. If we demonstrate our considerations in the following and represent our thoughts by means of an example, we focus at first verbal expressions and unique actions in the mathematical cognitive process that are examined as unique expressions for now. We will present an exemplary conversational situation of first-graders concerning a combinatorial problem. Our research perspective is guided by the super ordinate question about mathematical language and a potential mathematical language development in the process of learning mathematics on the part of young learners. In the present paper we want to focus on the following embedded questions:

What language do the here described pupils have at their disposal when handling a combinatorial problem in the conversational situation being presented?

What individual conceptions and mathematical ('target-consistent') concepts can be surmised behind the described children's verbal-linguistic activities in the examination with a combinatorial problem?

What patterns of actions are activated or what conceptions about 'to do mathematics' in the examination with an explicit structured combinatorial problem can be reconstructed by means of verbal activities?

The data and considerations used have emerged from our research within the framework of an initial exploratory pilot study. We conducted this exploratory study with a focus on designing and interpretating situations which could be analyzed in the view of mathematical concept development and the linguistic means of expression in discourse situations. This study could contain useful information and serves as a trial of such situations. It is embedded in the context of a current developed longitudinal study to investigate early steps in mathematics learning (related to the Centre for Research on Individual Development and Adaptive Education of Children at Risk (IDeA), a centre of DIPF (German Institute for International Educational Research) and the Goethe university, Frankfurt/Main in cooperation with the Sigmund-Freud-Institute, Frankfurt/Main).

THEORETICAL FRAMEWORK – MATHEMATICAL LANGUAGE ACTIVITIES OF CHILDREN IN ELEMENTARY SCHOOL

At the beginning of their time in school, young, monolingual, linguistically inconspicuous learners have at their disposal a fundamental passive and active vocabulary. Their language acquisition in the unique grammatical sub-systems can be termed basic. Now what becomes relevant in terms of language is the growth of special communication and action patterns to be ascribed to the institution of the school, such as the acquisition of a certain language of instruction (cf. “cognitive academic language proficiency,” according to Cummins 2000 after Gellert 2008, p. 140). For mathematics lessons in particular, a vocabulary and a specific language have to be acquired in which symbols are employed or terms from everyday speech adopt a different meaning (like ‘equal,’ ‘less,’ ‘greater’). Negotiation processes in the social context have to be mastered linguistically within the learning process so as to understand mathematical teaching contents and be capable of participation. Verbal expressions are thus embedded in the interaction process in which they are uttered. The process of analysis documented here represents an initial approach to a form of analysis yet to be developed, which would permit one to make statements about the applied forms of language in the context of mathematical cognitive processes. Beside that, the analytical method to be developed could be interlocked with other approaches like interaction, argumentation and participation analysis (Brandt & Krummheuer 2000; Krummheuer 2007).

The approach presented here in an initial outline bears a certain resemblance in several parts to Steinbring’s (2005, 2006) epistemological approach. In the epistemological triangle developed by Steinbring, the interactively constructed mathematical knowledge is of central importance. This knowledge, which is again based on pre-existing conceptual ideas, is generated by creating relations between the signs being utilized and reference context. In our approach the children used signs in the form of verbal, gestural and also written expressions to communicate their meaning or interpretation of the given mathematical content. In doing so, they needed to revert to their pre-existing conceptual ideas. Their expressions or signifier could only refer to the reference context or signified, whereas a common interpretation of this mathematical content has to be negotiated in interaction.

The question is how these mathematical pre-existing conceptual ideas and knowledge in Steinbring’s approach can be described. The point of departure of our analysis is the problem’s so-called mathematical content. While handling the ‘mathematical content,’ we try to describe the mathematical concepts or mental models (here in the meaning of Prediger 2008) that are of import for solving the problem. Mathematical concepts or mental models, according to Prediger (2008), are contrasted with the personal conceptions of the individual who is learning, which are reconstructed here by means of pupils' expressions. These individual "students' conceptions" (Prediger 2008, p. 6) which are comparable with Steinbring's pre-existing conceptual ideas (Steinbring 2006, 140), sum up the conceptions of the individuals who are learning, which could be developed up to now to handle similar mathematical problems. Any other structurally similar mathematical problem will re-activate these "individual models," which are then confirmed in the situation or may lead to irritations and potential expansions of these individual models. Mathematical experts and novices alike use individual mathematical models to be able to approach the abstract and immaterial mathematical objects and develop mental images for them: "[…] mathematical concepts are sometimes envisioned by help of 'mental pictures' […]. Visualization […] makes abstract ideas more tangible, […] almost as if they were material entities." (Sfard 1991, 6) Should a discrepancy arise between the individual model and the 'mathematical concept' relevant to the problem and prove to be too large to overcome, this may create learning opportunities that can be utilized more or less beneficially.

