CERME 6 – WORKING GROUP 4
ALGEBRAIC THINKING AND MATHEMATICS EDUCATION

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ALGEBRAIC THINKING AND MATHEMATICS EDUCATION

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In CERME–6 Working Group "Algebraic Thinking" we continued the work done in previous CERME conferences, both by following the discussions raised and by pointing out unanswered questions (Puig, Ainley, Arcavi & Bagni, 2007).

More particularly, in CERME–6, Working Group "Algebraic Thinking" was concerned with further discussion on historical, epistemological, and semiotic perspectives in research in the teaching and learning of algebra. The role of artifacts, technological or not, was also considered in this perspective. In general, Working Group "Algebraic Thinking" was interested in proposing to address the issue of the actual impact of research on curriculum design and development, and on practice.

In order to allow a detailed discussion of the contributions, we decided to split the working group into two subgroups:

- the subgroup A (co–ordinated by Lisa Hefendehl–Hebeker) included some contributions mainly focused on cognitive aspects. The Authors were W.F. Castro and J.D. Godino; M.–C. Croset; A. Cusi; J.–P. Drouhard; C. Fernández and S. Llinares; B. Gómez and C. Buhlea; M. Hoch and T. Dreyfus; J. Hodgen, D. Kuchemann, M. Brown and R. Coe; R. Oldenburg; I. Papadopoulos and M. Iatridou; F. Siebel and A. Fischer; I. Sinitsky and B.–S. Ilany; E. Söbbeke and C. Böttinger; A.M. Wille.
- the subgroup B (co–ordinated by Janet Ainley) included some contributions mainly focused on pedagogical aspects. The Authors were O. Akkus and E. Cakiroglu; M. Ayalon and R. Even; G.T. Bagni; A.B. Fyhn; S. Gerhard; M. Haspekian and E. Bruillard; I. Jones; J.–B. Lagrange and T.K. Minh; C. Marchini, A. Cockburn, P. Parslow–Williams and P. Vighi; M. Panizza; R.A. Rinvold, and A. Lorange; E. Robotti, G. Chiappini and J. Trgalova.

Posters presentations by R. Berrincha and J. Saraiva, Ç. Kiliç and A. Özdaş, B.M. Kinach, A. Matos, C. Monteiro and H. Pinto, A.I. Silvestre, I. Vale, T. Pimentel produced important contributions to our discussion.

In the following file, contributions are organised according to the alphabetic order of the corresponding authors.

GENERAL REFLECTIONS

The invention of the symbolic language of algebra influenced the development of mathematics in all domains. Symbolic language is used throughout all mathematics:

for instance, there is no possible calculus or analysis without solving inequalities, structures (groups, rings, …) are used to describe all parts of mathematics (Drouhard, 2009). This must be taken into account when considering early algebra.

Historically, algebra results from what evolution scientists call co–evolution. This co–evolution involves: first an art, then a science of resolution of numerical problems; first informal representation systems, then formal registers (semiotic representation systems); first a science of numbers, then a science of structures (Drouhard, 2009). So today algebra is a science of resolution of numerical problems, a family of semiotic systems (linguistic or not), and a science of numbers and structures.

In a passage of his *Questions Concerning Certain Faculties Claimed for Man*, Charles S. Peirce (1839–1914) suggests that it is impossible to "think without signs" (Peirce, 1868/1991, p. 49). In a Peircean perspective, algebraic language is based upon iconicity. Let us quote Peirce (1931–1958, 2.279, *MS* 787):

> Particularly deserving of notice are icons in which the likeness is aided by conventional rules. Thus, an algebraic formula is an icon, rendered such by the rules of commutation, association, and distribution of the symbols […]. For a great distinguishing property of the icon is that by direct observation of it other truths concerning its object can be discovered than those which suffice to determine its construction.

Two remarks must be taken into account. Firstly, every sign "contains" all the components of Peircean classification, although one of them (e.g. iconicity) is predominant. For instance, algebra is not characterised by the presence or absence of letters: algebra is characterised by the existence of a semiotic representational system, a system which allows us to solve numerical problems and to express number properties. So algebra is not but has got a language (Drouhard, Panizza, Puig, & Radford, 2006).

Secondly, Peirce's semiotics hardly explains the complexity of sign–based human thought processes and the manner in which they relate to their corresponding historical settings (Douek, forthcoming). The historical dimension of cognition and its cultural subbasement (see Bradford & Brown, 2005; D'Ambrosio, 2006) are a fundamental theme in recent sociocultural perspectives where cognition is conceptualized as "a cultural and historically constituted form of reflection and action embedded in social praxes and mediated by language, interaction, signs and artifacts" (Radford, 2008, p. 11). Sociocultural perspectives lead to both new conceptions of cognition and new views about knowledge and the cognizing subject: algebraic thinking can be framed into the mentioned perspective.

Algebraic language must be described by linguistic terms ("syntax", "semantics"). In terms of semantics, the power of algebra lies in the capability to judiciously "forget the meaning". From an educational viewpoint, it is worth noting that students must at the same time master the languages (natural and symbolic), their respective syntax and semantics and the semiotic aspects of these languages, and be flexible, so be able to work both with meaningless and meaningful expressions (see remarks in Puig, Ainley, Arcavi & Bagni, 2007).

COGNITIVE ASPECTS

As regards cognitive aspects (subgroup A), it is worth noting that the tension between the possibility of formal manipulation and the necessity of semantic understanding, which is typical for algebraic activities, causes particular cognitive demands for the learners. There are many partial abilities which should be learned and grow together to an interrelated system. Mental acts and ways of thinking (Harel, 2008) which are essential for algebraic thinking have to be activated on different layers:

- *Structuring:* The symbolic language of algebra is a tool to conceive arithmetical structures, and as a semiotic system it has a structure of its own. Comprehensive learning of algebra and successful manipulation of its language deserves "structure sense" in different respects.
- *Generalizing:* Generalizing belongs to the essence of algebra. It means to grasp something typical, which all cases under consideration have in common. Variables are tools to express indeterminacy and generality. To describe a sequence of geometrical patterns by a formula and to find a common form of a set of formulas (for example quadratic equations) are activities on different stages of generalization.
- *Representing:* The representation system of algebra in its final stage is symbolic and formal, that means, it allows context-free manipulation. This makes it difficult to grasp for learners, but for experts it gains a new kind of meaning and richness in itself.

Many contributions showed that there are previous stages in the development of these ways of thinking, which should be cultivated in the learning process. Such activities might help to reduce the "cognitive gap" between arithmetic and algebra:

- *Structuring and generalizing:* For example pre–service primary teachers experience structuring and thus develop "algebraic awareness" when they analyze, describe and continue patterns and structures in geometric and algebraic contexts. A fruitful interplay between arithmetic and geometric visual approaches can also be experienced on later stages.
- *Representing:* L. Radford demonstrated in his plenary address that alphanumeric symbolism is not the only way to express algebraic thinking. He pointed out that there is a conceptual zone before, where algebraic thinking is contextual and embodied in the corporeality of actions, gestures, signs and artefacts.

Nevertheless such approaches to teaching algebra have their own problems.

PEDAGOGICAL ASPECTS

In considering pedagogical approaches to teaching algebra (subgroup B) there is a potential tension between the need to focus on structure independently of context (for example to develop understandings of equality, equivalence), and the uses of context as ways to make structure visible (for example by means of metaphor, metonymy, allegory, artefacts, narratives, …). Teachers and pupils may be attending to different aspects of the activity: while the teacher is looking through a context such as a visual pattern in order to see generality, pupils may be looking at the stages of construction of the particular pattern.

