MATHEMATICAL REFLECTION IN PRIMARY SCHOOL EDUCATION
Theoretical Foundation and Empirical Analysis of a Case Study

Cordula Schülke & Heinz Steinbring,
Universität Duisburg–Essen, Campus Essen

Abstract. *The paper presents the theoretical construct "mathematical reflection" and elaborates its specificity with regard to the epistemological conditions of mathematical knowledge. This construct of "mathematical reflection" is the key concept in a wider research project. A conceptual grid with fundamental categories is developed that serves to carefully characterize the important components of "mathematical reflection" and that is used as an instrument for qualitatively analyzing students' mathematical collaboration in clinical interviews and for identifying different types of "mathematical reflection" in interaction.*

Key words: reflection, mathematical interaction, qualitative analysis, epistemology

1. INTRODUCTION: THE CENTRAL CONCEPT OF THE RESEARCH PROJECT – *MATHEMATICAL* REFLECTION

In several primary schools in Germany – also in North Rhine-Westphalia – teaching in grades 1 & 2 is organised comprehensively in the frame of experimental trials. It is assumed that "learning in grade-comprehensive groups [...][offers] a lot of opportunities of using the different learning potentials for the mutual stimulation and support for the students as a whole" (North Rhine-Westphalia State Ministry for School, Youth and Children 2004)

The research project presented here refers to age-mixed mathematics learning and is oriented on the paradigm of interpretative instruction research. On the basis of the interaction-theoretic perspective (developed by Bauersfeld 1994) and the specific research approach of social epistemology of mathematical knowledge (developed by Steinbring 2005), this project deals in particular with the socio-interactive learning of mathematics in grade-heterogeneous learning groups in the flexible entrance phase of elementary schooling. The analyses of mathematical interactions, elaborated in this project, refer in a complementary way to individual-psychological and social processes and at the same time to the particularity of mathematical knowledge as the object of the interaction.

The fundamental concept of the analyses attempts to theoretically capture the *reflective mathematical thinking* of the children. We proceed on the assumption that, by means of the collaboration of younger and older children on mathematical problems, particularly the older children receive manifold opportunities of reflecting *mathematically*. With his concept of *observed mathematics*, Freudenthal characterized the (reflective) moment of thinking, where mathematics carried out and used on a lower level becomes observed mathematics on a higher level (cf. Freudenthal 1978, 64). In addition, Nührenbörger and Pust (2006) pointed out that, in

the interaction with the younger children, the older children, already used to school, are challenged to "verbalize their own thoughts and insights. In this process, existing knowledge is reflected and newly organized before it is handed on to others, and becomes further differentiated during the explanation process. For the children who are already used to school, a possible retrospection onto a previous learning process opens up opportunities for reflection on the meta-level" (Nührenbörger/Pust 2006, 24).

But how can *reflective thinking* in *mathematical* interaction processes be identified and what can be understood by *reflective mathematical thinking* as a conceptual element of an epistemologically oriented interaction-theoretical point of view onto learning mathematics and the nature of mathematical knowledge?

An initial foundation of the concept of "reflection" took place on the basis of already existing descriptions of "reflection" within the existing research literature, particularly in (actual) mathematics education literature. The examination of the status of research clearly showed the necessity of a precision of the theoretical construct "*mathematical* reflection".

The elaboration of a broadened conceptual understanding of *mathematical* reflection is based on the (particular) epistemological nature and the conditions of the development of mathematical knowledge (cf. Steinbring 2005) as well as on the concept of reflection as a "change of standpoint", which Freudenthal has developed in his article "How does reflective thinking develop?": "The unfolding reflection shows different traits. One of them, I would like to call standpoint change – a mental standpoint change, where the standpoint itself can be local or mental, while the change can take place in space, time, or another, for instance mental, dimension" (Freudenthal 1983, 492).