RELEVANT MATHEMATICAL CONCEPTS IN SOLVING COMBINATORIAL PROBLEMS

Combinatorics involves the determination of the number of elements of finite sets. The point is to select elements from a given total (basic set) and re-combine and rearrange them according to specific criteria (cf. Krauter 2005/2006). The description "combining selected elements" refers to the formation of new combinations of sets. The description "arranging selected elements" focuses on the order and thus on the formation of variations (cf. Selter & Spiegel 2004, 291). Again the determination of the number, of the sets or lists that arise this way, will be of importance. Thus, combinatorics centers around counting. Although here we are moving in the context of discrete mathematics and hence in the range of countability, this will frequently take on a theoretical character and provoke mathematical methods that go beyond the act of counting. These arithmetical "counting methods" are documented as formulas that in a compressed form describe the appropriate algorithm. In addition to the formulas, instructions are described having the function of activating inner images with the learner. These images help to translate familiar situations into the unknown mathematical problem and encourage the utilization of a suitable formula (for instance, without regard to order and without replacement).

The conversational situation that our analysis is based on is a part of an explorative study in which a total of eight first-graders were under examination. We selected this particular situation because its progress is comparable with all other videotaped and transliterated situations. Furthermore we choose such situation with a combinatorial problem, because this requires from the pupils counting and manipulation with sequences in practice. For the explorative study we developed mathematical problems of different mathematical areas, e. g. combinatorics, and then presented one problem to a student-duad in a conversational situation. The setting which is important in the following descriptions was hence set up as follows: The researcher presents a combinatorial problem to two first-graders. The pupils had the joint task of solving the combinatorial problem. In the progress of the situation, the researcher simply joins the conversation of the children in an appropriate way. As material at their disposal the children had paper, pencils and a bag full of candies.

Problem: Emma has two red cherry candies and six green apple candies in her bag. She pulls four times from her bag and gives the candies that have been pulled to her brother Tom. What candies can Tom get? Find all the options that are not identical!

The problem describes precisely how the desired subsets – consisting of four elements – are to be generated. Four pullings in a row are to take place. Replacement does not make sense, as the generated subset is to be given away. This makes it quite explicit that one element of the initial set cannot be pulled more than once. Thus, the problem describes the combinatorial figure of pulling without replacement (a total of four pullings) of k elements from n. The second criterion of order is irrelevant to the problem (cf. set concept). Thus, the act can be translated into a pulling all at once, that is, without replacement and without regard to order (cf. Kütting & Sauer 2008, p. 93).

Cardinal number concept / set concept

The point of departure for the problem is an n-element set ($n = 8$), which is comprised of two subsets with the element numbers $r = 2$ and $g = 6$. In tangible terms, the problem is about the set of eight candies that differ in color (two subsets). In this way the cardinality of set or the subset comes to the fore. There are eight candies which consist of six green apple candies and two red cherry candies. Within these subsets, there exists no possible differentiation; hence no specific sequences that would be distinguishable are imaginable. For the subsets of four candies that are to be created anew, as well, the only thing that can be said is that each subset consists of candies that might be different in taste. A specific sequence is neither necessary nor would it make sense in the chosen everyday situation. Thus, all combinations of four candies that are distinguishable from one another have to be found from a set of eight candies.

Selection concept / combinatorial concept

Initially, all possible cases of distinguishable combinations according to the given assumptions of the problem have to be considered: With $k = 4$ pullings 0, 1 or 2 red candies and correspondingly 4, 3 or 2 green candies can be pulled. The following k-element sets are possible: $\{g, g, g, g\}$; $\{g, g, g, r\}$; $\{g, g, r, r\}$. The number of possible outcomes of the experiment could be found by a lexicographical counting of the combinations, following the formula of hypergeometric distribution (cf. Kersting & Wakolbinger 2008, p. 28) or by dint of a tree diagram. With the latter method, the doubles that are generated have to be discarded.

