3D GEOMETRY AND LEARNING OF MATHEMATICAL REASONING

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Teaching mathematical proof is a great issue of mathematics education, and geometry is a traditional context for it. Nevertheless, especially in plane geometry, the students often focus on the drawings. As they can see results, they don't need to use neither axiomatic geometry nor formal proof.

In this thesis work, we tried to analyse how space geometry situations could incite students to use axiomatic geometry. Using Duval's distinctions between iconic and non-iconic visualization, we will discuss here of the potentialities of situations based on a 3D dynamic geometry software, and show a few experimental results.

In mathematics education, resolving geometry problems is a usual way of teaching *mathematical proof*, and plane geometry is mainly used.

Nevertheless the students often focus on the properties of *drawings* — which are physical objects — instead of *figures* — the theoretical ones. In this case they may solve geometry problems by using empirical solutions, based on their own action on the drawing: *One can read the property on the drawing.* That is why using drawings as regards plane geometry is very confusing for many of them: since they are able to *see* results on the drawings, since they can work easily on it, mathematical proof seems to be useless, and may appear as a didactical contract effect (Parzysz, 2006).

On the contrary, in space geometry, it seems to be much harder for them to be certain of a visual noticing, and they may need new tools to study representations and to solve problems.

Our hypothesis is that it is possible, with specific situations, to make the students use tools concerning theoretical objects: working on figures, using geometrical properties… In order to control these new tools, mathematical proof is a very useful process the students can use to solve problems. This is why we assume that 3D geometry could be very helpful for proof teaching.

Nevertheless, formal proof is a complex process that not only involves hypothetico – deductive reasoning, but also (for instance) specific formal rules (Balacheff, 1999) we will not study here. Therefore, we will only focus in this paper on the first hypothesis we mentioned.

We will present here a preliminary study in order to illustrate and test our theoretical hypothesis.

THEORETICAL FRAMEWORK

Resolving problems of geometry

As it is said in Parzysz (2006):

> The resolution of a problem of elementary geometry consists of the successive working with G1 and G2, focusing on the "figure". The figure has a central part in the process: even if it is very helpful in order to make conjectures, it may be an obstacle to the demonstrating process, as the pupils don't know how to use data because of the "obviousness of the visual phenomenon".

Parzysz refers to Houdement&Kuzniak's geometrical paradigms, in so far as G1 is a "natural geometry" — where geometry and reality are merged — and G2 is a "natural axiomatic geometry", an axiomatic model of the reality, based on hypothetico-deductive rules (Houdement, Kuzniak, 2006).

As we can see, demonstrating is really meaningful when working with both G1 and G2, but the sensitive experience may encourage the pupils to work only with G1. In order to describe more precisely what can be this *sensitive experience*, and the ways it is related to using — or not — G2, we chose to use the distinctions that Duval (2005) makes between the different functions of the drawing, and the different ways of seeing it.

A first way of using representations is the *iconic visualization*: in this case the drawing is a true physical object, and its shape is a graphic icon that cannot be modified. All its properties are related to this shape, and so it seems to be very difficult to work on the constitutive parts of it — such as points, lines, etc. Then, the drawing does not represent the object that is studied, it *is* this object, and the results of geometrical activities inform on physical properties.

The other way is the *non-iconic visualization*, where the figure is analysed as a theoretical object represented by the drawing, using three main processes:

Instrumental deconstruction: in order to find how to build the representation with given instruments.

Heuristic breaking down of the shapes: the shape is split up into subparts, as if it was a puzzle.

Dimensional deconstruction: the figure is broken down into *figural units* — lower dimension units that figures are composed of —, and the links between these units are the geometrical properties. It is an axiomatic reconstruction of the figures, based on hypothetico-deductive reasoning.

These different possible ways of using the drawings lead us to two important consequences.

On the one hand, using G2 makes no sense with only iconic visualization, as geometry problems concern nothing but the drawings to the student's eyes.

