MULTIPLE SOLUTIONS FOR A PROBLEM: A TOOL FOR EVALUATION OF MATHEMATICAL THINKING IN GEOMETRY

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Based on the presumption that solving mathematical problems in different ways may serve as a double role tool - didactical and diagnostic, this paper describes a tool for the evaluation of the performance on multiple solution tasks (MST) in geometry. The tool is designed to enable the evaluation of subject's geometry knowledge and creativity as reflected from his solutions for a problem. The example provided for such evaluation is taken from an ongoing large-scale research aimed to examine the effectiveness of MSTs as a didactical tool. Geometry is a gold mine for MSTs and therefore an ideal focus for the present research, but the suggested tool could be used for different mathematical fields and different diagnostic purposes as well.

Introduction

The study described in this paper is a part of ongoing large-scale research (Anat Levav-Waynberg; in progress). The study is based on the position that solving mathematical problems in different ways is a tool for constructing mathematical connections, on the one hand (Polya, 1973, 1981; Schoenfeld, 1988; NCTM, 2000) and on the other hand it may serve as a diagnostic tool for evaluation of such knowledge (Krutetskii, 1976). In the larger study we attempt to examine how employment of Multiple-solution tasks (MSTs) in school practice develops students' knowledge of geometry and their creativity in the field. In this paper we present the way in which students' knowledge and creativity are evaluated.

Definition: MSTs are tasks that contain an explicit requirement for solving a problem in multiple ways. Based on Leikin & Levav-Waynberg (2007), the difference between the solutions may be reflected in using: (a) Different representations of a mathematical concept; (b) Different properties (definitions or theorems) of mathematical concepts from a particular mathematical topic; or (c) Different mathematics tools and theorems from different branches of mathematics.

Note that in the case of MSTs in geometry we consider different auxiliary constructions as a difference of type (b).

Solution spaces

Leikin (2007) suggested the notion of "solution spaces" in order to examine mathematical creativity when solving problems with multiple solution approaches as follows: *Expert solution space* is the collection of solutions for a problem known to the researcher or an expert mathematician at a certain time. This space may expand as new solutions to a problem may be produced. There are two types of sub-sets of expert solution spaces: The first is *individual solution spaces* which are of two kinds. The distinction is related to an individual's ability to find solutions independently. *Available solution space* includes solutions that the individual may present on the spot or after some attempt without help from others. These solutions are triggered by a problem and may be performed by a solver independently. *Potential solution space* include solutions that solver produce with the help of others. The solutions correspond to the personal zone of proximal development (ZPD) (Vygotsky, 1978). The second subset of an expert space is *a collective solution space* characterizes solutions produced by a group of individuals.

In the present study solution spaces are used as a tool for exploring the students' mathematical knowledge and creativity. By comparing the individual solution spaces with the collective and expert solution spaces we evaluate the students' mathematical knowledge and creativity.

MST and mathematics understanding

The present study stems from the theoretical assumption that mathematical connections, including connections between different mathematical concepts, their properties, and representations form an essential part of mathematical understanding (e.g., Skemp, 1987; Hiebert & Carpenter, 1992; Sierpinska, 1994). Skemp (1987) described understanding as the connection and assimilation of new knowledge into a known suitable schema. Hiebert & Carpenter (1992) expanded this idea by describing mathematical understanding as "networks" of mathematical concepts, their properties, and their representations. Without connections, one must rely on his memory and remember many isolated concepts and procedures. Connecting mathematical ideas means linking new ideas to related ones and solving challenging mathematical tasks by seeking familiar concepts and procedures that may help in new situations. Showing that mathematical understanding is related to connectedness plays a double role: it strengthens the importance of MSTs as a tool for mathematics education and it justifies measuring mathematics understanding by means of observing the subjects' mathematical connections reflected from one performance on MSTs..