Different perceptions of the nature of algebraic activity may become apparent when considering the role of, and need for, proof. Similarly, alternative perceptions of the nature of tools, artefacts and representations emerge from close study of the conversations in classrooms. This presents real challenges for teachers in their interactions with learners, and of their interventions in activities.

A continuing challenge is the design of tasks which may motivate a real need for algebraic thinking. There is clearly no single 'best' approach to algebra; many good approaches can support each other. It is important to interrogate each approach to identify what it may offer and for whom. The design of such tasks must take account of the rich variety which may be covered by the phrase 'algebraic thinking' and the ways in which such thinking may be expressed. Rather than focussing on differences between arithmetic and algebraic thinking, it may be powerful to see this as a continuum, or parallel development, rather than as a dichotomy. Generalisation may be embodied through gesture, including virtual gestures on a computer screen, or expressed through natural language as well as through symbolism. Variable is an algebraic idea that children must understand on their way to learning symbolic generalisation because it allows thinking about change, generalisation and structure. It is an idea which may be introduced and expressed in many ways: the design challenge is to find ways to engage learners in the real need for, and power of, algebra.

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THE EFFECTS OF MULTIPLE REPRESENTATIONS-BASED INSTRUCTION ON SEVENTH GRADE STUDENTS' ALGEBRA PERFORMANCE

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The purpose of this study was to investigate the effects of multiple representation-based instruction on seventh grade students' algebra performance. The study was conducted on four seventh grade classes from two public schools lasting eight weeks. For assessing algebra performance, three instruments called translations among representations skill test, objective based achievement test, and Chelsea diagnostic algebra test were used. The analyses were conducted by using multivariate covariance statistical model. The results pointed out that multiple representation-based instruction had a significant effect on students' algebra performance compared to the conventional teaching. In addition to this, students from experimental groups found this way of teaching fruitful.

INTRODUCTION

Various meanings can be given to the concept of "representation" in connection with the teaching and learning of mathematics. Seeger, Voight, & Werschescio (1998) summarized some of those definitions in very general terms as follows: "…representation is any kind of mental state with a specific content, a mental reproduction of a former mental state, a picture, symbol, or sign, symbolic tool one has to learn their language, a something "in place of" something else".

Multiple representations can be generally defined as providing the same information in more than one form of external mathematical representation by Goldin and Shteingold (2001). The usage of multiple representations in mathematical learning was investigated in depth by Janvier who defined it "understanding" as a cumulative process mainly based upon the capacity of dealing with an "ever-enriching" set of representations (Janvier, 1987, p. 67). There are two important key terms in a theory of representation that are; "to mean or to signify, as they are used to express the link existing between external representation (signifier) and internal representation (signified)" (Janvier, Girardon, & Morand, 1993, p. 81). External representations were defined as "acts stimuli on the senses or embodiments of ideas and concepts", whereas internal representations are regarded as "cognitive or mental models, schemas, concepts, conceptions, and mental objects" which are illusive and not directly observed (Janvier, et. al., 1993, p. 81).

Another approach to the theory of multiple representations which is called Lesh Multiple Representations Translations Model (LMRTM) has been suggested by Lesh (1979). His theory draws the theoretical framework of this study since he improved a

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model involving translations among representational modes and transformation within one representational mode. According to Lesh, Post and Behr (1987), representations are crucial for understanding mathematical concepts. They defined representation as “external (and therefore observable) embodiments of students’ internal conceptualizations” (Lesh, et al. 1987, p. 34). This model suggests that if a student understands a mathematical idea she or he should have the ability of making translations between and within modes of representations. According to this view, a good problem solver should be able to “sufficiently flexible” in using variety of representational systems. He claimed further, “As a student’s concept of a given idea evolves, the related underlying transformation/translation networks become more complex; and teachers who are successful at teaching these ideas often do so by reversing this evolutionary process; that is, teachers simplify, concretize, particularize, illustrate, and paraphrase these ideas, and imbed them in familiar situations” (p. 36).

A MULTIPLE REPRESENTATIONS TRANSLATION MODEL

After reviewing a number of theories about multiple representations, this study emphasizes investigating particularly students’ ability to use the given representational mode for solving problems, and to make translations among the representational modes. A multiple representational translations model combined from the models belonging to Lesh and Janvier would seem to be perfect modeling for this research study. The five distinct representational modes; namely, manipulatives, real-world situations, written symbols, spoken symbols, and pictures or diagrams in LMRTM were directly included in the model of this study. Some of those representational modes were named differently referring the Janvier Representational Translation Model (JRTM). Instead of “written symbols” from LMRTM, wording of “formulas” from JRTM was included in this study. Besides in lieu of the combination of “situations, pictures, and verbal descriptions”, the researcher decided to use those representational modes separately. Therefore instead of “situations, pictures, and verbal descriptions” in JRTM, “manipulatives,” “pictures or diagrams,” and “spoken symbols” were taken from LMRTM. “Tables” and “graphs” were taken separately from JRTM. Janvier’s Representation Translation Process was revised in light of the Lesh (1979) ideas as appeared in Table 1.

Table 1: The combined model of Lesh and Janvier for translations among representation modes

From \ To	Spoken Symbols	Tables	Graphs	Formulas (Equations)	Manipulatives	Real Life Situations	Pictures
Spoken Symbols	_	Measuring	Sketching	Abstracting	Acting out	Acting out	Drawing
Tables	Reading	_	Plotting	Fitting	Modeling	Modeling	Visualiz-ing
Graphs	Interpre-	Reading Off	_	Fitting	Modeling		

	tation						
Formulas (Equations)	Reading	Computing	Sketching	_	Concretizing	Exemplifying	Localizing
Manipulatives	Describing	Exemplifying	Concretizing	Symbolizing	_	Simplifying, Generalizing	Drawing
Real Life Situations	Describing	Exemplifying	Plotting	Modeling, Abstracting	Particularizing	_	Modeling
Pictures or Diagrams	Describing	Describing	Sketching	Abstracting	Constructing	Situationalizing	_

SIGNIFICANCE OF THE STUDY

The issue of what instructional approaches should be used in algebra classes does not have a single and clear answer. No matter which instructional approach is used, the primary goal of mathematics instruction should be to help students in forming conceptual understanding. Janvier (1987) mentioned that if teachers enrich their algebra classrooms by placing multiple representations, the students can more efficiently make connections between the meaning of algebraic concepts and the way of representing them, therefore they simply “go for the meaning, beware of the syntax” which results in conceptual understanding.

The improvement of mathematical understanding and representational thinking of students require flexible use of multiple representations and the interaction of external and internal representations (Pape & Tchoshanov, 2001). Since making meaningful translations in representational modes plays a crucial role in acquisition of mathematical concepts and there are still unanswered questions about the instructional outcomes of using multiple representations, we believe that it would be worth to investigate the multiple representations in this respect.

Since this study focuses on the effects of multiple representation-based environments in mathematics classroom, its results should help mathematics educators who seek alternative pedagogical instructions in classroom settings. Furthermore, if a teacher is aware of his/her students’ understanding of the multiple representations and what kind of learning is supported by multiple representation-based environments, s/he can better choose and utilize appropriate type of methods, manipulatives, or activities to meet the needs of students. Moreover, providing students with a multiple representation-based algebra instruction would promote a conceptual shift to thinking algebraically. Therefore, receiving such kind of instruction makes students more competent in the area of algebra.