Thus, by *mathematical* reflection, we understand a *cognitive activity, a process of thinking*, in the sense of a *change of standpoint* or *perspective*, on the basis of which *processes of re-interpretation* take place. Old, common mathematical knowledge and familiar ways of proceeding are thought through again intentionally, they are scrutinized and *newly* or *re-interpreted*. The construct "reflective mathematical thinking" corresponds with the epistemological character of mathematical knowledge as pattern-like, relational structures. With the assumption that stimulating reflective thinking aims at the development of mathematical knowledge, *mathematical* reflective thinking is not merely a repeated consideration, a remembrance, or a reference to familiar contents.

This specific characterization of *mathematical* reflection requires to take into consideration the following essential issues when trying to analyze whether one can observe within a mathematical interaction this kind of *mathematical* reflection. First, when analyzing a *change of standpoint* or *perspective* (in the sense of Freudenthal) within an observed mathematical interaction, we use the epistemological analysis and apply the epistemological triangle (see Steinbring 2005) to figure out whether one

can speak in a *proper epistemological* sense of a change of standpoint that introduces new mathematical relations or that generalizes mathematical relations. The second analysis instrument is the "analysis grid" that tries to characterize the specific *type* of change of standpoint; this basic instrument is developed in the following section.

2. A GRID FOR THE ANALYSIS OF *MATHEMATICAL* REFLECTIONS WITHIN INTERACTION PROCESSES

The analysis grid (see Fig. 1) is divided into four fields, labelled "trigger", "response", "reaction" and "reflective level" together with sub-categories. The two fields "trigger" and "reaction" are *descriptive* elements in the analysis grid, and the fields "reaction" and the central category of the "reflective level" are characterized as *interpretative* elements.

In an interaction sequence, the question to which extent a new or re-interpretation of a mathematical content on the basis of a standpoint change becomes apparent, can only be examined in an exclusively interpretative way. In the frame of a sequential analysis of the scope of possible interpretation hypotheses, the convincing possibilities of interpretation, which can be justified by the direct reference to the transcript, are elaborated.

<table>
<tr><td>descriptive element</td><td>interpretative element</td><td>descriptive element</td><td>interpretative element</td></tr>
<tr><td>Trigger</td><td>Response</td><td>Reaction</td><td>Reflective Level</td></tr>
<tr><td rowspan="2">Question Exercise Remark
(Interviewer /Teacher)</td><td rowspan="2">No Irritation</td><td>No Remark</td><td rowspan="2">Standpoint Change "foreign Perspective"</td></tr>
<tr><td>Imitation</td></tr>
<tr><td rowspan="2">Question Discovery Way of proceeding Remark
(Partner child)</td><td rowspan="3">Moment of Irritation</td><td>Recapitulation</td><td rowspan="2">Standpoint Change "Context"</td></tr>
<tr><td rowspan="2">Construction</td></tr>
<tr><td>Own way of proceeding/remark</td><td rowspan="2">Standpoint Change "Retrospection"</td></tr>
</table>

Fig. 1: Analysis grid

The allocation to the descriptive elements of the analysis grid is exclusively oriented on the linguistic format of a remark and has a purely *descriptive* character.

The grid serves for the purpose of being able to focus on the central research questions and it allows on the basis of an epistemological analysis to examine the interactive processes taking place during a partner interview in a purposive and careful way. Even if, at first sight, the analysis grid might seem to present a chronological sequence of the fields, it is expressly not the aim of the grid to simply be used for the description of a temporal sequence.

During the real interaction proceedings, different sub-categories can overlap. For instance, a mathematical remark, which on the basis of its linguistic format is allocated to the sub-category of recapitulation, can at the same time contain a hint

towards a moment of irritation. The following more detailed explanations of the categories and sub-categories will further clarify the analysis grid.

The different elements of the analysis grid

• *The element "trigger":* On a descriptive level, several possible triggers for reflection or thinking activities can be identified in the interviews. Examples: a question, a discovery or a way of proceeding can represent a triggering moment.

For the research it is important *which person* stimulates reflections. Is this rather true for the remarks by the interviewer, for one's own discoveries and ways of proceeding, or the remarks of a cooperating partner child? This relevant aspect is allowed for by the distinction of the three sub-categories.

• *The element "response":* A first central research question concerns the identification of possible clues in the analysis of interactive processes, which suggest reflective thinking. When does a question or a mathematical problem not only initiate recapitulation or imitation, but a *reflective* process?

