IN SEARCH OF ELEMENTS FOR A COMPETENCE MODEL IN SOLID GEOMETRY TEACHING. ESTABLISHMENT OF RELATIONSHIPS

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ABSTRACT

In this paper we present part of the analysis of a Teaching Model for the geometry of solids of an initial Education Plan for elementary school teachers, and its implementation in the University School of Teaching of the Universitat de València, Spain. We have focused our attention on how the establishment of relationships among geometric concepts have been worked on. For this analysis we considered theoretical contents related to geometric contents (concepts, mathematical processes and different types of relationships). This study is part of a more extensive work that tried to elaborate the competent conduct features for a teacher teaching solid geometry in elementary school.

PRESENTATION

This work is part of a more extensive research project which uses as a methodological framework the theory of the "Modelos Teóricos Locales" (MTL) (Local Theoretical Models) (Filloy, 1999). According to Filloy and col. (1999), to be able to take into account the complexity of the phenomena that take place in the educative systems, the MTL incorporate several interrelated theoretical components: 1) Competence Model; 2) Teaching Model; 3) Cognitive Processes Model, and 4) Communication Processes Model. Our work is focused on the first component in relation with the training process of elementary school teachers in the subject of solid geometry.

De Ponte and Chapman (2006) point out that this research line has given priority to the analysis of teachers knowledge or practice paying less attention to the analysis of the programs for their training. In our work we analyze a solid geometry training Program for elementary school teachers and its putting into practice; we want to establish some elements for the Initial Competence Model (ICM) in relation with the training of elementary education teachers in the geometry of solids. In previous papers we have presented elements of this competence model that show a competent conduct for teaching mathematical processes related with describing, classifying, generalizing and particularizing. In the present paper we focus on the elements related to the establishment of relationships among geometrical contents.

BACKGROUND AND FRAMEWORK

The analysis we present in this paper is part of a more extensive work - González (2006)[1]-, which had the purpose of elaborating the elements for an ICM that can be used as a reference to interpret the teaching models proposed for teaching solid geometry in training programs for elementary school teachers. This work belongs to a project that aimed for the creation of a "Virtual Library"[2] that could help to teachers' permanent education.

In previous works (González and col. 2006, 2008; González, E. and Guillén, G. 2008) we have presented some results of the analysis. To group these results we have followed the distinction made by Climent and Carrillo (2003), who take into consideration teacher's knowledge and distinguish as different components the mathematical content knowledge (in our case contents *of* and *about* geometry) and the knowledge of the subject for its teaching.

In previous papers above mentioned we refer to results that have to do with the contents of "solid geometry" related to mathematical processes of classifying, describing, generalizing, and particularizing. We show how the attempt of organizing the surrounding objects and their construction, by means of different procedures, provides very rich contexts to develop these mathematical processes. We also present some of the reflections encouraged by the teacher concerning the learning process of both children and teachers, questions having to do with preparing the lesson, are related to the use of language, or the way to respond to the appearance of misconceptions.

The observations we present in this paper belong to the first group of contents *of* and *about* geometry, and complete the study; these observations refer to relationships among geometric objects of the same and different dimension; that is, relationships among solids, among their elements or among plane and space elements.

As we advanced in the presentation, we follow the Theory of the MTL as experimental methodological framework. We have commented that in this Theory four interrelated theoretical components can be distinguished. What differences each component from the others is, among others, the phenomena taken into account

[1] Work carried out to obtain the "Diploma de Estudios Avanzados" (Certificate of Avanced Studies) of the PhD program of Mathematics Education. Universitat de València, Spain.

[2] Project "Procesos de transferencia de resultados de investigación al aula: el caso del bajo rendimiento escolar en matemáticas". Research project, co-financed by the Consejo Nacional de Ciencia y Tecnología (CONACYT-G37301-S) (Nacional Council for Science and Technology). México.

http://www.pernodis.com/ptria/index.htm. In the site dedicated to geometry, section *"Descubrir y matematizar a partir del mundo de las formas"*, chapter *¿Cómo enseñan otros?* we present extracts of the class sessions with the corresponding analysis (http://hipatia.matedu.cinvestav.mx/~descubrirymat/).

regarding the concept subject of analysis. In this work in particular, the ICM includes elements of the knowledge of an ideal person, capable of carrying out tasks related to the teaching of solid geometry at elementary school level. This is, it includes the elements which should be part of the competent conduct of elementary school teachers when teaching the geometric topics regarding solids in their classes. We have already pointed out that the elements commented in this work refer to the establishment of relationships among geometric contents.