RESEARCH QUESTION

The purpose of this study is to examine the effects of a treatment based on multiple representations on seventh grade students’ performance in algebra, and this study attempted to answer the following research question;

"What are the effects of the multiple representations-based instruction compared to conventional teaching method on seventh grade students' algebra performance when students' gender, mathematics grade of previous semester (MGPS), age, prior algebra level are controlled?"

METHOD

The research question was examined through a quasi-experimental research design since this study did not include the use of random assignment of participants to both experimental and control groups. The target population of this study consists of all seventh grade students from public schools in Çankaya district in Turkey. There were 103 public schools in this region. However, two schools from this district were determined as the accessible population. There were 2 seventh grade classes in School A, and 7 in School B. One experimental and one control group were selected from both schools. There were 15 girls and 13 boys in experimental group and 16 girls and 13 boys in control group taken from School A. On the other hand, the experimental group from School B consists of 17 girls and 21 boys and in the control group the number of girls and boys were equal, that is 18. The participants in this study ranged in age from 11 years to 14 years old.

INSTRUMENTS

To assess algebra performance, three distinct instruments namely Algebra Achievement Test (AAT), Translations among Representations Skill Test (TRST), and Chelsea Diagnostic Algebra Test (CDAT) were used. The rational of using combined instruments is to perceive algebraic learning in a multi dimensional way. It includes procedural, conceptual, and translational knowledge and skills in its nature (Lesh, Landau & Hamilton, 1983). By utilizing three instrument it was aimed to assess algebraic learning within its all dimensions and each instrument was tried to assess different aspect of algebra learnings. It can be claimed that when a student gets higher scores from three instruments s/he can be called as successful in algebra since getting high score means that s/he can use procedural algebra knowledge in problem solving, understand algebra conceptually, and also make simple translations among representations.

Among three instruments Algebra Achievement Test (AAT) was administered to analyze students' computation skills in algebra intensively. 10 essay type questions were used in this instrument which combines traditional school algebra test items including symbolic manipulations and computations in algebra. The items which are related with the procedural skills in school algebra are criterion-referenced tasks addressing key learning goals specified in the Mathematics Curriculum for Elementary Schools, published by Turkish Ministry of National Education (MEB, 2002). The required time for this instrument was 30 minutes. The internal reliability value of Cronbach alpha was calculated as .90. To score the students' responses to each question in AAT four-point rubric was used. The highest point of 4 indicated a complete understanding of underlying mathematical concepts and procedures while the lowest point of 0 was given for irrelevant or no responses. The minimum and maximum possible scores from the test items are 0 and 40 points, respectively. Students who got scores above mean score of the group was accounted as high achievers.

Another instrument for assesing students' algebra performance was the Translations among Representations Skill Test (TRST). The purpose of this test was to obtain data about students' abilities in making translations among different representational modes. TRST contains 15 open-ended items which were designed to measure skills of translation among representations, use of certain representations, and creating new representations. The items in TRST required a translation from one representation type to another, such as from tabular representation to graphical one. In the last two items all type of representations were required to solve the problem. Duration of the test was 40 minutes. It was scored by using a three-point holistic scoring rubric. The highest point of 3 was awarded for responses showing that the problem was solved correctly and that the appropriate translations among representations were used. The lowest point of 0 indicated if the response is completely wrong or immaterial to the. The possible minimum score was 0, and the possible maximum score was 36. The internal reliability estimate of TRST was found to be .79 by calculating the Cronbach alpha coefficient.

The last instrument was Chelsea Diagnostic Algebra Test (CDAT) which was developed by the Concepts in Secondary Mathematics and Science Team (Hart, Brown, Kerslake, Küchemann, & Ruddock, 1985) to determine 13-15 years old children's algebraic thinking levels. This test was designed to measure the conceptual knowledge of elementary algebra. In CDAT there are six different categories of interpreting and using the *"letter"*. Apart from these six categories, four levels of algebra understanding were developed with respect to the children's responses and the items themselves. In Level 4, children can deal with the items that require specific unknowns and which have a complex structure (Hart, et al., 1989) and they can be accounted as successful in algebra. The students answered the items in this test approximately in 60 minutes. The discrimination power of the items ranged from 0.20 to 0.60. Reliability measure as based on KR-20 coefficient was found to be 0.93. There were 53 items in the adapted version of CDAT. The possible minimum and maximum scores were 0 and 53 respectively. Besides, CDAT was used as a pretest to find out experimental and control group students' conceptual algebraic knowledge before the intervention. It was considered that seventh grade students' algebra knowledge coming from their previous mathematics background might affect the experiment therefore CDAT as pretest was also taken to MANCOVA statistical model as a profounding variable.

TREATMENT BASED ON MULTIPLE REPRESENTATIONS

For this study, the instructional design for experimental groups consists of daily lesson plans in which several activities took place. There were 21 activities which were involved in the lesson plans of the instructional unit in order to aid in teaching of a unit of algebra. All 21 lesson plans which had distinct contexts and problem situations were developed in order to reflect the procedure of translations among representations, transformations within a specified representation, usage of any representational mode in dealing with algebraic situation. In particular, students were required to learn constructing the multiple representations of algebraic situations, including expressing them in tables, graphs, and symbols. Instead of teaching these representation skills in isolation, it was anchored within meaningful thematic situations. Instead of direct instruction in how to construct and use mathematical representations in algebra, students were only guided in the activities to explore different representations and to develop their understanding of each one. In experimental groups students were frequently given tasks that require them to make translations among different representations. This approach was used to present and develop concepts from verbal, algebraic, graphical, and tabular standpoints. To illustrate, for instance, a concept first introduced a numerically intuitive approach in which tables were used to collect and work on data. Then a verbal representation was used to verbally complement the relationship among numbers in the tabular representation. Finally, a transition was made to the algebraic representation. The usage of multiple representations varied for each activity presented in this treatment. For instance, for the topic of equations, first the tabular representation then the verbal representation were constructed; however, for conceptualizing the concept of graph, first, the algebraic representation, and then the other representations were used.

The actualization of treatment can be illustrated in one activity namely; *"Inequalities"*. In this activity students were responsible to find out the main characteristics of inequalities using the tabular representation. At the beginning the activity sheets were given to the students, and then they examined the activity. They filled the given table by required numbers, and then the translation from one representation to another came. For this, the daily life situations and the algebraic representational modes were selected. Students were required to give one daily life example to the inequality of "x–3<7". Students' examples were like;

"There are x number of teachers in one school, then 3 of them are appointed to another school, and the number of the remaining teachers was less than 7".

"Let us say that the number of the desks in our class was x, we get rid of 3 of them, then there are less than 7 desks in our class".

After getting students translations among representations, all of them were discussed in class. It is compulsory for the students to keep the activity sheets in the folder that the researcher gave them, since they did all the works on those papers. They were also responsible to bring their folder to the class every mathematics lesson.

In the treatment, particularly the translations among representational modes were stressed and valued by the researcher. In conventional algebra teaching, however, translation among representations might occur only when the students are required to draw a graph. In this case, instead of constructing a table to represent the given equation, they only identified two points where the line passess through. Then, by the help of this information, a graph could be drawn. However, the multiple representations-based instruction emphasizes the translations from variety of representational modes to the other modes. Therefore, students could have the opportunity to notice that one mathematical concept can be represented in several ways and these ways can be complementary to understand this concept. The same task of drawing the graph of a linear equation is taken in a way that, students analyze the equation through daily life situations, plain language, tables, and graphs. In that sense, drawing the graph of an equation is not an end but it is a means of interpreting the existing mathematical situation. The treatment lasted eight-weeks. Each week experimental groups received four lesson hours, with each session lasting 40 minutes.