The research results up to now show that irritation or a moment of surprise is an important indicator in this context. If, for example, an exercise cannot be done spontaneously, if one does not agree with the previous proceeding of the answer or with the ways of proceeding, ideas or remarks by another participant, and if one shows irritation or surprise, that means that it is not possible to simply resort to common knowledge or familiar ways of proceeding. An irritating exercise can challenge to engage in a foreign perspective.

• *The element "reaction" (descriptive element):* Children can react differently to the different triggers. In this regard, we distinguish between the sub-categories "no remark", "imitation", "recapitulation" and "construction".

Besides "not remarking", a possible reaction is "imitation", which means the literal repetition of one's own or someone else's remarks or the direct imitation of familiar ways of proceeding or the partner child's strategies.

By "recapitulation", we understand resorting to knowledge or ways of proceeding already familiar from the previous context, or the reference to remarks and strategies of a partner child in one's own words.

If the children also refer to mathematical knowledge, which had not been introduced by any of the interaction participants in the previous contexts, the category of "construction" is fulfilled.

The allocation of the children's reaction to one of the given categories takes place depending on the format of the remark and is oriented on the linguistic elements used, on a purely descriptive level.

If the children only refer to common knowledge or familiar ways of proceeding in phases of cooperation, the interaction remains on the level of reaction. But if new or

re-interpretations of old knowledge or new constructions take place, the level of "mathematical reflection" is addressed as well.

• "*Reflective level" (interpretative element)*: The question whether new or re-interpretations are carried out within interactions or if new mathematical knowledge is constructed, can only be examined interpretatively. In order to do so, the epistemological triangle (Steinbring 2005) is used in the analysis.

The identification of the standpoint changes, which might follow, takes place with the help of the developed characteristics and features of differentiation.

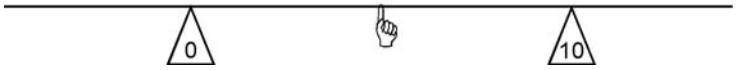
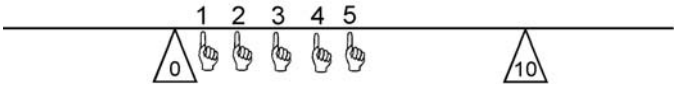
Three levels of changes of standpoint or perspective: The point of view developed by Freudenthal about reflective thinking as a standpoint or perspective change made it possible to characterize and distinguish three different forms of possible standpoint changes from the data material. Besides the theoretical clarification of the concept *mathematical* reflection, these represent an essential result of this research.

An important feature of the three levels of standpoint changes consists in the new or re-interpretation of a mathematical exercise, a mathematical content or a mathematical sign / symbol. A distinction is made with regard to the different possibilities or ways of changing one's own standpoint.

- Standpoint change "foreign perspective": The children take a foreign perspective, someone else's standpoint, for instance they relate the ways of proceedings, discoveries and views of their partner child to their own points of view and ways of proceeding, test and evaluate these and are stimulated to newly or re-interpret their own mathematical knowledge.
- Standpoint change "context": A mathematical challenge is put into and observed within another context and thus is subject to a new or re-interpretation. In contrast to the standpoint change "foreign perspective", no concrete possibility of interpretation is given, which then might be followed, but rather the change of context *allows for* a new point of view. If, by means of such a context change, one of the participants develops a new interpretation perspective, a mathematical reflection according to the standpoint change "context" has taken place.
- Standpoint change "retrospection": If there is an *intentional* resort to common knowledge and familiar ways of proceeding from a previous context in order to thus new or re-interpret a mathematical content, a standpoint change "retrospection" has taken place. Such a standpoint change can only be spoken of if a remark by a participant presents a way of proceeding or a mathematical context as familiar and relates this with the current exercise.