Why geometry

The fact that *proving* is a major component of geometry activity makes work in this field similar to that of mathematicians. The essence of mathematics is to make abstract arguments about general objects and to verify these arguments by proofs (Herbst & Brach, 2006; Schoenfeld, 1994).

If proving is the main activity in geometry, *deductive reasoning* is its main source. Mathematics educators claim that the deductive approach to mathematics deserves a prominent place in the curriculum as a dominant method for verification and validation of mathematical arguments, and because of its contribution to the development of logical reasoning and mathematics understanding (Hanna, 1996; Herbst & Brach, 2006). In addition to these attributes of geometry, which make it a meaningful subject for research in mathematics education, geometry is a gold mine for MSTs and therefore an ideal focus for the present research.

Assessment of creativity by using MST

Mathematical creativity is the ability to solve problems and/or to develop thinking in structures taking account of the peculiar logico-deductive nature of the discipline, and of the fitness of the generated concepts to integrate into the core of what is important in mathematics (Ervynck ,1991, p.47)

Ervynck (1991) describes creativity in mathematics as a meta-process, external to the theory of mathematics, leading to the creation of new mathematics. He maintains that the appearance of creativity in mathematics depends on the presence of some preliminary conditions. Learners need to have basic knowledge of mathematical tools and rules and should be able to relate previously unrelated concepts to generate a new product. The integration of existing knowledge with mathematical intuition, imagination, and inspiration, resulting in a mathematically accepted solution, is a creative act.

Krutetskii (1976), Ervynck (1991), and Silver (1997) connected the concept of creativity in mathematics with MSTs. Krutetskii (1976) used MSTs as a diagnostic tool for the assessment of creativity as part of the evaluation of mathematical ability. Dreyfus & Eisenberg (1986) linked the aesthetic aspects of mathematics (e.g., elegance of a proof/ a solution) to creativity. They claim that being familiar with the possibility of solving problems in different ways and with their assessment could serve as a drive for creativity. In sum, MSTs can serve as a medium for encouraging creativity on one hand and as a diagnostic tool for evaluating creativity on the other.

According to the Torrance Tests of Creative Thinking (TTCT) (Torrance, 1974), there are three assessable key components of creativity: fluency, flexibility, and originality. Leikin & Lev (2007) employed these components for detecting differences in mathematical creativity between gifted, proficient and regular students in order to explain how MSTs allow analysing students' mathematical creativity, and thus serve as an effective tool for identification of mathematical creativity.

Fluency refers to the number of ideas generated in response to a prompt, *flexibility* refers to the ability to shift from one approach to another, and *originality* is the rareness of the responses.

In order to assess mathematical thinking in the Hiebert & Carpenter (1992) and Skemp (1987) sense, while evaluating problem solving performance of the participants on MSTs, we added the criterion of connectedness of mathematical knowledge which is reflected in the overall number of concepts/theorems used in multiple solutions of a MST.

In this paper we outline the use of MSTs as a research tool for evaluation of mathematical knowledge and creativity in geometry.

Method

Following MST instructional approach, three 60 minutes tests were given to 3 groups of 10th grade, high-level students during geometry course (total number of 52 students). The first test was admitted in the beginning, the second in the middle and the third in the end of the course. Each test included 2 problems on which students were asked to give as many solutions as they can.

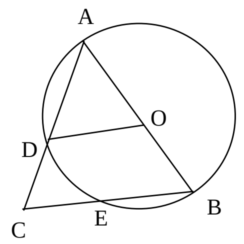
Example of the task

The following is one of the MSTs used for the tests

TASK:

AB is a diameter on circle with center O. D and E are points on circle O so that DO||EB .
C is the intersection point of AD and BE (see figure).