On the other hand, carrying out the dimensional deconstruction means isolating subparts of the drawing and, at the same time, describing how these subparts are linked: this last part has no sense when using only G1. Therefore this operation implies a more axiomatical point of view, and the figure — described by the dimensional deconstruction — is likely to be used.

Finally, we assume that dimensional deconstruction would become an efficient tool if the iconic visualization weren't reliable any longer, as the pupil would have to make up for the lack of information in order to solve geometry problems. Using graphic representations is much more complex in space geometry, and then it seems to be an appropriate environment for the teaching of axiomatic geometry.

3D geometry

Using physical representations is very different in space geometry: there are various ways of representing figures, such as models or plane projections, and each kind of representation has specific properties and constraints. As the physical models are too restricting — for instance, adding new lines is generally impossible, and constructing models takes much time —, cavalier perspective representations are generally used. Then, visual information is no longer reliable: for instance, it is impossible to know whether two lines intersect or not, or whether a point is on a plane, without further information.

So in space geometry iconic visualization fails, and it is necessary to analyse the drawings in other ways. The problem is that using drawings is generally too difficult for the pupils. Chaachoua (1997) mentions that this involves the students' interpretation, based on their mathematical and cultural knowledge. They have to break down the drawing into various components, so that they can imagine the shape of the object. In fact, they would have to carry out dimensional deconstruction *before* any visual exploration. Therefore they are unable to understand that iconic visualization is not sufficient to solve geometry problems, as they only think that they *see* nothing.

Using 3D geometry computer environments may balance these difficulties, since the students could get more visual information, for instance by using various viewpoints as if the representations were models. It has to be noticed that, even in this kind of environment, visual information is usually not reliable, so that iconic visualization remains inadequate to solve geometry problems.

Hypothesis about Cabri 3D

With Cabri 3D, the user can watch the representation as if they were models. It is possible to adjust viewing angles by turning around the scene, to look at the drawing from various viewpoints, and then to be more easily conscious of the visual issues. For instance, it becomes possible to see that a point belongs to a plan, when the point *visually* belongs to it. Actually the user can get visual information to determine the shape and some properties of the figures, but generally this information is not sufficient to carry out geometrical works. For instance, as the representations are not infinite in Cabri 3D, two secant lines could have no intersection point on the screen, then it would be impossible to determine visually whether these lines are secant or not. Some operations are almost impossible too, like moving a point to reach a given line with no other tools than visual perception.

Then, the feedback from a Cabri 3D - based *milieu* — as described in Brousseau (1997) — may emphasize that, even if visual information is available, this information is partial. A Cabri3D drawing does not permit to *see* all the specificities of the object the student has to study – which is clearly not the drawing itself.

It seems that a problem any student would have to deal with, when using Cabri3D, is "How can I get information from the drawing, and how may I use it in order to deduce information I cannot see, and solve geometry problems?". We showed that there are two main kinds of answers: the iconic visualization based ones, and the non-iconic visualization based ones.

Our first hypothesis is that with Cabri 3D it is much easier for the students to get information about the drawings, and then to start a research process, even if they only use iconic visualization. This research process may evolve because of the dynamic geometry software properties of Cabri3D.

Cabri 3D not only produces representations, it is a dynamic geometry software. In this way it is possible to use *hard* geometric constructions: these drawings are based on geometric properties, and keep it when the user drags a part of it. As an example, a hard square remains to be a square — with different size — when one of its vertexes is dragged. Therefore, the students may assume that the reason of simultaneous movements of figural units is the relation between them: if a point moves when another one is dragged, it may seem that they are *linked*, in a way that has to be elucidated by the students.

We can guess that this point is stressed in 3D dynamic geometry situations, since other visual information is generally not reliable: one can be sure of the simultaneous movement of two figural units, even if it can be quite difficult to determine *how* these units are linked. These links are in fact invariant properties when points are dragged, and then direct results in Cabri3D of geometrical properties (Jahn, 1998).

Our second hypothesis is that with dynamic geometry it is possible to stress the inefficiency of iconic visualization, and to support experimental studies of the properties of the figure. Therefore dimensional deconstruction and axiomatic geometry would become very efficient tools for the students to design research processes, to study a given representation and to solve geometry problems.