3. Analysis of an Exemplary episode: Gina & Sharon discuss a "Number line"–Problem

1	Int	(*places the number cards 0 and 10 at the number line*) I am placing the number cards at the number line, …
2	S	[incomprehensible]
3	Int	Put this card, (*places the number card "5" between Gina and Sharon onto the table*)

4	G	*(Gina takes the number card "5" with her left hand)*
5	Int	at the number line.
6	G	# (*leans forward / holds the number card "5" with both hands / looks at the number line*)
7	S	# In fact the zero belongs in front # (*places her left hand onto the left end of the number line*)
8	G	# (*holds the number card in her right hand / looks at Sharon"s left hand*)
9	S	or shall we now, well shall that now be like that the number line begins with this? (*puts the edge of her left hand on the left of the number card zero on the table*)
10	Int	Think about it together, how you can do that now.
11	S	You now certainly have (*looks at I.*) well. (…) (*turns to Gina*) She probably has chosen such a place (*points over the section of the number line which is marked by the number cards 0 and 10 / Gina looks at the number line*) where one could add that, so that we well that this, that this is supposed to be the beginning (*places the edge of her right hand to the left of the number card "0" on the table*) that this piece is then practically gone, (*moves her right hand in the direction of the left edge of the table over the number line*) in your mind, right? (*looks at Gina / Gina continues to look at the number line*) Well such a place, then the five would go here, right? (*puts a finger between the numbers 0 and 10 onto the number line, see below. / looks at Gina / Gina continues to look at the number line*) (…)
		
		because one two three four five. (*while counting the numbers, she points at the spots marked in the diagram, see below*)
		

This short episode originates from an interview about the topic "number line", which was conducted with Sharon and her classmate Gina in the second project year. For Sharon, this was the fifth interview during the research project, for Gina it was the first.

Before the children were introduced to the number line, which they had never used as means of visualisation. This scene of positioning of "5" takes 5 minutes.

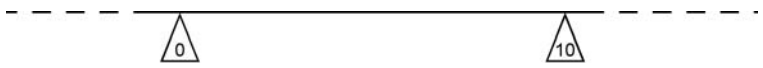


Fig. 2: Section of the number line

On the children's desk, a string was attached as a number line. The interviewer had positioned the "0" and "10" (cf. Fig. 2) when asking the exercise question.

Analysis of the interview sequence

The exercise is opened by the interviewer. She positions the "0" and "10" thus providing the initial situation. This *task of the interviewer* is emphasised by the remark ("I am placing the number cards at the number line" (1)).

Sharon directly reacts to this action or remark (2). Maybe she already shows a first reaction to the positioning of the number cards. As Sharon's remark is incomprehensible, therefore this guess cannot clarified definitively.

Gina immediately takes up the number card "5" and at the same time watches the number line (4, 6). While doing this, she shows that she is engaging with the exercise question and is considering where to put the number card "5".

Sharon exclusively refers to the current position of the number card “0” and wonders about the position of the “0” and “10” at the number line in her following remarks (7, 9, 11).

Sharon's remarks are of essential importance for the central research question and the identification of reflective mathematical thinking and thus represent the main focus and the starting point of the following interpretations. The *clarification of the position of the number card “0”* as an element of the number line (by Sharon) is at the centre of analysis.

In her first remark after the exercise question, Sharon points at the left end of the number line and explains that the “0” should be placed directly at the beginning of the number line (7 “In fact the zero belongs in front”). The positioning of the “0” by the interviewer does not correspond with her idea of the “correct position”. Her remark suggests that, according to her previous point of view, the position of the “0” on the number line is fixed and cannot be chosen freely.

The possible previous consideration of changing the position of the number card in the frame of the work on the exercise can be seen in particular in remark (9) “or shall we now, well shall that now be like that the number line begins with this?”. This is supported by the use of the words “in fact”, which underlines the discrepancy between the current and Sharon's “correct positioning” of the “0”.

The interviewer gives the question raised by Sharon back to the two students (10 “Think about it together, how you can do that now.”).

Sharon's remark (11) suggests that she now assumes an intentional positioning of the “0” by the interviewer and is challenged to find an explanation for the “unusual position” of the number card at the number line (“She probably has chosen such a place where one could add that, so that we well that this, that this is supposed to be the beginning”).