Prove ***in as many ways as you can*** that CB=AB



Examples of the solutions

Solution 1:

$DO = \frac{1}{2}AB$ (Equal radiuses in a circle) ⇒ DO is a midline in triangle ABC (parallel to BC and bisecting AB) ⇒ $DO = \frac{1}{2}AB = \frac{1}{2}BC$ ⇒ AB=BC

Solution 2:

DO=AO (Equal radiuses in a circle) ⇒ $\angle AOD = \angle ABC$ (Equal corresponding angles within parallel lines) ⇒ $\angle A = \angle A$ (Shared angle) ⇒ $\Delta AOD \sim \Delta ABC$ (2 equal angles) ⇒ AB=BC (a triangle similar to an isosceles triangle is also isosceles)

Solution 3:

DO=AO (Equal radiuses in a circle) ⇒ $\angle ADO = \angle A$ (Base angles in an isosceles triangle)

$\angle ADO = \angle ACB$ (Equal corresponding angles within parallel lines), $\angle ACB = \angle A$ ⇒

AB=BC (a triangle with 2 equal angles in isosceles)

Solution 4:

Auxiliary construction: continue DO till point F so that DF is a diameter. Draw the line FB (as shown in the figure)

DO=AO (Equal radiuses in a circle) ⇒ $\angle ADO = \angle A$ (Base angles in an isosceles triangle)

$\angle F = \angle A$ (Inscribed angles that subtend the same arc) ⇒

$\angle F = \angle ADO$ ⇒ CD|| BF (equal alternate angles)

