THE EXISTENCE OF MATHEMATICAL OBJECTS IN THE CLASSROOM DISCOURSE

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In this paper we are interested in the understanding of how the classroom discourse helps to develop the students' comprehension of the non ostensive mathematical objects as objects that have "existence". First, we examine the role of the objectual metaphor in the understanding of the mathematical entities as "objects with existence", as well as in some of the conflicts that the use of this type of metaphor can provoke in the students' interpretations. Second, we examine the mathematics discourse from the perspective of the ostensives representing non ostensives that do not exist.

INTRODUCTION

In this report we present some findings from our current research on the role of objectual metaphors in the interpretation of the existence of non ostensive mathematical objects within the classroom discourse. We illustrate these findings with a reinterpretation of data from Acevedo (2008). In particular we analyze certain remarks of different teachers that have in common the use of metaphors in their teaching practices. In that study, the fourth author presented an analysis of some teachers' discourses while teaching the graphic representation of functions in Spanish high schools. The focus was on the teachers' discourses and practices when interacting with the students in certain lessons. The main data was gathered by means of video and audio tapes, together with written tests, students' work and filed notes.

We organize the report from theory to example in order to deal with language and communication issues in mathematics classrooms from a semiotic point of view. We begin by briefly reviewing part of the literature on metaphors and presenting the notions of image schema and conceptual metaphor, which are drawn on the theories of the embodied cognition. When introducing some findings, we show how the use of metaphorical expressions of the objectual metaphors in the teachers' discourses leads the students to understand the mathematical entities like "objects with existence". Finally, we show how the mathematics discourse on ostensives representing non ostensives that do not exist and on the identification of mathematical objects with some of its representations, leads the students to separately interpret the mathematical objects and its ostensive representations.

IMAGE SCHEMAS AND METAPHORICAL PROJECTIONS

In recent years, several authors (see, for instance, Bolite, Acevedo & Font, 2006; Lakoff & Núñez, 2000; Núñez, Edwards & Matos, 1999; Pimm, 1981, 1987;

Presmeg, 1997) have pointed to the role of metaphors in the teaching and learning of mathematics, and some of them have reflected on the embodied cognition theory. Sriraman and English (2005), in their survey of theoretical frameworks that have been used in mathematics education research, talk about the importance of the embodied cognition theory. On the other hand, the discursive emergence of mathematical objects is interpreted as a research focus within that theory. Sfard (2000, p. 322) has stressed some of the metaphorical questions concerning the existence of the mathematical objects:

> To begin with, let me make clear that the statement on the existence of some special beings (that we call mathematical objects) implicit in all these questions is essentially metaphorical.

We argue that the use of objectual metaphors in the mathematics classroom discourse leads to talk about the existence of mathematical objects. Our notion of objectual metaphor is highly related to the notions of image schema and metaphorical projections (Johnson, 1987; Lakoff & Johnson, 1980). The image schemas are basic schemas, in the middle of the images and the propositional schemas that help to construct the abstract reasoning by means of metaphorical projections. These schema are constituted by multiple corporal experiences experimented by the subject. Some of these experiences share characteristics that are incorporated within the image schema. Both the experiences and the shared characteristics are a consequence of situations that have been physically and repeatedly lived.

Lakoff and Núñez (2000) claim that the cognitive structure for the advanced mathematical thinking shares the conceptual structure of the non mathematical daily life thinking. The metaphorical projection is the main cognitive mechanism that permits to structure the abstract mathematical entities by means of corporal experiences. We interpret the metaphor as the comprehension of an object, thing or domain in terms of another one. The metaphors create a conceptual relationship between an initial or source domain and a final or target domain, while properties from the first to the second domain are projected. In relation to the mathematics, Lakoff and Núñez distinguish two types of conceptual metaphors:

- Grounding metaphors: they relate a target domain within the mathematics to a source domain outside them.
- Linking metaphors: they maintain the source and the target domains within the mathematics and exchange properties among different mathematical fields.