RESULTS

To test the null hypothesis related to the research question, the statistical technique of Multiple Analysis of Covariance (MANCOVA) was used for comparing the mean scores of control and experimental groups separately on the AAT, TRST, and CDAT. MANCOVA was carried out by putting experimental groups together as a one experimental group and control groups as one control group as well (Cohen & Cohen, 1983).

Initial descriptive analysis revealed that the experimental groups had the higher scores on all the instruments compared to the control groups. Before conducting MANCOVA the assumptions called normality, multicollinearity, homogeneity of regression, equality of variances, and independency of observations were verified (Green, Salkind, & Akey, 2003).

The MANCOVA results revealed that, there was a significant effect of two methods of teaching on the population means of the collective dependent variables of seventh grade students' scores on the AAT, TRST, and CDAT after controlling their age, the MGPS, and PRECDAT scores. 37% of the total variance of MANCOVA model for the collective dependent variables of the AAT, TRST, and CDAT was explained by group membership of the participants. Using the Wilks' Lambda test, significant main effects were detected between the groups experimental group and control group (λ = .63, p = .000). Therefore, the results of this study were of practical significance. The significant finding of a group effect from MANCOVA, allowed further statistical analysis to be done in order to determine the exact nature of significant differences found in main effect. Therefore univariate analyses of covariance (ANCOVA) were carried out on each dependent variable in order to test the effect of the group membership. From the analyses, it can be stated that, multiple representation based instruction has a significant effect on the dependent variable scores of CDAT [$F(1,125) = 38.005$, $p = .000$)], TRST [$F(1,125) = 25.942$, $p = .000$], and AAT [$F(1,125) = 18.271$, $p = .000$]. Furthermore, for the observed treatment effects, it was obvious that the values of eta squared for the scores of the CDAT, TRST, and AAT were .233, .172, and .128 respectively which are equal to the medium effect size. This explains 23% of the variance in CDAT, 17% of the variance in TRST, and 13% of the variance in AAT related with the treatment. Power for the scores of the CDAT, TRST, and AAT were found as 1.00, .92, and .78 respectively. Step-down analysis was carried out as significant MANCOVA follow up analysis. By the help of this analysis, the unique importance of dependent variables which were found as significant in the MANCOVA analysis was investigated. Since there are three significant dependent variables namely, CDAT, TRST, and AAT, three step-down analyses were conducted. By doing so, any possible variance overlap among the dependent variables was planned to be detected. According to these results, the effect of multiple representation-based instruction had still significant effect on each dependent variable.

DISCUSSION

This research study has documented that, compared to conventional instruction, multiple representations-based instruction did make a significant influence on the algebra performance of seventh grade students. There might be various reasons to this result. Visualization of algebraic objects, connections among algebraic ideas, and the improvement of translational abilities in algebra problem solving (Lesh, Post, & Behr, 1987) can be counted as what multiple representations-based instruction provide for students. By the help of this instruction, students avoid memorization in algebra learning, and understand concepts meaningfully. As suggested in Swafford and Langrall`s (2000) study; multiple representations-based instruction promotes conceptual understanding of algebra and makes students conceptualize algebraic objects. The results of this study are supported in the literature by numerous studies. One of them is Brenner`s (1995) and her colleagues study. They conducted only 20 days multiple representations unit including variables and algebraic problem solving. After treatment they implemented four instruments related to algebra learning to the seventh and eight graders. Significant difference was found between experimental and control group of students in favor of the students in experimental groups. The findings of this study are also consistent with the findings of previous studies (Ozgun-Koca, 2001; Pitts, 2003) that provided evidence for the effectiveness of multiple representations-based instruction in engaging students in meaningful algebra learning. Additionally, in Herman`s (2002) study similar results were found. It was stated that after multiple representation based instruction in college algebra course, students were better able to establish connections between varieties of representational modes.

This study confirmed the need for considering other kinds of representations, such as; representations used in graphic calculator and computer programs or representations that students create and unique for them. As it was suggested by Ozgun-Koca (2001), computer-based applications can be used to provide linked and semi-linked representations, and graphical form of representations. These applications can make students to abstract mathematical concepts from virtual world. Besides, allowing students to create their own representations for solving algebra problems makes them more creative and flexible in mathematics (Piez & Voxman, 1997). In this study it was observed that, students were mainly restricted by four types of representations which are tabular, graphical, algebraic, and verbal. This can be due to the activities or researcher's emphasize on those representation types. However, students should be given an opportunity that they can use representations that they invent or create. Moreover, it can be suggested that future research can focus on teachers and teaching strategies in algebra classrooms. All of the data for this study was collected from students. Future research could combine data from students and their teachers, because teachers have also impact on shaping students' representation preferences. What teaching strategies and representation types are used within algebra classrooms by teachers and how those representations are conceptualized by the students seems to be worthwhile to investigate.

According to the researcher, mathematics educators ought to recognize making establishment between concepts for the mathematics instruction for all students. Nowadays, many attempts can be observed to improve mathematics instruction. Multiple representation-based instruction for conceptual algebra understanding is just the one that the researcher implemented and appreciated the benefits of using this method. Giving opportunity to new instructional methods like multiple representation based instruction in mathematics classrooms enables students better mathematics learner. As Klein (2003) implied; *`Learning to create and interpret representations using specific media such as texts, graphics, and even videotapes are themselves curricular goals for many teachers and students` (p. 49).* As a three-year experienced mathematics teacher before, the researcher could say that in traditional mathematics classroom, there is a need to encourage students to think more deeply on mathematical concepts, to intrinsically motivate for learning, to make students appreciate the nature of mathematics by getting rid of rote memorization, and to avoid overemphasizing mathematical rules and algorithms. In fact, new instructional methodologies like multiple representation-based instruction can address this need.

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OFFERING PROOF IDEAS IN AN ALGEBRA LESSON IN DIFFERENT CLASSES AND BY DIFFERENT TEACHERS

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This paper analyzes the ways proof ideas in an algebra lesson were offered to students (1) by two different teachers, and (2) in two different classes taught by the same teacher. The findings show differences between the two teachers, and between the two classes taught by the same teacher, regarding the proof ideas made available to learn in the lesson.

Keywords: Proof ideas, algebra, classroom, curriculum, teacher.

INTRODUCTION

Research suggests that getting students to understand what a mathematical proof is and the role that proofs play in mathematics is not an easy task (de Villiers, 1990; Dreyfus & Hadas, 1996; Harel & Sowder, 2007). However, most of the research on proof focuses on the individual student's cognition and knowledge. There is an absence of studies that focus on the complexity of teaching and learning proof in the classroom (Mariotti, 2006), and on the role of the content and sequencing of the curriculum on the quality of teaching proof (Holyes, 1997; Stylianides; 2007). Moreover, research related to proof is commonly conducted in the context of geometry, and examination of proof in algebra is sparse. This study addresses this shortcoming of current research. Its aim is to examine the enactment of a written algebra lesson, which centers on determining and justifying equivalence and non-equivalence of algebraic expressions. The study focuses on ways important proof ideas were offered to students, the extent to which they were explicit in the lessons, and the contributions of the teacher and the students to their development. Two of these ideas are general: refutation by a counter example as mathematically valid, and supportive examples for a universal statement as mathematically invalid – two ideas that are difficult for students (e.g., Balacheff, 1991; Fischbein & Kedem, 1982; Jahnke, 2008). Another idea is algebra specific: the use of properties and axioms in proving that two algebraic expressions are equivalent as mathematically valid.