Applying the analysis grid “mathematical reflection” to the episode

The element trigger: The exercise question given by the interviewer as well as the given positioning of the number cards 0 and 10 at the number line (1, 3, 5) represents the trigger for the following cognitive activities by the two students.

The element response: Sharon makes a remark about the current position of the number card “0” at the number line directly after the explanation of the exercise question by the interviewer. The position of the number card does not correspond with her idea and she is probably surprised or irritated by the interviewer's way of proceeding. A clue for a possible moment of irritation becomes apparent in her remark (7): “In fact the zero belongs in front”. Sharon points out an alternative possibility of positioning the number card. Her remark “In fact” can be seen as an indicator for her not agreeing with the current position of the number card.

The element reaction: In her reaction to the triggering moment, which is the exercise question and the localization of the section of the number line to be observed, Sharon

refers to the positioning of the number card “0” and discusses this (not verbally expressed) action of the interviewer with her own words. Thus Sharon's reaction can be allocated to the sub-category recapitulation.

The levels of mathematical reflection

The question to which extent Sharon performs a change of view and carries out a new or re-orientation of her mathematical knowledge regarding the positioning of the “0” at the number line is examined with the epistemological triangle (Steinbring 2005) as an analysis instrument of relations between signs, reference contexts and concept.

If a change of standpoint or perspective can be identified, this will be allocated to one of the three levels of mathematical reflection on the basis of the characteristics described in the presentation of the analysis grid.

The analysis instrument “epistemological triangle”

Conventional interpretation: The *sign* to be clarified in the present interview sequence is the position of the number card “0” at the number line. In this first representation the original, conventional interpretation by Sharon regarding the position of the number card is made clear by referring to a familiar *reference context*.

In her remark (7) “In fact the zero belongs in front”, Sharon probably refers to the known “familiar” position of the number card “0” at the beginning of the number line. Maybe she remembers the positioning carried out previously to the interview and points at the left end of the number line as the only possible position for the number card up until this point. Two different aspects become manifest in her remarks. On the one hand, there seems to be a fixed position for the number card at the number line for Sharon, on the other hand the number card “0” belongs to the beginning of the number line, i. e. left of this number, neither does the number line continue nor can there be further number cards.

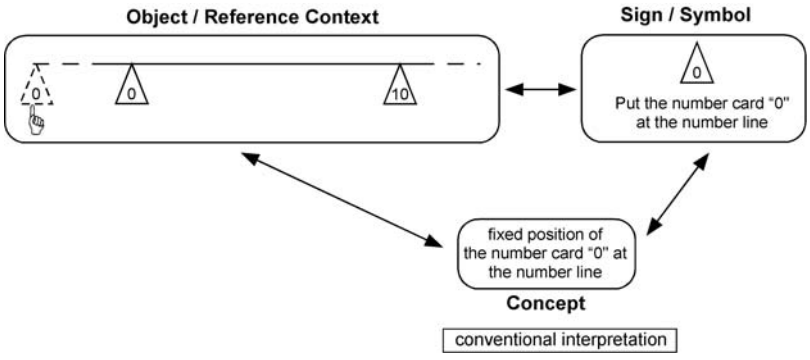


Fig. 3: Epistemological triangle: The original interpretation of the position of the number card “0”

Beginning of a relational interpretation: Besides the originally conventional view concerning the position of the number card “0”, a beginning mentally more flexible interpretation becomes apparent in this scene. Sharon tries to conciliate her previous point of view with the current position of the number card. In doing so, she refers to the reference context presented in Fig. 4. She explains the – for her point of view – still unfamiliar position of the number card “0” by placing her hand to the left of the

number card and remarking in the one hand (9): “or shall we now, well shall that now be like that the number line begins with this?”, on the other hand (11): (“She probably has chosen such a place where one could add that, so that we well that this, that this is supposed to be the beginning”.

The mentally changed number line thus forms the reference context, i. e. the current position of the number card is interpreted by referring to the theoretical picture of the number line, which Sharon has developed and in which the sequence in front of the number line is mentally ignored.