Within the group of grounding metaphors, there is the ontological type, where we find the objectual metaphor. The objectual metaphor is a conceptual metaphor that has its origins in our experiences with physical objects and permits the interpretation of events, activities, emotions, ideas... as if they were real entities with properties. This type of metaphor is combined with other ontological classical metaphors such as that of the “container” and that of the “part-and-whole”. The combination of these types leads to the interpretation of ideas, concepts... as entities that are part of other entities and are conformed by them. This interpretation is clear in the axioms of existence and link, as they are mentioned in a classical Spanish textbook on Geometry (Puig Adam, 1965, p. 4):

> Ax. 1.1. We recognize the existence of infinite entities called <points> whose set will be called <space>.
>
> Ax. 1.2. The points of the space are considered grouped in partial sets of infinite points called <planes> and those from each plane in other partial sets of infinite points called <lines>.

METAPHORICAL EXPRESSIONS OF OBJECTUAL METAPHOR

We consider it necessary to make a distinction between the metaphorical expressions and the conceptual metaphors, as highly interrelated but different ideas. This distinction permits to establish generalizations that, otherwise, would remain invisible. The metaphorical expressions may be grouped into conceptual metaphors, and seen as isolated, they can be thought of as individual cases of particular conceptual metaphors.

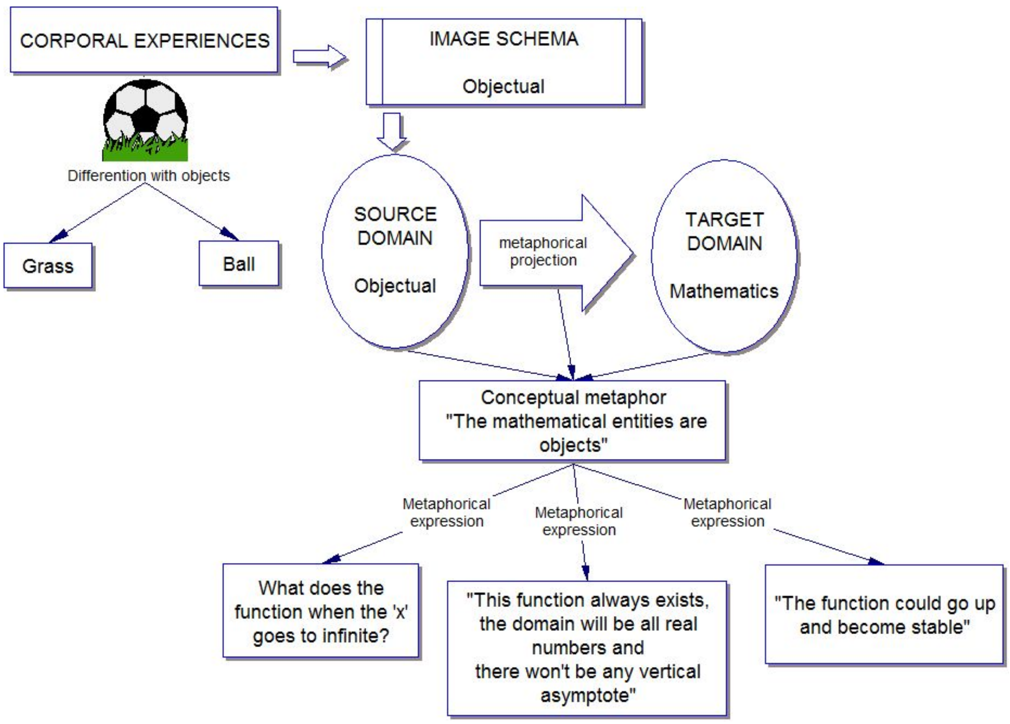


Figure 1. A representation of the objectual metaphor

The conceptual metaphor “The mathematical entities are physical objects” is a grounding ontological metaphor. Figure 1 (Acevedo, 2008, p. 138) illustrates the metaphorical projection with the different metaphorical expressions that appear when using this conceptual metaphor in a mathematics classroom where the graphical representation of functions is being taught to students in high school. Figure 1 shows our experiences in the world of things and the interpretation of the physical objects as separated from this world context; these experiences generate the “objectual” image schema that become the source domain that is projected into the world of the mathematical objects. Table 1 refers to the source and target domains that intervene in the interpretation of this metaphor.