Recent research suggests that different teachers enact the same curriculum materials in different ways (Manouchehri & Goodman, 2000), and that the same curriculum materials may be enacted differently in different classes taught by the same teacher (Eisenmann & Even, in press). Thus, we chose to focus here on the ways the proof ideas in the algebra lesson were offered to students (1) by different teachers, and (2) in different classes taught by the same teacher. This study is part of the research program *Same Teacher – Different Classes* (Even, 2008) that compares teaching and learning mathematics in different classes taught by the same teacher as well as classes taught by different teachers. Various aspects are examined, with the aim of gaining insights about the interactions between mathematics teachers, curriculum and classrooms.

PROOF IDEAS IN THE WRITTEN LESSON

The lesson appears in a 7th grade mathematics curriculum program developed in Israel in the 1990s (Robinson & Taizi, 1997). The curriculum program used by the teachers in this study is intended for heterogeneous classes and includes many of the characteristics common nowadays in contemporary curricula. One of its main characteristics is that students are to work co-operatively in small groups for much of the class time, investigating algebraic problems situations. Following small group work, the curriculum materials suggest a structured whole class discussion aimed at advancing students' mathematical understanding and conceptual knowledge. The curriculum materials include suggestions on enactment, including detailed plans for 45-minute lessons.

The lesson "Are they equivalent?", which is the focus of this paper, is the 6^{th} lesson in the written materials. Prior to this lesson, equivalent expressions were introduced as representing "the same story", e.g., the number of matches needed to construct a train of *r* wagons. The use of properties of real numbers (e.g., the distributive property) was mentioned briefly as a tool for moving from one expression to an equivalent one, but it was not yet presented explicitly as a tool for proving the equivalency of two given expressions.

Based on an analysis of the textbook and the teacher guiding, three proof-related ideas were found as being explicit in this lesson:

<u>Idea 1</u>: Substitution that results in different values proves that two expressions are not equivalent (a specific case of refutation by a counter example as mathematically valid).

<u>Idea 2</u>: Substitution cannot be used to prove that two given algebraic expressions are equivalent[3] (a specific case of supportive examples for a universal statement as mathematically invalid).

<u>Idea 3. It</u> addresses the problem that emerges from idea 2: the use of properties in the manipulative processes is a mathematically valid method for proving that two expressions are equivalent.

The lesson is planned to start with small group work aiming at an initial construction of Ideas 1 and 2. Students are given several pairs of expressions; some equivalent and some not. They are asked to substitute in them different numbers and to cross out pairs of expressions that are not equivalent. After each substitution they are asked whether they can tell for certain that the remaining pairs of expressions are equiva-

[3] Students were not familiar at that stage with the properties of linear expressions.

lent. Finally, students are instructed to write pairs of expressions, so that for each number substituted, they will get the same result.

Then small group work continues, asking students to write equivalent expressions for given expressions. The aim is to direct students' attention to the use of properties in relation to equivalence of algebraic expressions, which is relevant to idea 3.

The whole class work returns to idea1, and moves, through idea 2, to idea 3, aiming at consolidating these ideas, by discussing questions, such as: How can one determine that expressions are not equivalent? that expressions are equivalent? By substituting numbers? If so, how many numbers are sufficient to substitute? If not, what method is suitable? Finally, the teacher guide recommends that the teacher demonstrate the use of properties for checking equivalence, and together with the students implement this method on several pairs of expressions in order to check their equivalency.

Ideas 1, 2, and 3 are connected to three other ideas, none of which appears explicitly in the first six lessons in the written materials:

Idea 4 justifies Idea 2: There may exist a number that was not substituted yet, but its substitution in the two given expressions would result in different values, thus showing non-equivalence.

Idea 5 justifies Idea 3: The use of properties of real numbers in the manipulative processes guarantees that any substitution in two expressions will result in the same value, thus showing equivalence.

Idea 6 is the underpinning for Ideas 1, 2, and 3, as well as for Ideas 4 and 5. It defines equivalent algebraic expressions: Two algebraic expressions are equivalent if the substitution of any number in the two expressions results in the same value.

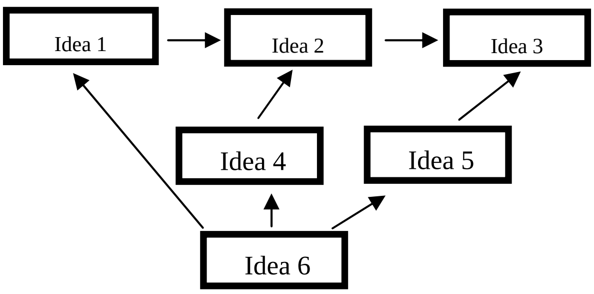

Figure 1: Connections among the proof-related ideas in the lesson

Ideas 4 and 6 are implicit in the written lesson, and Idea 5 does not exist.

METHODOLOGY

The primary data source include video and audio tapes of the enactment of the written lesson in four classes, each from a different school (i.e., four different schools). One teacher, Sarah, taught two of the classes, S1 and S2; another teacher, Rebecca, taught the other two classes, R1 and R2 (pseudonyms). The talk during the entire class work was transcribed. The transcripts were segmented according to focus on the six ideas, yielding 3-4 more or less chronological parts in each class. Next, the collective discourse in the classroom was analyzed by examining the contributions of the teacher and the students to the development of the proof ideas in each enacted lesson. We compared how the teachers structured and handled the proof ideas in each lesson, and what was available to learn in different classes of the same teacher and in the classes of the two teachers.

PROOF IDEAS IN THE ENACTED LESSONS

Idea 1

In line with the written curriculum materials, the whole class work in all four classes included an overt treatment of Idea 1. However, contrary to the recommendations in the written materials, in none of the classes did the whole class work begin with the question, how can one determine whether algebraic expressions are not equivalent. Instead, the students performed substitutions in pairs of algebraic expressions from Problem 1 because the teacher requested them to do so, and not as a way of addressing a problem. When the substitutions resulted in different values, the classes concluded that the two expressions were not equivalent. In all four classes, it was the teacher who eventually presented Idea 1 explicitly, attending only to the specific context of non-equivalence of expressions, with no reference to the general idea of refutation by using a counter example as mathematically valid.

Idea 2

After working on non-equivalence, the four classes proceeded to work on equivalence of algebraic expressions. In both of her classes Sarah presented Idea 2, that substitution cannot be used to prove that two given algebraic expressions are equivalent. She explicitly incorporated in the presentation of this idea its underlying justification (which does not appear explicitly in the written materials) that possibly there exists a number that was not yet substituted, but its substitution in the two given expressions would result in different values (idea 4). For example, Sarah said in class S1:

> We saw that with substitution, it is always possible that there is a number that I will substitute, and it will not fit. We can substitute ten numbers that would fit, and suddenly we will substitute one number that will not fit, and then the expressions are not equivalent… We have to find some way other than substitution, which will help us determine whether expressions are equivalent.

Contrary to the recommendations in the written materials, the students in Sarah's classes did not participate in constructing Idea 2 in class. Sarah merely presented it as motivation for finding a method to show equivalence, and immediately proceeded to work on using properties in the manipulative processes as a means to prove equivalence (Idea 3).

The idea that substitution cannot be used to prove that two given algebraic expressions are equivalent was dealt with differently in Rebecca's classes. In general, in both classes Rebecca pressed on finding a method that works, rather than evaluating the method of substitution, which does not work. However, the issue of substitution continued to be raised. In class R1, following the students' suggestion, the initial focus was on rejecting substitution because of the inability to perform substitution of all required numbers (an infinite number), as the following excerpt illustrates:

Rebecca: When will I be sure that these three [points to the pairs on the board] are indeed equivalent? That each pair is equivalent? When will I be sure?
S: When you check all the numbers.
…
S: There is an infinite number of numbers so you will never finish.
Rebecca: So I am not going to substitute infinite numbers. I need to find some other trick.