Nevertheless, these theoretical tools are not sufficient: any experimental work in Cabri 3D has to involve Cabri 3D's tools. Therefore we have to study their role and the way they could interact with the theoretical ones.

First, many tools are very linked to visual perception: changing viewpoint tools, drawing and measuring tools. If they are not used with other tools, there is no need for the student to control her/his work with G2. S/he can measure drawings, watch their shape and construct objects as *soft*, and not hard constructions. When a part of such a drawing is dragged, the shape changes and so do the geometric properties the user can see. Then the feedback from Cabri 3D invalidate this kind of construction to the user's eyes (Laborde, Capponi, 1994).

Secondly, other tools are more strongly linked to a theoretical control of the constructions: construction primitives — intersection, parallel, perpendicular, tetrahedron, etc. — and transformations. Even if using axiomatic geometry is not necessary to control the use of these tools, an empirical control may be very difficult in many situations (for instance, in order to use a transformation, the user generally has to choose the values of several arguments before any visual control). So using G2 would become an economical way of controlling it. Furthermore, these tools would be very helpful for the process of instrumental deconstruction, as they are designed with axiomatic definitions. Actually, for this reason, instrumental and dimensional deconstructions would be very linked in this case.

Eventually, we have to point out that the designer of a situation (teacher, researcher…) can choose the toolset available in Cabri 3D. This is a way for him to delete specific tools in order to design feedbacks. For instance, if the students have to construct hard squares, there is no feedback about the hardness of constructions when using the "square" tool. Therefore choosing the available toolset is often a very important choice for this *didactical variable*, to make strategies inefficient or impossible.

Then, our third hypothesis is that in some specific situations, with a specific Cabri 3D toolset, it is possible to provoke a particular instrumental deconstruction, strongly linked to dimensional deconstruction.

Research problem

As a consequence of our theoretical framework, it is now possible to make the problem mentioned in the introduction clearer and more accurate: is it possible to design adidactical situations with Cabri 3D that make iconic visualization inefficient and in which dimensional deconstruction can be a tool to analyse figures and solve problems? Then we have to wonder whether using dimensional deconstruction could be liable to make the students using G2.

The following example is a situation we designed in order to test our hypothesis, in which a student has to analyse a Cabri3D-drawing in order to explain to another student how to construct the same object with Cabri 3D.

AN EXAMPLE OF A RECONSTRUCTION SITUATION

Methodology

We used a qualitative approach to analyse the students dealing with this task. We refered to our theoretical study in order to distinguish different strategies they were likely to use. It was possible to foresee how they would analyse the drawings, as shapes or as geometrical constructions... Moreover we had to analyse how they design their construction strategies. For instance, anticipating the properties of the object constructed would reveal G2-based strategies. We will only detail below the three main kind of strategies we distinguished.

In order to analyse the students' work, we used a screen-recorder software (Camtasia), microphones, and a video camera. Then we could observe at the same time their dialog, their gestures (for instance to describe physical objects), and the way they used Cabri 3D.

The situation.

This situation involves 10^{th} French graders (15 to 16 year-old students), working in pairs. Each student works on a computer. The first one (S1) has to analyse a model, a Cabri3D-drawing, and describe orally to the second student (S2) a way of reconstructing it. Using S2's computer is forbidden to S1, and S2 cannot see S1's screen.

There are four distinct phases, from the simple to the complex one (see Fig.1): first a prism with a rhombus as a base, and then are successively added its symmetrical with respect to a vertex, an edge and a lateral face. All these prisms are constructed from three directly movable points: a and b are in the base plane, and c is on the line perpendicular to the base plane at point O (the centre of the bottom face of the prism). All the other points are constructed using symmetries, so that the constructions are robust ones.

S2 is given a file with the three points, a, b and c, and the two students have to validate their constructions by themselves. The only condition is that the behaviour of the new object has to be the same as the model's one when point a, b or c are moved.

S2 doesn't see the prism and the polyhedron tool is not available, so it is much harder to solve empirically the three last problems by constructing symmetricals of the first prism.