"The mathematical entities are physical objects"	
Source domain: Image schema	Target domain: Mathematical entities
Physical object	Mathematical object
Properties of the physical object	Properties of the mathematical object

Table 1. Domains of the metaphorical projection

THE OBJETUAL METAPHOR IN THE TEACHERS' DISCOURSE

The objectual metaphor is always present in the teachers' discourse because here the mathematical entities are presented as "objects with properties" that can be physically represented (on the board, with manipulatives, with gestures, etc.). In Acevedo (2008), metaphorical expressions of the objectual metaphor occur when the mathematics teacher refers to the graphic of a function as an object with physical properties. When he talks about the application of mathematical operations in order to obtain the first derivative of a function, the teacher uses verbal expressions and gestures that suggest the possibility of manipulating mathematical objects as if they were things with a physical entity (Acevedo, 2008, p. 137):

> Teacher1: The derivative of the numerator, no! You multiply by the denominator as it is, minus the numerator multiplied by the derivative of the denominator. Ok. Now you divide it by the denominator... square, it is. (...) This is the first derivative. Now, what's next? To operate, to manipulate... What's left?

The use of the objectual metaphor facilitates the transition from the ostensive representation of the object –written on the board, drawn with the computer, etc.– to an ideal and non ostensive object. Hence, the use of this type of metaphor leads to talk in terms of the "existence" of the mathematical objects. This use may lead the students to interpret that the mathematical objects exist within the mathematical discourse (internal existence) and, sometimes, may lead them to interpret that they exist like chairs and trees do (external existence, physical or real). In Acevedo (2008, pp. 136-137), we first find a classroom discussion on the domain of the logarithm function and later a discussion on the domain of the square root function, during the instruction of the graphical representation of functions. Here the "existence" is considered within the language game of the mathematical discourse, in comparison to the former teacher's comments on the existence of the first derivative of a function:

Teacher2: The domain goes from zero to infinite because logarithms of *negative numbers do not exist*, logarithm *of minus one does not exist*. Shall we say with the zero included?

Teacher2: Not the negative… because *the square root of a negative number does not exist.* We could also say the real numbers without the negatives, or even easier, all the positive numbers, we can write it like this, with an interval, from the zero to the infinite, now the zero is included.

If the teacher is not careful enough with the way of using (or not using) the verb "exist" in his discourse, the students in this class may not remain within an "internal existence" position. Instead, they may change the "language game" (Wittgenstein, 1953) and assume the "external existence" of the mathematical objects. In the following paragraph, a third different teacher explains the graphical representation of functions to the students in the class and explicitly mentions the idea of existence, although he does so in a rather controversial way (Acevedo, 2008, p. 137):

Teacher3: Then...this function always exists, the domain will be all real numbers and there won't be any vertical asymptote.

We observe a deviation in the "expected" use of the word "exists" within the language game of the mathematics discourse. It would be reasonable to affirm that the images of the values in the domain exist or are defined. When attributing the existence to the whole function instead of talking about its images, the teacher is making a use of the word "exists" that can lead to the understanding of the function as a "real" object with properties, like a chair or a person. Moreover, by doing so, the teacher can promote the movement from the mathematical internal existence of the object to a physical external existence.

DIFFERENTIATON BETWEEN OSTENSIVES AND NON OSTENSIVES

We draw on the theoretical distinction between ostensive and non ostensive objects as established by the onto-semiotic approach to mathematics education (Godino, Batanero & Font, 2007, p. 131):

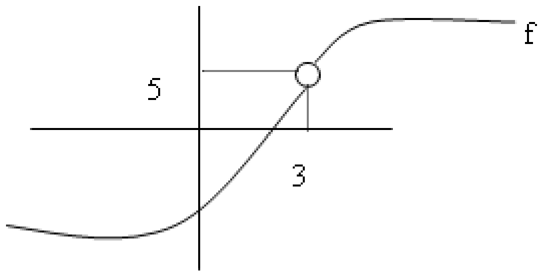
> Ostensive–non-ostensive Mathematical objects (both at personal or institutional levels) are, in general, non-perceptible. However, they are used in public practices through their associated ostensives (notations, symbols, graphs, etc.). The distinction between ostensive and non-ostensive is relative to the language game in which they take part. Ostensive objects can also be thought, imagined by a subject or be implicit in the mathematical discourse (for example, the multiplication sign in algebraic notation).