Idea 2, that supportive examples (i.e., substitution) could not be used to prove a universal statement (i.e., that two given algebraic expressions are equivalent), was not dealt with in class R1. Rather, it seemed to be taken as shared. Repeatedly, after substituting numbers in pairs of expressions and receiving the same value, the class concluded that the pairs appeared to be equivalent but that it was impossible to know for certain. For example,

Rebecca: OK, we are told to check another number, four.
S: Right.
Rebecca: You checked four. What did you get?
S: That they are equivalent.
Rebecca: I got the same result, right?
S: Yes, right.
S: All is well so far.

By stating, "I got the same result" following the statement "they are equivalent" Rebecca signaled that they did not yet know whether the latter claim was correct. Students then agreed, "All is well *so far* (emphasis added)". Later in the lesson, a similar conversation took place,

Rebecca: So, does it mean that they are equivalent?
S: Yes.
S: Yes. Ah, no, not necessarily.
Rebecca: Why? Do you have a counter example?
S: We don't know that they are equivalent.

Still, there was no explicit rejection of substitution for proving equivalence, as a specific case of supportive examples for a universal statement as mathematically invalid. Instead, Rebecca changed the focus of the activity to looking for a connection between the two algebraic expressions in each pair, as a transitional move towards Idea 3.

In contrast with class R1, class R2 embraced the idea that substitution is a valid means of determining equivalence of algebraic expressions. Unlike R1, where after several substitutions that resulted in the same value, students claimed that they still could not conclude that the two expressions were equivalent, in similar situations R2 students claimed that the expressions were equivalent because all the numbers they substituted resulted in identical numerical answers. This happened even after Rebecca offered idea 4, that there may be a number, which was not yet substituted, but its substitution in the two given expressions would result in different values. For example,

Rebecca: So, what do you say, what should I do, check all the numbers; maybe there is a number that won't fit here?

S: No [interrupts the teacher]

Rebecca: Or will it always fit?

S: Always.

…

Rebecca: Why are they equivalent? Why do I say that these are equivalent…?

S: Because we checked at least thirty.

Rebecca: We didn't check thirty, but I am asking: Why are these equivalent, in your opinion?

…

S: Because we checked.

Rebecca: Because you checked, but we said that maybe there is one number that you did not check.

S: But we checked almost all the [inaudible].

Eventually, Rebecca changed the focus of the activity to looking for algebraic expressions that are equivalent to given expressions, aiming at Idea 3. Thus, unlike Sarah, who used the brief mention of Idea 2 (and 4) as a motivational transition from Idea 1 to Idea 3, in R2, Rebecca did not motivate the search for a method different from substitution.

Idea 3

Led by Sarah, in line with the written materials, S1 and S2 searched for properties that show that the expressions produced when working on Problem 1 (S1), or given in Problem 3 (S2), were equivalent. Sarah then stated that the use of properties is the way to show equivalence, not substitution. When introducing Idea 3 in S2, Sarah explicitly connected with Ideas 5 and 6, which underpin and justify Idea 3. However, no such connections were made then in S1. Only later on, in her concluding remarks in S1, when summarizing both ways of proving equivalence and non-equivalence of expressions, did Sarah explicitly propose Idea 6.

Class R1 started to work on Idea 3 by searching for connections between pairs of expressions from Problem 1 that remained as potentially consisting of equivalent expressions. The class then quickly embraced the discovery that by using properties, it was possible to move from one expression to another, by indicating equivalence. Rebecca then introduced explicitly Idea 3. However, in R1, like in S1, no connections were made then to Ideas 5 and 6. Nevertheless, Idea 6 was introduced explicitly at the beginning of the lesson, when a student asked for the meaning of equivalence expressions.

Class R2 had a different starting point than R1 for treating Idea 3 because the students were confident that based on the substitutions they performed they could infer that the remaining pairs of expressions from Problem 1 were equivalent. Rebecca then slightly deviated from the written materials' suggestions and asked the students to find new expressions that would be equivalent to the given ones. Eventually, R2 embraced the idea that equivalence can be determined by manipulating the form of expressions, using properties. In R2, too, no connections were made with Ideas 5 and 6. Moreover, Idea 6 was not proposed at all.

Figure 2 depicts the teaching sequences of the proof-related ideas as offered during the whole class work, in the written materials, as well as in the four classes.

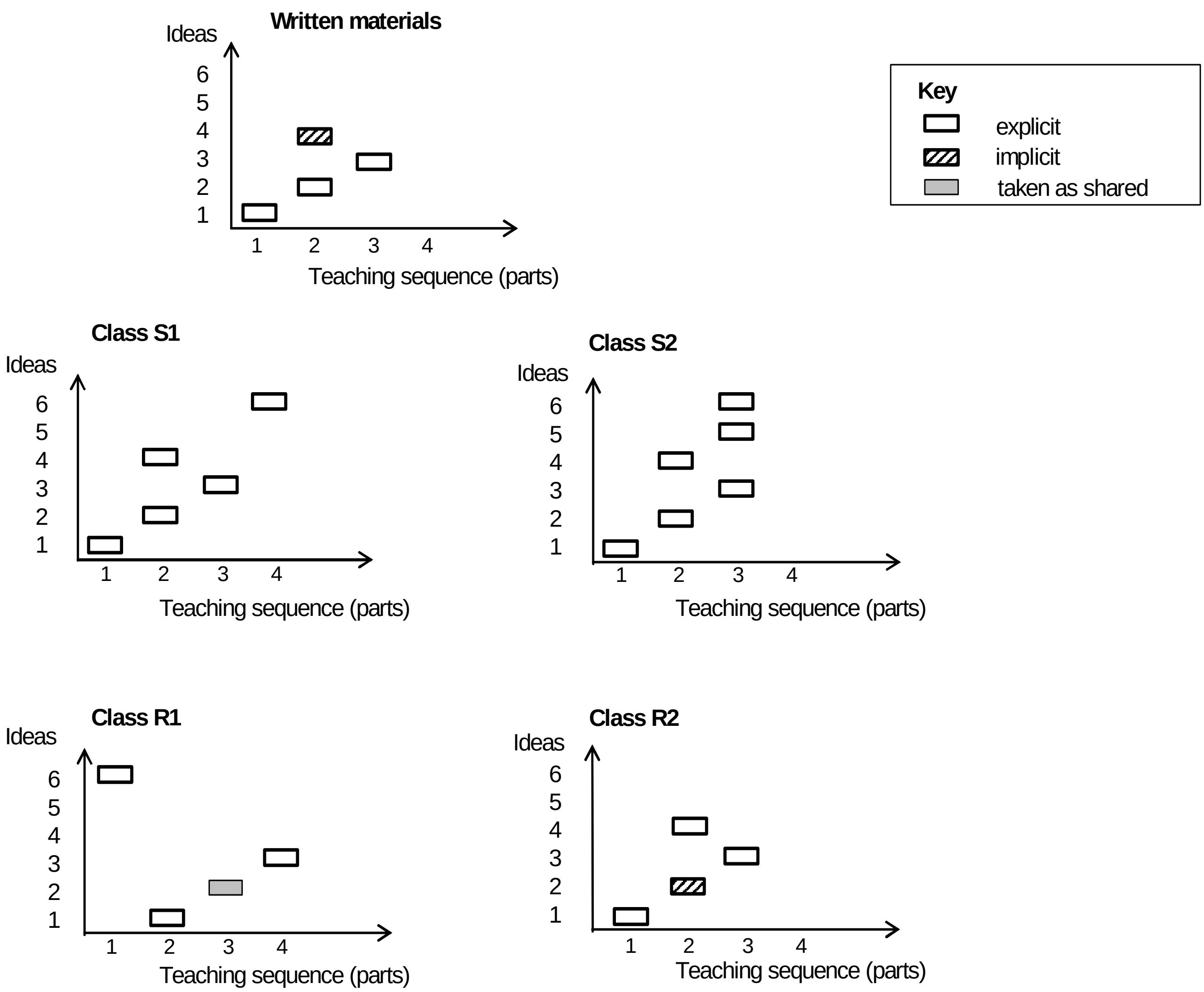

Figure 2: Teaching sequences of the proof-related ideas, as offered in the whole class work, in the written materials, as well as in the classes