When we focus on solids, our theoretical framework is based on the studies made in Didactics of solid geometry (Guillén, 1991, 1997; Guillén and Figueras, 2005), we continue reorganizing these contents as referred to: a) geometric concepts, b) mathematical processes (to analyze, to describe, to classify, to generalize, etc.), c) relations among geometric contents. When we studied how these geometric contents were taught, we also paid attention on how the skills are used (to construct, to modify, to transform) to work the mathematical processes indicated or to develop skills (to communicate and/or to represent forms). The reorganization of the school contents has leaded to organize the observations as related to the teaching/learning of concepts, of mathematical processes, or of the establishment of relationships among different geometric contents. The observations made are detailed in Guillén (1991, 1997). These works take into account, on the one hand, relationships among solids and/or families of solids. These refer to inscription and duality relationships among families of solids, to composition or decomposition relationships, or to inclusion, exclusion or overlapping relationships among different classes established with different classification types (hierarchic partitions or classifications) taking into account several universes and criteria for classifying. On the other hand, we stand out the relationships among the solids elements that can be either of parallelism and perpendicularity or numerical relationships among them. Also were taken into account the relationships among geometric contents of several dimensions that emerge when solids truncate or during the construction of models parting from a plane surface. Moreover, attention has been paid to the establishment of relationships by analogy. In the work of González and Guillén (2006) the inclusion, exclusion or overlapping relationships among families of solids were studied. The rest of types of relationships are the ones that have been taken as reference to organize the observations that this report presents.

The studies above mentioned have been developed taking as a reference the works of Freudenthal (1973, 1983) and others, that have been carried out at the Freudenthal Institute (for example Treffers 1987). These works are the theoretical basis for our concepts over geometry and its teaching, over the relationships among the different geometric contents, and also provides us with information to organize the solids geometry teaching. In this framework one of the aims of geometry teaching is the development of mathematization through mathematical practice.

To carry out the analysis we have also taken as a reference other studies about the appropriate contents for the teachers training plans, emphasizing on the different contents that should be discussed on a reflective level (Shulman, 1986; Climent y Carrillo, 2003; De Ponte y Chapman, 2006; González et. al. 2006).

DATA COLLECTION AND ANALYSIS

To create the MCI, we analyzed the available literature related to the mathematical content analysis and observation of the learning process for mathematical processes and the literature related to teachers' education, this enabled us to elaborate the Theoretical Framework of the work and define the criteria used to analyze the design and implementation of a Teaching Model of the teacher of Teaching with an extensive experience in introducing to the study of geometry having as a support solids geometry.

The work has been developed in several stages. In the first one, we examine theoretical works of the research lines we mentioned in the previous section and the teachers' training plan of the teacher who constitutes the study scope of our work (Guillen, 2000). In a second stage we analyzed the implementation of this training plan.

The data for this experimental study was obtained during the 2005-2006 school year. We attended and took notes of 22 class sessions the training teacher dedicated to solid geometry during the course she gave to a group of students belonging to the foreign language specialty at the University School of Teaching of the Universitat de València (Spain). Each session lasted 50 minutes approximately.

To control all the information that emerged during the teaching, the sessions were recorded in video and audio. These recordings were transcribed and from them, together with the notes taken during the classes, were obtained the extracts to carry out the analysis. These were considered the essential element and were defined taking as a reference the theoretical analyses performed during the first stage. They could be a sentence or a set of sentences that not necessarily had to match the answers or individual interventions of the teacher or of the students.

These extracts were organized in groups as it follows: i) On geometry and its teaching. Student and teacher; ii) On geometric contents; iii) How do some of those students learn? What for?; iv) The class planning; v) Interacting in the class and ... vi) What about language? In Gonzalez et al. (2006) we briefly detail observations related to each of them.

The school contents organization we carried out, mentioned in the previous section, show the distinction we made in the observations we included in group ii). We separated them as follows: ii.1) relative to concepts learning; ii.2) relative to mathematical processes; ii.3) relative to the establishment of relationships. We have already mentioned that in the following section we will refer to group ii.3).

To analyze the corresponding extracts for the establishment of relationships we used, on one hand, the diagram presented by Olvera (2007) and showed in figure 1. This diagram was constructed starting from the characteristics of Van Hiele levels for solid geometry determined in the study by Guillén (1997). On the other hand, in its organization the families of solids and polygons implicated and the relations among flat geometric objects and space geometric objects were taken into account. Also different representations of the solids used as a context were considered and numerical relations were also underlined.