In the mathematics discourse, it is possible to talk about ostensives representing non ostensives that do not exist. For example, we can say that f'(a) does not exist because the graphic of f(x) has a pointed form in $x = a$. This gives us another example of the semiotic and discursive complexity of the classroom discourse when referring to the existence of mathematical objects. In Acevedo (2008, p. 320) we find the following remark made by a teacher in his classroom discourse:

> Teacher4: As you can see, the one-sided limits are not the same and then the limit does not exist… or the limit is infinite, I mean it is more or less infinite.

In García (2008, appendix 2, p. 8), we find a teacher who uses a discourse with ostensives (*f*(3)) that represent non ostensives that do not exist. He does not say that they do not exist but literally says that “we cannot have them”. The instances from García’s research were obtained in a similar methodological setting –in regular high school classrooms focused on functions and graphs–, to that constructed for the study that was developed by Acevedo.

> Teacher 5: […] Let’s imagine this function:



> What is the domain of *f*? [He answers on the board $\Re - \{3\}$]. And *f*(3)? Don’t make the mistake of saying five, because it is not in the domain and we cannot have an image. We are not worried about *f*(3), but about going as closer as possible to three, before and after the three. Attention, where are the images? Now I don’t have a formula.
>
> Students: Near the five.
>
> Teacher 5: And now if I get closer to three on the right, where are its images?
>
> Students: Over the five.
>
> Teacher 5: Yes we can say limit of *f*(*x*) when *x* goes to three.
>
> Students: But *f*(3) does not exist.
>
> Student: But the asymptote does not touch it either.
>
> Teacher 5: It is curious but $\lim_{x \to 3} f(x) = 5$ [on the board]. It is not defined in three but its limit does exist. That limit exists without having the analytical expression and without having *f*(3).

In order to talk about the existence of certain non ostensives, we have to use a discourse with ostensives constituted in accordance to the “grammar” that regulate the construction of the well-established formulas. This type of discourse is frequently used by many students, as the following remark shows (Acevedo, 2008, p. 368):

Student: Then you do the same here, well you first put the zero here because it is… you look for it, it is the number that you have obtained and the derivative is zero. Then in minus one and in one, you also have to write a zero, but as you have vertical asymptotes you can say that the derivative does not exist, neither does it exists the function. Then you do it with minus one and zero and you get a negative, with the same procedure, and then with the zero and the one you get a positive. As it is positive, it means that you have a minimum here because you have this drawing and it is a minimum.

The use of ostensives that represent non ostensives that do not exist may create confusions in the students' thinking, although it also can turn into philosophical implicit reflections for them. This is the case with a student (Acevedo, 2008, p. 213) that makes a distinction between "to be" and "to exist". He misunderstands the vertical asymptote and makes a mistake:

Teacher5: Could you explain a bit more about the vertical asymptote?

Student: I understand that the vertical asymptote is the value that does not exist in the function.

The existence of well-established ostensives that represent non ostensives that do not exist facilitates the consideration of the non ostensive object as something different from the ostensive that represents it. Duval's work (2008) has pointed to the importance of the different representations and transformations between representations in the students' understanding of the mathematical object as something different from its representation.

Many textbooks of mathematics, implicitly or explicitly make the students observe that an object has many different representations and it is needed to distinguish the object from its representation. In a popular Catalan textbook (Barceló et al., 2002, p. 89), for instance, the following is written:

> In all the activities made, you have been able to observe the different ways of expressing a function: as a statement, as a table of values, as a formula and as a graphic. You always have to remember these four forms of representation and know how to go from one to another.

However, these textbooks frequently tend to identify the mathematical object with one of its representations. In the same Catalan textbook (Barceló et al., 2002, p. 90), it is said "Given the function f(x) = 1/x …" The explanation is that the representation is identified with the object or differentiated from it depending on the purpose. Peirce (1978, §2.273) mentions this idea in his work:

> To stand for, that is, to be in such a relation to another that for certain purposes it is treated by some mind as if it were that other. Thus a spokesman, deputy, attorney, agent, vicar, diagram, symptom, counter, description, concept, premise, testimony, all represent something else, in their several ways, to minds who consider them in that way.