The figure clearly demonstrates that Sarah was the only one who explicitly proposed the sequence of the three proof-related ideas (1, 2, and 3) that were explicit in the written lesson, whereas Rebecca explicitly proposed only Ideas 1 and 3. Moreover, any connections between these three ideas and the other three ideas (4, 5, and 6), which did not appear explicitly in the written lesson, were made only in Sarah's classes: Idea 2 was connected to its underlying justification, Idea 4 in both of Sarah's classes, whereas Idea 3 was connected to its underlying support by Ideas 5 and 6 in S2 only. Nevertheless, Idea 4 was offered by Rebecca in R2 with no explicit connection to Idea 2, and Idea 6 was offered in S1 (at the end of the lesson) and in R1 (at the beginning of the lesson), with no explicit connections to the other ideas.

FINAL REMARKS

Sarah and Rebecca taught the written lesson “Are they equivalent?” using the same written materials, which included a detailed lesson plan. Thus, it is not surprising that the mathematical problems enacted in class were similar in all four classes. However, the ways the proof ideas in the lesson were offered to students differed to some degree from what was recommended in the written materials. There were also differences between the two teachers, and between the two classes of the same teacher, in what was available to learn in the lesson. One of the main differences is related to offering Idea 2. This idea is central in the written materials. However, Sarah only briefly mentioned it in her classes, just as a transition to Idea 3. In R1 this idea was taken as shared, never made explicit, as was the case in R2, which strongly embraced the opposite idea. Another central idea in the written materials is Idea 3. The way that the written materials deal with Idea 3, without making Ideas 5 and 6 explicit, seemed to make teaching it a challenge. Eventually, each teacher handled this idea somewhat differently in each of her two classes.

These differences seem to be related to differences in teaching approaches. Sarah tended to make clear presentations of important ideas. Rebecca hardly made presentations, but instead, attempted to probe students, expecting them to explicate these ideas. Thus, some ideas were never made explicit, in one class more than the other, because of differences in students’ mathematical behaviour and performance.

These initial findings illustrate the complexity of the interactions among teachers, curriculum and classrooms (Even, 2008). Rebecca faced serious challenges in her attempts to make students genuine participants in the construction of mathematical ideas, as was recommended in the written materials – more so in one of her classes – challenges that lie at the meeting point of the specific teacher, specific curriculum and specific class. Sarah, who chose to make clear presentations of the mathematical ideas, faced different challenges, even though she used the same materials.

The mere fact that different teachers offer mathematics to learners in different ways, even when using the same written materials, is not entirely surprising, and has been documented by empirical research (e.g., Manouchehri & Goodman, 2000). Nonetheless, the nature of the differences is important because what people know is defined by ways of learning, teaching, and classroom interactions, as documented by Boaler (1997). Consequently, Sara'h and Rebecca's students were offered somewhat different proof-related ideas that are central in algebra and in mathematics in general, and that are known as not being easy for students. Furthermore, when instead of focusing solely on the comparison between teachers, different classes taught by the same teacher were also compared, important information was revealed about the interactions among curriculum, teachers and classrooms**.**

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RAFAEL BOMBELLI'S ALGEBRA (1572) AND A NEW MATHEMATICAL "OBJECT": A SEMIOTIC ANALYSIS

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In the theoretical framework based upon the ontosemiotic approach to representations, some reflections by Radford, and taking into account Peirce's semiotic perspective, I proposed to a group of 15–18 years–old pupils an example from the treatise entitled Algebra (1572) by Rafael Bombelli. I conclude that the historical analysis can provide insights in how to approach some mathematical concepts and to comprehend some features of the semiosic chain.

INTRODUCTION

In this paper I shall examine a traditional topic of the curriculum of High School and of undergraduate Mathematics that can be approached by historical references. The introduction of imaginary numbers is an important step of the mathematical curriculum. It is interesting to note that, in the Middle School, pupils are frequently reminded of the impossibility of calculating the square root of negative numbers. Then pupils themselves are asked to accept the presence of a new mathematical object, "$\sqrt{-1}$", named *i*, and of course this can cause confusion in students' minds. This situation can be a source of discomfort for some students, who use mathematical objects previously considered illicit and "wrong". The habit (forced by previous educational experiences) of using only real numbers and the (new) possibility of using complex numbers are conflicting elements.

Although the focus of this paper is not primarily on the analysis of empirical data, I shall consider an educational approach based upon an historical reference that can help us to overcome these difficulties. More particularly, I shall consider the semiotic aspects of the development of the new mathematical objects introduced (imaginary numbers) and I shall ask: can we find an element from which the semiosic chain is originated? Can we relate the early development of the semiosic chain to the objectualization of the solving procedure of an equation?

THEORETICAL FRAMEWORK

Radford describes "an approach based on artefacts, that is, concrete objects out of which the algebraic *tekhnē* and the conceptualization of its theoretical objects arose. […] They were taken as *signs* in a Vygotskian sense" (Radford, 2002, § 2.2). In this paper I shall not consider concrete objects. Nevertheless Radford's remark about the importance of "signs in a Vygotskian sense" can be considered as a starting point of my research.

When we consider a sign, we make reference to an object, and in the case of mathematical objects, to a concept. However my approach does not deal only with "concepts". Font, Godino and D'Amore (2007, p. 14) state that although "to understand representation in terms of semiotic function, as a relation between an expression and a content established by 'someone', has the advantage of not segregating the object from its representation, […] in the onto–semiotic approach […] the type of relations between expression and content can be varied, not only be representational, *e.g.*, 'is associated with'; 'is part of'; 'is the cause of/reason for'. This way of understanding the semiotic function enables us great flexibility, not to restrict ourselves to understanding 'representation' as being only an object (generally linguistic) that is in place of another, which is usually the way in which representation seems to us mainly to be understood in mathematics education".

In my research I shall consider the ontosemiotic approach to mathematics cognition. It "assumes socio-epistemic relativity for mathematical knowledge since knowledge is considered to be indissolubly linked to the activity in which the subject is involved and is dependent on the cultural institution and the social context of which it forms part" (Font, Godino & D'Amore, 2007, p. 9, Radford, 1997).