What is important for this concept is that there be combinations of selection distinguishable from one another that are created in a specific way, namely without replacement. In addition, a selection of candies may occur consisting of only one kind, since there are only two of the other kinds in the initial set. Moreover, fictitious combinations are generated mentally, of which only one will actually occur (cf. randomness concept). For that reason the initial situation (eight candies in the bag) has to be restored after each pulling, although there must be no replacements for each four-time pulling. For the discovery of all possibilities, it is advisable to compare the combinations that have been found and written down, thus eliminating doubles. Hence, this approach provokes a certain kind of documentation, since the process of pulling has to be repeated until all the various combinations have been discovered. Furthermore, written documentations often indicate a certain order, which in this context is unimportant, though.

Randomness concept / combinatorial concept

Which of all the possible combinations will occur cannot be definitively predicted. All imaginable possibilities can be pulled, but the pulling does not lead automatically to all the different combinations. It is possible that the same combination is pulled several times. Hence, a situation has to be considered that will only possibly occur. With the facet of the randomness concept that is relevant here, it is less the probability of particular combinations than the determination of all possible events that is in the foreground. The combination of the four candies that have been pulled is random. The missing combinations have to be added by thought experiment.

TECHNICAL TERMINOLOGY – MATHEMATICAL COGNITIVE PROCESS – PUPILS' EXPRESSIONS

Mathematical cognitive processes take place between the poles of mathematical and individually formed concepts. Mathematical as well as individual concepts are expressed in signs in form of the respective language culture (mathematical technical language, mathematical language of instruction, mathematical everyday speech). In this paper, we define the mathematical technical language as a language, which is used in the conversation between mathematical experts with a focus on formalization in verbal and written contexts in support of an agreed form of communication over a particular issue. The mathematical technical language is hence the result of many discursive negotiation processes that lead to a formal presentation. The mathematical everyday speech displays a discursive, processual character and serves more for individual formation of concepts and the approach to mathematical concepts.

Using the example presented above, figures 1 and 2 (see below) illustrate mathematical and individual pre-concepts, which, at best and naturally individually formed, approach one another. Verbal orientated signs that would be used by an expert (e.g. mathematician) are listed in the category of mathematical technical language and expresses mathematical knowledge which is adequate for the given problem. This mathematical knowledge and the expression of it also emerged in discursive negotiation processes and in build a relation between signs and reference context and aim at an agreed form of communication – language culture among mathematicians (Morgan 1998). The pupils' expressions specific to the situation are listed in the right column and are conceptually oral as well.

ANALYSIS

At first glance, the language of the pupils is molded by phrases taken from everyday speech and child-like action patterns like "which should I take [using a counting-out rhyme]" as well as by terms from the text of the posed problem. In Steinbring's words you can reconstruct out of these expressions the children's given pre-existing conceptual ideas or in Prediger's words their individual concepts. These conceptions are tried to communicate by dint of signs or signifiers which should convey the children's interpretation of the meaningful mathematical content.

<table>
<tr><th rowspan="2">mathematical technical language ⇔</th><th>mathematical concepts</th><th>individual mathematical concepts</th><th rowspan="2" colspan="2">⇔ verbal expressions by students</th></tr>
<tr><th colspan="2">mathematical cognitive process</th></tr>
<tr><td rowspan="2">There is a finite set, called A, with n elements. This set is composed of two disjoint sets, called A1 and A2, with r respectively g elements. From the finite set A should be removed a set of k elements, in this case four elements.</td><td colspan="2">set concept /
cardinal number concept</td><td rowspan="2">Student 1:
Student 1:

Student 1:

Student 2:
Student 1:</td><td rowspan="2">Emma has six candies
yes two cherry candies and em six apple candies
she wants to give her brother four of them
and we must pull them
Yes, two plus two</td></tr>
<tr><td>formally abstract conceptions of sets</td><td>additive conception, tangible objects – candies</td></tr>
<tr><td rowspan="2">There are four different combinatorial cases. In view of this task the case can be described as pulling without replacement and without regard to order. You can use the formula of n over k and adhere to the lamination.</td><td colspan="2">selection concept /
combinatorial concept</td><td rowspan="2">Student 1:
Student 1:

Student 1:</td><td rowspan="2">how much possibilities
which should I take [using a counting-out rhyme with one hand in the sac]
at first we must always pull them and later then we have to lay all of them back into the sac</td></tr>
<tr><td>'inner action images':
for instance, urn model, 0-1 sequence</td><td>playful action</td></tr>
</table>

Figure 1: Mathematical cognitive process, exemplified by the set concept / cardinal number concept and the selection concept

The technical-language part described here uses phrases that are more typically mathematical: "There is a finite set, called A, [...]". It can be determined that the pupils utilize terms like "pull" or "replacement," which they probably connect with their everyday conception of pulling situations (pulling lottery tickets, for instance). In the situation that is presented, the pupils skillfully focused mathematical facets: not the taste or type of candies (cherry, apple) but the number, the color as a differentiator, and the possible combinations under the given assumptions constitute the focus of their consideration. "At first we must always pull them and later then we have to lay all of them back into the bag," is the description of the combinatorial figure of pulling without replacement and, in addition, something actually in contrast to that: the restoration of the initial situation after pulling four times. Here, the close connection of context and mathematical conception (→urn model) – intended by the text of the problem – is presumably taking hold.

In terms of the technical language, mathematical terms are used also as typical formulations like “as pulling without replacement and without regard to order” for modeling, which are applied in a way relevant to the problem. The students are still in the process of model discovery, which is displayed in such comments about possible combinations: “Ah we can’t red, red, red, red we can’t because there are only two red,” which presents an interactive verbal negotiation of this cognitive process and suggests mathematical concepts that are still developing but are already target-consistent and are moving within the domain relevant to combinatorics. The production of relations between signs and reference context here therefore generate new mathematical knowledge. While the children at the beginning of the situation seem to utilize more operational and process-oriented dynamic concepts (they pull, put down, count by dint of a counting-out rhyme), they use in the proceeding of the situation more and more also structural descriptions: “We have red, red, green […].” Sfard (1991, p. 5) said, that seeing both “[…] a process and […] an object is indispensable for a deep understanding of mathematics [...].”

mathematical technical language ⇔	**mathematical concepts**	**individual mathematical concepts**	⇔ **verbal expressions by students**	
	mathematical cognitive process			
[in continuation of figure 1] With the different given sets taking into account, you can hence say, that r is less than k, which is less than g, which is finally less than n. Accordingly you can suggest, that k could be equal g or g plus r but never just r.	selection concept / combinatorial concept	Negotiation, what combinations are possible? ⇩ selection concept / combinatorial concept	Student 1:	[Looking at the subscription]: we have red, red, green / red, green, green, red / green, green, green, red / green, green, green, green
			Student 2:	we can also take red, red red/
			Student 1:	Ah we can’t red, red, red, red we can’t because there are only two red
			Student 2:	I would say I take three red and a green but it doesn’t work because we have only two red
The result set could be defined as gggg, gggr or ggrr. You have to use the classic propability formula and divide the number of well cases by all possible cases.	randomness concept		Student 2:	Don’t look while pulling
			Student 1:	perhaps we will pull the same
			Student 1:	[By pulling]: please, please, please
	stochastic model: classic probability	random access by dint of desire, luck, bad luck	Student 1:	please no red, no red please
			Student 1:	[By pulling]: if we get a green

Figure 2: Mathematical cognitive process, exemplified by the selection concept

The pupils’ randomness concept is molded by child-like pre-conceptions where events are attempted to be ‘wished’ to come true, which becomes implicit in expressions like “please no red, no red please.” Mathematically, randomness becomes comprehensible by dint of the formula about classic probability. Nonetheless, the students already have at their disposal the skill that is crucial for handling combinatorial problems: being capable of mental imagining the configurations of possible combinations and they can also communicate this by dint of verbal signs: "Perhaps we will pull the same." It becomes manifest that the pupils have a concept of mathematizing at their disposal. Certainly in part guided by the setting, the students activate action patterns, which focus on those facets of the problem that are relevant to combinatorics and express mathematical thought processes verbally. Individual conceptions converge with mathematical concepts.