In the mathematical practices, we constantly identify the object with its representations and, on the other hand, we make a distinction between the object itself and some of its representations. The rules of this language game, where the objectual metaphor is crucial, may be difficult to learn for some students. When we deal with physical objects, we can differentiate the sign from the object (for instance, the word "watch" and the physical object "watch"). The objectual metaphor as it is used in the mathematics discourse permits to transfer this differentiation to the mathematical objects and, therefore, we also differentiate the "representation" from the "mathematical object". Moreover, the type of discourse that we produce within the mathematics classroom, leads us to infer the "existence" of the object as something independent from its representation. This situation let us conclude about the existence of a mathematical object that can be represented by means of different "representations".

FINAL REMARKS

In this report we have argued that the objectual metaphor plays a central role in the pedagogical process in the classroom, where teachers (and, consequently, the students) talk about mathematical objects and physical entities. We have shown how the use of metaphorical expressions of objectual metaphors in the mathematics classroom discourse leads the students to interpret the mathematical entities like "objects with existence". On the other hand, the mathematics discourse about ostensives representing non ostensives that do not exist and about the identification (differentiation) of the mathematical object with one of its representations leads the students to interpret the mathematical objects as being different from its ostensive representations. As a consequence, the classroom discourse helps to develop the students' comprehension of the non ostensive mathematical objects as objects that have "existence".

REFERENCES

Acevedo, J. I. (2008). *Fenómenos relacionados con el uso de metáforas en el discurso del profesor. El caso de las gráficas de funciones.* Unpublished Doctoral Dissertation. Barcelona, Spain: Universitat de Barcelona.

Barceló, R., Bujosa, J. M, Cañadilla, J. L., Fargas, M. & Font, V. (2002). *Matemàtiques 1*. Barcelona, Spain: Castellnou.

Bolite, J., Acevedo, J. & Font, V. (2006). Metaphors in mathematics classrooms: Analyzing the dynamic process of teaching and learning of graph functions. In M. Bosch (Ed.), *Proceedings of the Fourth Congress of the European Society for Research in Mathematics Education* (pp. 82-91). Barcelona, Spain: Universitat Ramon Llull.

Duval, R. (2008). Eight problems for a semiotic approach in mathematics education. In L. Radford, G. Schubring & F. Seeger (Eds.), *Semiotics in mathematics* *education: Epistemology, historicity, clasroom, and culture*. Rotterdam, (pp. 39-62) The Netherlands: Sense Publishers.

García, M. (2008). *Significados institucionales y personales del límite de una función en el proceso de instrucción de una clase de primero de bachillerato.* Unpublished Doctoral Dissertation. Jaén, Spain: Universidad de Jaén.

Godino, J. D., Batanero, C. & Font, V. (2007). The onto-semiotic approach to research in mathematics education. *ZDM-The International Journal on Mathematics Education, 39*(1-2), 127-135.

Johnson, M. (1987). *The Body in the mind: The bodily basis of meaning, imagination, and reason*. Chicago, IL: Chicago University Press.

Lakoff G. & Johnson, M. (1980). *Metaphors we live by*. Chicago, IL: Chicago University Press.

Lakoff, G. & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York, NY: Basic Books.

Núñez, R., Edwards, L. & Matos, J. F. (1999). Embodied cognition as grounding for situatedness and context in mathematics education. *Educational Studies in Mathematics, 39*, 45-65.

Peirce, C. S. (1978). *The Collected Papers of Charles Sanders Peirce.* Cambridge, MA: The Belknap Press of Harvard University Press.

Presmeg, N. C. (1997). Reasoning with metaphors and metonymies in mathematics learning'. In L. D. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 267-279). Mahwah, NJ: Lawrence Erlbaum Associates.

Pimm, D. (1981). Metaphor and analogy in mathematics. *For the Learning of Mathematics, 1* (3), 47-50.

Pimm, D. (1987). *Speaking mathematically*. New York: Routledge and Kegan, Paul.

Puig Adam, P. (1965). *Curso de geometría métrica. Tomo I. Fundamentos*, Madrid, Spain: Nuevas Gráficas.

Sriraman, B. & English, L. D. (2005). Theories of mathematics education: A global survey of theoretical frameworks/trends in mathematics education research. *Zentralblatt für Didaktik der Mathematik, 37*(6), 450-456.

Sfard, A. (2000). Steering (dis)course between metaphors and rigor: Using focal analysis to investigate an emergence of mathematical objects. *Journal for Research in Mathematics Education, 31*(3), 296–327