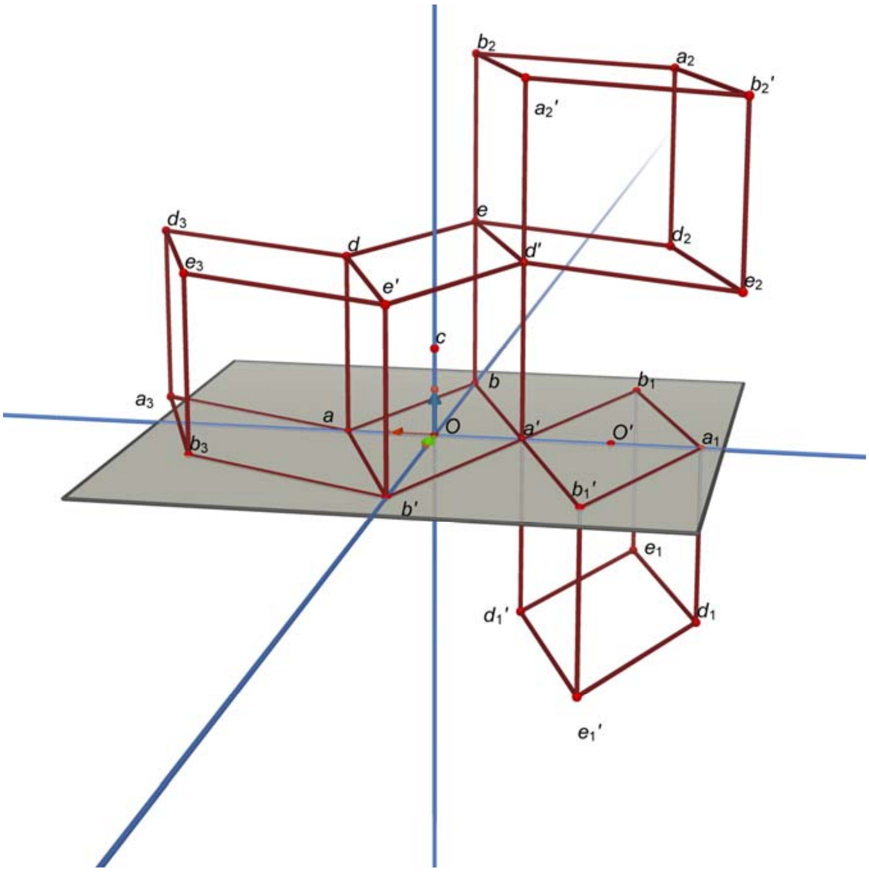

Fig. 1: Figure to analyse and reproduce in phase 4 (in previous phase, parts of the figure have been reconstructed)

Three strategies

First, if they worked using only G1, they would analyse the shapes and sizes of the models, and try to reproduce it by creating points and dragging it to the right positions. This is very difficult in a 3D space represented in 2D, and we can guess that construction primitives may be used as stands on which a visual control of the positions is possible. This is a basic strategy, and it fails in Cabri 3D whereas it wouldn't in a paper/pencil environment. We call it R1.

The second strategy (R2) is based on the use of construction primitives controlled by knowledge about "basis configurations" (Robert, 1998) learnt before. For instance, point O may be recognized as the centre of symmetry of the bottom rhombus not because a and a' seem to be symmetrical with respect to it, but because the student already know that the "centre" of a rhombus is its centre of symmetry. Therefore the students may use locally plane transformations (on some planes). But in space, as they have no previous knowledge about symmetry in a prism, their strategy may be similar to R1. We expect that in this case, in the model analysis phase and in the interaction with S2 phase, S1 may focus at the same time on geometric properties and on size information. This strategy does not necessarily require dimensional deconstruction. The result of it is a partial failure, as the dynamic properties exist in planes, but not in space.

The third strategy (R3) may be based on transformations. In this case, we assume that the student use axiomatic geometry and dimensional deconstruction, then we can guess that their analysis would focus on invariant properties when they drag points, and their reconstruction strategy would be designed in order to reproduce these properties.