CONCLUSIONS

With our initial attempts at analysis, preliminary insights in the mathematical utterances of first-graders can be described. Concerning our introductorily questions we can summarize the following conclusions:

1. In view of the presented analysis of this exemplary situation we suggest that there is first evidence that children, who just starting school obviously have at their disposal manifold forms of expression in terms of mathematical problems. They convey these forms of expression by dint of everyday speech as well as of first technical language, e. g. in using mathematical terms like "possibilities" or abstract from the given context in using "red, red, red" rather than the concrete objects (here: candies). Terms belonging to combinatorics are utilized in a meaningful and productive way during the process of handling the problem and suggest mathematical concepts that have been already acquired or are developing.
2. In reference to the problem's core question, language is dominant for action steps that are in need of explanation, or when considering an action result (here the combinations of candies that have been pulled). Concepts are verbalized that have to be tested or that only develop in – and through – the process of verbalization. In doing so, the individual mental concepts converge with mathematical concepts, which can be partially considered as already acquired.
3. The young learners in the presented situation utilize process-oriented and structural concepts, which indicate they are focusing on what doing mathematics means to them in the context of the specific combinatorial problem.

These initial conclusions have to be examined in further research to follow, in other mathematical areas or different problem arrangements, for instance. Moreover, it is essential to approach the analytical procedures mentioned above and, for one, to examine more closely the construction of mathematical knowledge in the focus of interaction. In our further investigations we want to deepen this analysis and adopt it to other comparable situations in which children solve problems in different mathematical areas. In this context we plan to investigate the mathematical development in the age of kindergarten children in a longitudinal study (a study inside IDEA, in front explained). This could enable us to describe over the period in which the children visit the kindergarten the development of mathematical thinking. The project of research is applied as a cooperation study with researchers of language acquisition, which should enable us to investigate in particular the coherency of mathematical development and language acquisition. Furthermore it is possible to broaden the perception of language from purely verbal expressions to other activities such as gesture or body language as well as written and presented mathematical facets and also focus on interaction processes for an implication of a social perspective.

REFERENCES

Brandt, B. & Krummheuer, G. (2000). Das Prinzip der Komparation im Rahmen der Interpretativen Unterrichtsforschung in der Mathematikdidaktik. Journal der Mathematikdidaktik, 21, 193-226.

Gellert, U. (2008). Cognitive academic language proficiency in primary mathematics classrooms. In: J.F. Matos, P. Valero & K. Yasukawa (Hrsg.) Proceedings of the Fifth International Mathematics Education and Society Conference (pp. 140-143). Lissabon: Centro de Investigação em Educação, Universidade de Lisboa.

Kersting, G. & Wakolbinger, A. (2008): Elementare Stochastik. Basel, Boston, Berlin: Birkhäuser Verlag AG.

Krauter (2005/06). Einführung in die Elementare Kombinatorik (Vorlesungsskript). Pädagogische Hochschule Ludwigsburg.

Krummheuer, G. (2007): Argumentation and participation in the primary mathematics classroom: Two episodes and related theoretical abductions. Journals of Mathematical Behaviour, 26, 60-82.

Kütting, H. & Sauer, M.J. (2008). Elementare Stochastik. Mathematische Grundlagen und didaktische Konzepte. Berlin, Heidelberg: Springer Verlag.

Morgan, C. (1998). Writing mathematically. The discourse of Investigation. London: Falmer Press.

Pimm, D. (1989): Speaking mathematically. Communication in Mathematics Classrooms. London: Routledge.

Prediger, S. (2008). The relevance of didactic categories for analysing obstacles in conceptual change: Revisiting the case of multiplication of fractions.

Selter, C. & Spiegel, H. (2004). Elemente der Kombinatorik. In: G. N. Müller, Heinz Steinbring & E. Ch. Wittmann (Eds.): Arithmetik als Prozess (pp. 291 – 310). Seelze: Kallmeyersche Verlagsbuchhandlung GmbH.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. In: Educational Studies in Mathematics, 22, 1-36.

Steinbring, H. (2005). The Construction of New Mathematical Knowledge in Classroom Interaction. An Epistemological Perspective. New York: Springer.

Steinbring, H. (2006). What Makes a Sign a Mathematical Sign? An Epistemological Perspective on Mathematical Interaction. Educational Studies in Mathematics, 61, 133-162.