In this interaction of sign and reference context the beginning of a detachment from a purely empirical point of view concentrated on the concrete, towards a stronger mental use and change of the number line becomes apparent. The following remarks by Sharon can serve as concrete indicators of this more flexible point of view “in your mind” (11) and “would” (11: “then the five would go here, right?”). The positioning of the “5” which she suggests takes place depending on the current position of the “0” and “10”.

While at the beginning of the interview sequence Sharon still allocates a fixed position at the beginning of the number line to the number card “0”, she ultimately takes a more flexible point of view about this: By means of the possibility of putting the number card “0” at a random position of the number line, sections of the number line can be realized variably.

Still, the number card “0” remains the first card for Sharon, however, thus left of this number card there can be no other number cards. Furthermore, her way of proceeding when positioning the number card “5” (11) indicates that she continues to pay attention to the sequence and distance of the number cards.

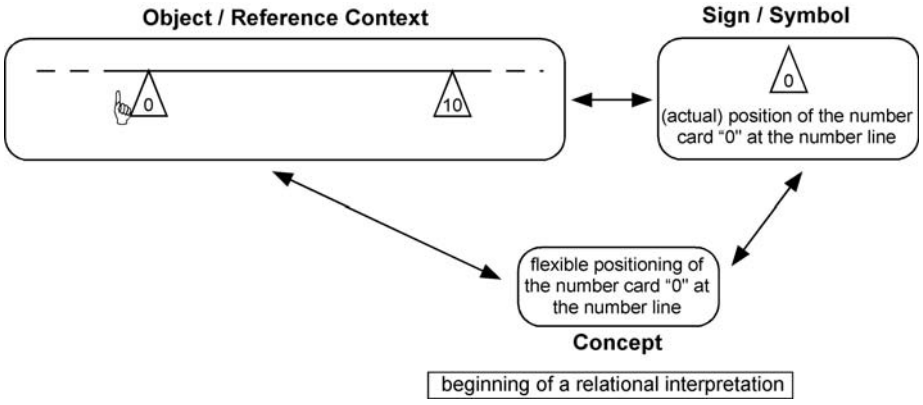


Fig. 4: Epistemological triangle: Beginning of a relational interpretation

Characterization of the standpoint change

As has already become clear in the first step of the analysis, Sharon performs a new or re-interpretation of the number line regarding the positioning of the number card “0” during the course of the interaction.

As previously to the present interview sequence, the number card was always placed at the beginning of the number line, its current position represents a *changed context* in this regard.

Concerning the position of the number card "0", Sharon develops a new interpretation perspective and thus carries out a *standpoint change "context"* on the basis of this changed context given by the interviewer.

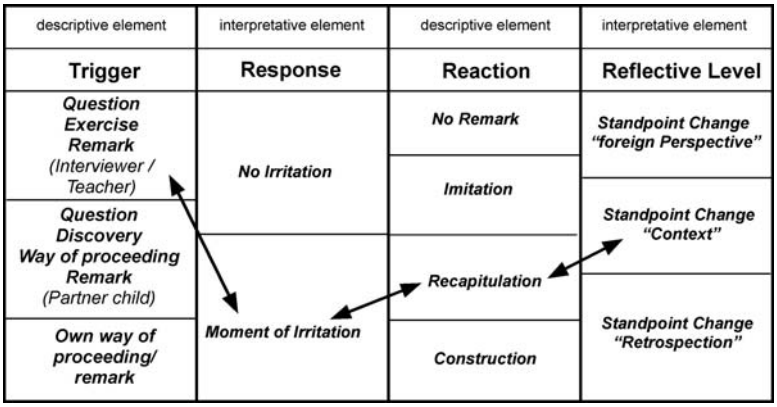


Fig. 5: Application of the analysis grid

4 SHORT RÉSUMÉ

The analysis grid developed in the course of the research project offers the possibility of presenting the results of the analyses and interpretations cohesively. The central element of the grid is the "reflective level". The distinction of the three categories of standpoint changes is a fundamental result of the research up until now and allows for the analysis to pursue the question which specific form of a standpoint change provokes and stimulates the process of new interpretation of mathematical knowledge, which is essential for the learning of mathematics.

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