Experimental results

We experimented this task with three pairs of 10th French graders, who had been just introduced to Cabri 3D before. Our following analyse will mainly focus on the "reconstruction phases", and not on S1's analysis of the drawings.

First of all, it seems that the students could get information about the drawings by manipulating it. They were able to determine, visually, shapes and basic physical properties, and to try to find a solution to the problem. For instance, the Group 3 students only used iconic visualization, and they could construct the prism shape – but a soft construction, based on the length of the edges. They tried something, and their failure was not the consequence of the too high complexity but was linked to the expected properties: some points *"don't move"*.

Secondly, all the students realized that iconic visualization was not sufficient to carry out the expected construction. We have to distinguish to main cases.

Groups 1 and 2 first used only R1, but they realised that this strategy was no longer efficient in 3D geometry. As they were able to use – more or less easily – non-iconic visualisation, they tried other strategies and could reproduce the dynamic properties. It has to be noticed that they used R2 and R3 because it was easier that R1, and not in order to make *hard* constructions (even if this was a consequence).

On the contrary, at the beginning, Group 3 students were not able to use anything but iconic visualisation. They constructed the first prism with R1, which led them to a failure: the points "didn't move". Iconic visualisation couldn't help them to analyse this:

S1: Try to make the point move

S2: I can't, there is no line [on which the point could move]

Then they started to use iconic and non-iconic visualisation at the same time, depending on their aim. For instance, they first tried to make b', b_1 and b_1' while dragging b, but didn't care about a, a'... They kept constructing a, a', a_1, etc., by measuring lengths, but constructed b_1 and b_1' by using geometrical properties, such as “parallel”, instead of adjusting positions. This second case underlines that using non-iconic visualisation can be strongly linked to the dynamic properties of the drawing.

Eventually, we have to point out that the students didn’t use easily dimensional deconstruction, and then they first tried to use it as little as possible. For instance, it seemed to Group 2 students that ded’e’ and ed_2’e_2’d’ (see Fig. 1) were linked, and that (ed’) had something to do with this link: “*a rotation*”. They tried to use the tool without any further analysis (basic instrumental deconstruction), and couldn’t succeed. Then, they analysed more precisely the link, and discovered that they had to use “*symmetry*”. Actually, as instrumental deconstruction was not precise enough, thay used dimensional deconstruction in order to control more precisely the way they used the tools.

CONCLUSION

Finally, our experimental results have a global consistency with the three hypothesis we mentionned.

The students used the representations as if they were models, and could get information from it. Even if they wanted to draw shapes, without any dynamical properties, they were able to get enough information by *looking* and *measuring* the models. Moreover, we could observe that, even to draw shapes, non-iconic visualization led them to more efficient strategies (Groups 1 and 2).

Nevertheless, because of the dynamic geometry, this process was inefficient, and they had to find a way of reproducing dynamical effects. With this new research process, they had not only to use iconic visualization but to find something else. Depending on the students’ knowledge, most of them tried to use dimensional deconstruction and an axiomatical point of view, as the most efficient strategy – efficient for analysing, giving oral information, reconstructing, arguing... In every group, the strategies used by the students evolved and dimensional deconstruction was more and more involved, so that they were able to give an interpretation to dynamical effects.

It seems that Cabri3D’s tools were very important in the evolution of strategies. Using of transformations appeared to be a way of solving the problems, but an empirical control was very difficult in most cases. Then, the students changed their strategies, and tried to find new ways of controlling it, by using dimensional deconstruction.

Therefore, these results give us informations about our research question: iconic visualisation failed, and dimensional deconstruction was necessary to solve the problem. Moreover, even the weakest students started using dimensional deconstruction, whereas they were unable to do so at the begining of the exercise. Then we could ask two new questions, more accurate. One the one hand, how did dimensional deconstruction appear, and how is it related both to the task and to instrumental deconstruction? On the other hand, we will have to study wether using dimensional deconstruction is liable to make the students use G2 in geometry, and not only in 3D geometry.

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