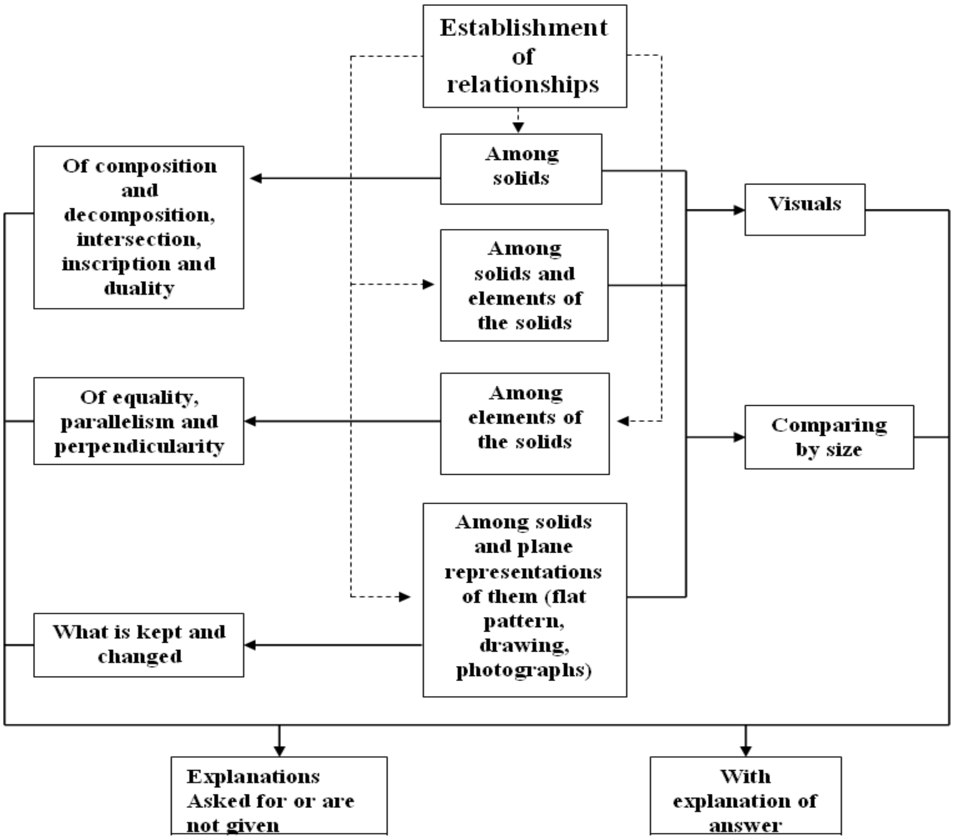


Figure 1

IMPLEMENTATION OF A TEACHING MODEL FOR SOLID GEOMETRY. OBSERVATIONS RELATED TO THE RELATIONSHIPS AMONG THE GEOMETRIC CONTENTS

In Figure 1 we show how the observations of relationships among geometric contents during the implementation of the analyzed training plan are grouped. Following, we present some examples.

Establishment of relationships

The observations that we present in this section have been organized taking into account, on the one hand, the solid families used as a support to develop the activity. On the other one, that the context can also consist of the different representations of solids. It is also necessary to take into account that the relations established could also be numerical.

1. Relations of inscription and duality among regular polyhedrons. When numerical relations are exposed in a table as shown in Figure 2, in which the number of faces, vertexes, edges, order of the vertexes and number of sides of the polygons of the faces have been registered, it leads to the establishment of a wide variety of relationships.

For example, it comes to express that the number of faces of the dodecahedron is equal to the number of vertexes of the icosahedron; or that the number of vertexes of the octahedron is equal to the number of faces in the cube. From this type of relationships, it can be concluded that some polyhedrons can be inscribed in others. For example, the cube can be inscribed in a octahedron in such a way that the vertexes of the cube are in the center of the faces of the octahedron, or vice versa.

	C	V	A	O de V	Forma de C
Tetraedro	4	4	6	3	3
Cubo	6	8	12	3	4
Octaedro	8	6	12	4	3
Dodecaedro	12	20	30	3	5
Icosaedro	20	12	30	5	3

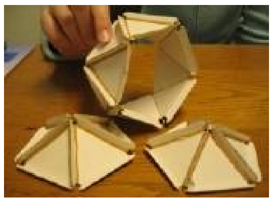
Figure 2

Octahedron inscribed in the cube

There are also relations established among elements of the dual regular polyhedrons when instead of considering models of pairs of dual regular polyhedrons inscribed, compound models are considered, which are intersections of pairs of dual polyhedrons. For example, the cube and the octahedron.

After encouraging students to imagine in a dynamic way how to pass from the inscribed model to the compound model when the size of the inscribed polyhedron is increased, the attention is focused on the fact that the edges of both polyhedrons cut perpendicularly at their midpoint.

2. Relations among regular polyhedrons and other solid families. When trying to analyze regular polyhedrons, they have repeatedly been studied in relation to other families. For example, in the analysis of the icosahedron it is emphasized that it can be seen, on the one hand, as the composition of two pentagonal pyramids of regular faces and a pentagonal antiprism of regular faces or as the fitting of two caps that correspond, each of them, to a pentagonal bipyramid of regular faces, in which one of the pyramids has been opened.



3. Cylinders and Prisms. Cones and pyramids. Immersed in the situation of generating models with different procedures, in first place, the family of straight prisms was introduced through the truncation of a straight cylinder.