DFBC is a parallelogram (2 pairs of parallel sides) ⇒

DF=CB (opposite sides of a parallelogram), DF=AB (diameters) ⇒ AB=BC

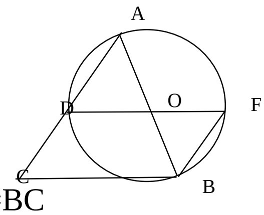


Figure 1: Example of MST

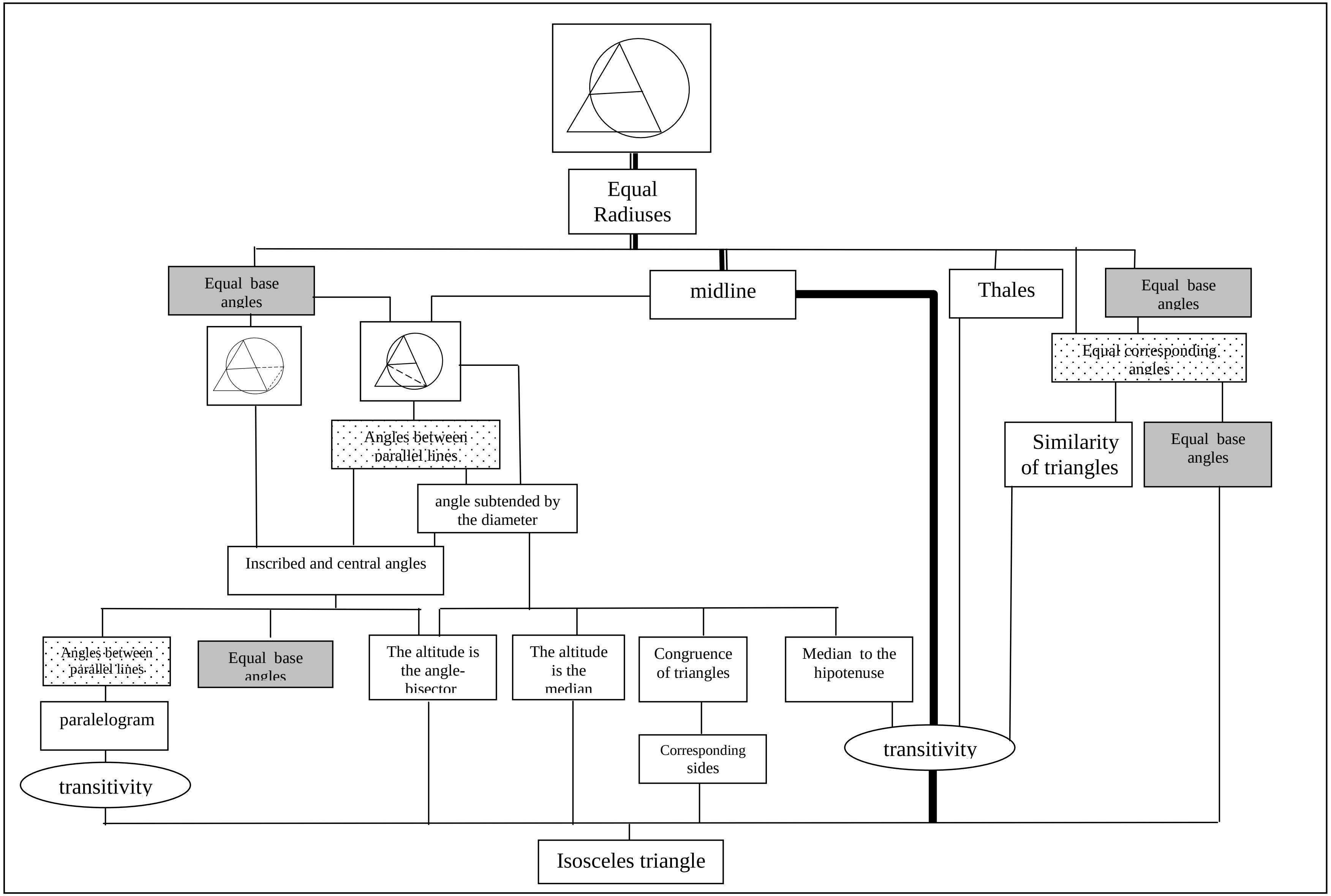


Figure 2: The map of an expert solution space for the task (see Figure 1)

Figure 1 presents an example of a task used in this study. Figure 2 depicts a map of the expert solution space for this task. The map outlines concepts and properties used in all the solutions as well as the order of their use in each particular solution (for additional maps of MSTs see Leikin, Levav-Waynberg, Gurevich and Mednikov, 2006).

The bold path in the map (Figure 2) represents Solution 1 of the task (see Figure 1).

Data analysis

	Correctness	**Connectedness**	**Creativity**		
			Fluency	Flexibility **groups of solutions should be defined**	Originality According to (**P**) frequency (conventionality) of a strategy used
Scores per solution	1-100	--	1	Flx=10 for the first solution Flxi=10 solutions from different groups of strategies Flx_i=1 solutions from the same group – meaningfully different subgroups Flxi=0.1 solutions from the same group-similar subgroups	$Or_i=10$ P<15% $Or_i=1$ 15%<P<40% $Or_i=0.1$ P>40%
Final score for a solution group (strategy used by a student including its repetitions)	--	$\frac{t}{T}*100$		$Flx = \sum Flx_i$	$Or=\sum Or_i$
			$Flx_i \times Or_i$		
Score per individual solution space for a problem (SPI)	*Correctness*= Max score among the solution for a problem	$Con = \frac{t}{T}*100$	$n \times \sum (Fl_i \times Or_i)$		

n: *number of solutions in the individual solution space*
N: *number of the students in a group*
T: *number of concepts and their properties used in the expert solution space*
t: *number of concepts and their properties used in the individual solution space*

m_i: *the number of students who used the strategy i*

$$P=\frac{m_i}{N}100\%$$

Figure 3: Scoring scheme for the evaluation of problem-solving performance on a particular MST based on Leikin (forthcoming)

The analysis of data focuses on the student's individual solution spaces for each particular problem. The spaces are analyzed with respect to (a) Correctness; (b) Connectedness; (c) Creativity including fluency, flexibility, and originality.

The maximal correctness score for a solution is 100. It is scored according to the preciseness of the solution. When solution is imprecise but lead to a correct conclusion we consider it as *appropriate* (cf. Zazkis & Leikin, 2008). The highest correctness score in an individual solution space serves as the individual's total correctness score on the task. This way a student who presented only 1 correct solution (scored 100) does not get a higher correctness score than a student with more solutions but not all correct. Connectedness of knowledge associated with the task is determined by the total number of concepts and theorems in the individual solution space. Figure 3 depicts scoring scheme for the evaluation of problem-solving performance from the point of view of correctness, connectedness and creativity. The scoring of creativity of a solutions space is borrowed from Leikin (forthcoming). In order to use this scheme the expert solution space for the specific MST has to be divided into groups of solutions according to the amount of variation between them so that similar solutions are classified to the same group. The number of *all the appropriate solutions* in one's individual solution space indicates one's fluency while flexibility is measured by the *differences among acceptable solutions* in one's individual solution space. Originality of students' solution is measured by the rareness of the solution group in the mathematics class to which the student belongs. In this way a minor variation in a solution does not make it original since two solutions with minor differences belong to the same solution group.

Note that evaluation of creativity is independent of the evaluation of correctness and connectedness. In order to systematize the analysis and scoring of creativity and connectedness of one's mathematical knowledge we use the map of an expert solution space constructed for each problem (see Figure 2).

Results – example

In the space constrains of this paper we shortly exemplify evaluation of the problem-solving performance of three 10^{th} graders – Ben, Beth and Jo -- from a particular mathematics class. The analysis provided is for their performance on Task in Figure 1. Their solutions are also presented in this figure. We present these students' results because they demonstrate differences in fluency, flexibility and originality. Solutions 1, 2 and 3 are classified as part of the same solutions group whereas solution 4 which uses a special auxiliary construction is classified as part of a different group.

Ben performed solutions 1, 3 and 4, Beth produced solutions 1, 2 and 3, and Jo succeeded to solve the problem in two ways: solutions 1 and 3 (Figure 1). Figure 4 demonstrates connectedness and creativity scores these students got on the Task when the scoring scheme was applied (Figure 3). Their correctness score for all the solutions they presented was 100.

We observed the following properties of the individual solution spaces for Ben and Beth: they were of the same sizes; they included the same number of concepts and theorems and contained two common solutions (solutions 1 and 3). However Ben's creativity score was much higher then Beth's one as a result of the originality of Solution 4 that was performed only by Ben, and his higher flexibility scores.

Beth and Jo differed mainly in their fluency: Beth gave 3 solutions and Jo only 2. Since their solutions had similar flexibility and originality scores their creativity scores are proportional to their fluency scores.

		Solution Type (in order of presentation in the test)	**group**	**Connectedness**	**Creativity**			
					Fluency	Flexibility	Originality	creativity
Ben	Scores per solution	1	1			10	0.1	1
		3	1			1	0.1	0.1
		4	3			10	10	100
	Final			50	3			303.3
Beth	Scores per solution	2	1			10	0.1	1
		3	1			1	0.1	0.1
		1	1			1	0.1	0.1
	Final			50	3			3.6
Jo	Scores per solution	3	1			10	0.1	1
		1	1			1	0.1	0.1
	Final			30	2			2.2

Figure 4: Evaluation of the solutions on the task for three students

Concluding remarks

MSTs are presented in this paper as a research tool for the analysis of students' mathematical knowledge and creativity. The tasks are further used in the ongoing study in order to examine their effectiveness as a didactical tool. The larger study will perform a comparative analysis of students' knowledge and creativity along employment of MST in geometry classroom on the regular basis. The scoring scheme presented herein can be considered as an upgrading of the scoring scheme suggested by Leikin and Lev (2007). Correspondingly we suggest that the scoring scheme presented herein can be used for examination of individual differences in students' mathematical creativity and students' mathematical knowledge in different fields. We are also interested in employment of this tool for the analysis of the effectiveness of different types of mathematical classes in the development of students' mathematical knowledge and creativity.

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