For example, questions raise such as: What form do we obtain if we cut perpendicularly the base? How many cuts, perpendicular to the base, should be done for the circle of the base to turn into a 5-sided polygon? What does the cylindrical surface turn into? How are the cylinders obtained with parallel to the base cuts? Can we also obtain oblique prisms? And this problem extends to the establishment of relations between cones and pyramids.

Likewise, comparisons among naive ideas and properties of both families are established. For example, it is pointed out that with parallel cuts to the bases in both families (cylinders and prisms) the shape of the sections is maintained (same form of the bases), and these cuts divide the corresponding solid into other solids with the same form, with the same bases as the original one; and, when adding the corresponding heights, the original solid height is obtained. Immersed in this matter, it is concluded that some prisms can be inscribed in cylinders raising the question of which polygons can be inscribed in a circumference?

4. Comparing cylinders and cones. Prisms and pyramids. When considering a dynamic transformation of one family into another, this transformation is profited to establish relations among the elements of the families of implied solids. For example, when the attention is focused on the transition from a prism into a pyramid, one of the bases of the prism is reduced to a point in the pyramid and it results in the transformation of the lateral faces of the prism into triangles, or that the number of faces in prisms is reduced by one in the number of faces of pyramids, etc.

5. Families of solids and flat shapes. When we focus on counting the elements of regular polyhedrons paying attention to their layout in space, relations are established among this layout and the form of the cuts sections equidistant from opposite faces, vertices or edges. The study is completed with the determination of the different types of planes of symmetry and axes of rotation of each regular polyhedron and the number of planes and axes of each type.

In a context of truncation in cylinders, cones, spheres, prisms and pyramids the relations among the direction of the cut and the form of the sections are established. The process is also considered in a dynamic way; that is, it starts with the observation of a section shape and this is compared with the other sections obtained by parallel cuttings done to the original.

6. Different representations of the solids as a starting point. This situation enables setting relations among different representations or among the corresponding models and their representation. For example, when disassembling the straight cylinder model, the cylinder edges are related to the sides of the rectangle in the flat pattern, and to the length of the circumferences of the bases.

When comparing a model with its flat pattern, problems arise such as the following: To which vertex of the model corresponds a given vertex of its flat pattern? Observing the flat pattern of a cube, can we know the number of faces? Observing at the flat pattern of a solid, can we know the number of faces? How many cuts do we need to make to a model to obtain the flat pattern? Which sides of the flat pattern form an edge in the model?

In order to work on the establishment of relations among the different representations the teacher compares the model properties maintained and the properties that “are broken” in each of them. For example, in a perspective representation of a cylinder, the property of bases being circles is “broken”, or in a perspective representation of the cube, the property of all edges being equal and all angles being equal is “broken”, property that does show on the corresponding flat patterns.

7. Numerical relations. These types of relationships are studied in several contexts. For example, when finding the numerical characteristics of the prisms, we obtain certain relations such as: the number of edges of a n-agonal prism is equal to 4 times the number of lateral faces plus 2 times the number of sides of the polygon of the base; for regular polyhedrons: the number of edges (sides of polygons of the faces) is equal to number of polygons of the sides of faces multiplied by the number of faces and divided into two.

CONCLUSIONS

In Gonzalez et. al. (2006; 2008) we already pointed out that solids constitute a very important context for the development of mathematical activity and we have presented some features that characterize a competent conduct to teach solid geometry in primary school. These results complement those deduced from observations that we will refer to in the following paragraphs. To introduce the study of geometry in primary school, the competent conduct implies putting into practice the different contents recommended in a training plan for teachers related to the establishment of relationships among geometric contents:

- The use of different contexts with all the possibilities they offer for the establishment of relations among geometric contents of the same and different dimension.

- The establishment of relations among geometric contents of one, tow and three dimensions.

- To emphasize about the multitude of relations among geometric contents. For example, those that arise when considering different solids families and/or their elements: i) cylinders and prisms, cones and pyramids; ii) some polyhedra families (prisms, pyramids); iii) solids families and flat figures, etc; iv) regular polyhedrons and other solids families; v) relations of inscription and duality among regular polyhedrons.

- To work on the transformation of some solid families into others with different objectives, such as: i) focusing attention on seeing them in a more dynamic way; ii) discovering the properties maintained and lost along the transformation; iii) discovering new knowledge; iv) using knowledge that we already have in order to discover new; v) working on the same geometric content in different contexts and times.

- To present the contents regarding the subject knowledge for its teaching without overlooking the contents of the subject itself. For example, to propose different questions with the intention of generating mathematical activity, emphasizing on the relations expressed and paying attention to the type of language used for this purpose; the use of different materials, diagrams and tables with the aim of facilitating the discovery and verbalization with a each time more specific geometric language of the relationships that arise.

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