My framework is also linked with some considerations about semiotic aspects, based upon a Peircean approach (although, for instance, the relationship between Vygotsky and Peirce is not trivial: Seeger, 2005). According to Peirce we cannot "think without signs", and signs consist of three inter–related parts: an object, a proper sign (*representamen*), and an interpretant (in Peirce's theory sign is used for both the triad "object, sign, interpretant" and the representamen, in late works). Peirce considered either the immediate object represented by a sign, or the dynamic object, progressively originated in the semiosic process. As a matter of fact, an interpretant can be considered as a new sign (*unlimited semiosis*). The limit of this process is the *ultimate logical interpretant* and it is not a real sign, which would induce a new interpretant. It is an *habit–change* ("meaning by a habit–change a modification of a person's tendencies toward action, resulting from previous experiences or from previous exertions of his will or acts, or from a complexus of both kinds of cause": Peirce, 1931–1958, § 5.475. I shall cite paragraphs in Peirce's work).

The sign determines an interpretant by using some features of the way the sign signifies its object to generate and shape our understanding. Peirce associates signs with cognition, and objects ("mathematical objects" will be considered as "objectualized procedures": Sfard, 1991, Giusti, 1999) "determine" their signs, so the cognitive nature of the object influences the nature of the sign. If the constraints of successful signification require that the sign reflects some qualitative features of the object, then the sign is an *icon*; if they require that the sign utilizes some physical connection between it and its object, then the sign is an *index*; if they require that the sign utilizes conventions or laws that connect it with its object, then it is a *symbol*.

According to Peirce, the formulas of our modern algebra are icons, i.e. signs which are mappings of that which they represent (Peirce, 1931–1958, § 2.279). Nevertheless pure icons, according to Peirce himself (1931–1958, § 1.157), only appear in thinking, if ever. Pure icons, pure indexes, and pure symbols are not actual signs. In fact, every sign "contains" all the components of Peircean classification, although one of them is predominant. So our algebraic expressions are *complex icons* (Bakker & Hoffmann, 2005). Moreover, it is worth noting that a sign in itself is not an icon, index or symbol. From the educational viewpoint, the identification of signs is not just a question of classifying a sign as *e.g.* an icon, but it is a question of showing their cognitive import (Bagni, 2006).

Frequently Peirce underlined the importance of iconicity. He argued (1931–1958, § 3.363) that "deduction consists in constructing an icon or diagram the relations of whose parts shall present a complete analogy with those of the parts of the object of reasoning, of experimenting upon this image in the imagination, and of observing the result so as to discover unnoticed and hidden relations among the parts". (Peirce distinguished three kinds of icons: images, metaphor, and diagrams). According to Radford (forthcoming), since the epistemological role of "diagrammatic thinking" rests in making apparent some hidden relations, it relates to actions of *objectification*, and a diagram can be considered a *semiotic means of objectification*.

HISTORY OF MATHEMATICS AND IMAGINARY NUMBERS

History of mathematics can inform the didactical presentation of topics (although the very different social and cultural contexts do not allow us to state that ontogenesis recapitulates phylogenesis: Radford, 1997). Let us consider the resolution of cubic equations according to G. Cardan (1501–1576) and to N. Fontana (Tartaglia, 1500–1557). R. Bombelli (1526–1573), too, is one of the protagonists of history of algebra. His masterwork is *Algebra* (1572), where we find some cubic equations, and sometimes their resolution makes it necessary to consider imaginary numbers.

The resolution of the equation $x^3 = 15x+4$ leads to the sum of radicals $x = \sqrt[3]{2+11i} + \sqrt[3]{2-11i}$ where $2+11i = (2+i)^3$ and $2-11i = (2-i)^3$. So a (real) solution of the equation is $x = (2+i)+(2-i) = 4$. In the following image (Fig. 1) I propose the original resolution on p. 294 of Bombelli's *Algebra*.

$x^3 = 15x+4$

$[x^3 = px+q]$

$(4/2)^2-(15/3)^3 = -121$

$[(q/2)^2-(p/3)^3 = -121]$

$x = \sqrt[3]{2+11i} + \sqrt[3]{2-11i}$

$x = (2+i) + (2-i) = 4$

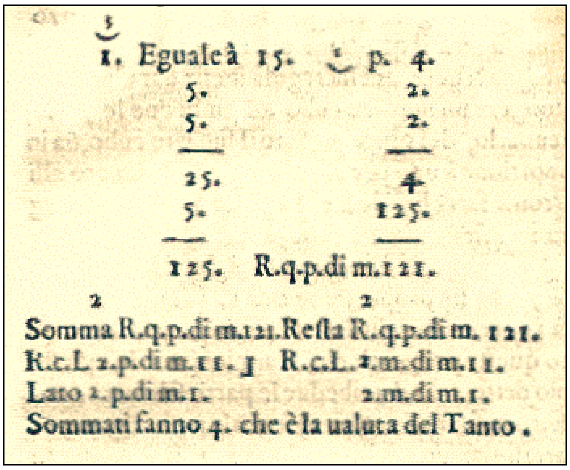

1. Eguale à 15. p. 4.
5. 2.
5. 2.
25. 4
5. 125.
125. R.q.p.di m.121.
Somma R.q.p.di m.121. Resta R.q.p.di m. 121.
R.c.L 2.p.di m.11.⌋ R.c.L.2.m.di m.11.
Lato 2.p.di m.1. 2.m.di m.1.
Sommati fanno 4. che è la ualuta del Tanto.

Fig.1

Bombelli justified his procedure using the two–dimensional and three–dimensional geometrical constructions (1966, pp. 296 and 298, Fig. 2 and Fig. 3 respectively). (Space limitations prevent a detailed discussion of these. The reader is referred to Bombelli: Bombelli, 1966).

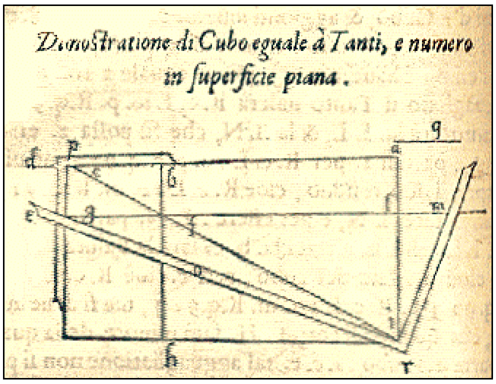


Fig. 1

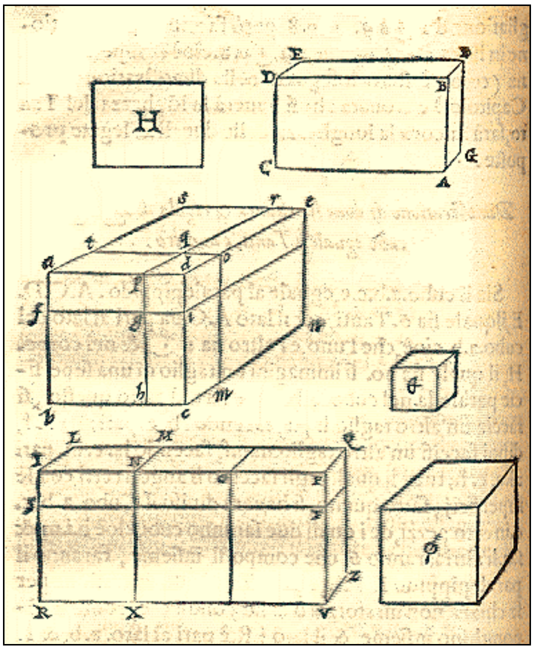


Fig. 2

From the educational point of view, Bombelli's resolution can help our pupils to accept imaginary numbers. As a matter of fact, its effectivity supports Bombelli's rules for *pdm* and *mdm* ("più di meno" and "meno di meno" respectively, today written as *i* and *–i*. In the image see the original "rules" as listed on p. 169 of Bombelli's *Algebra*, Fig. 4).

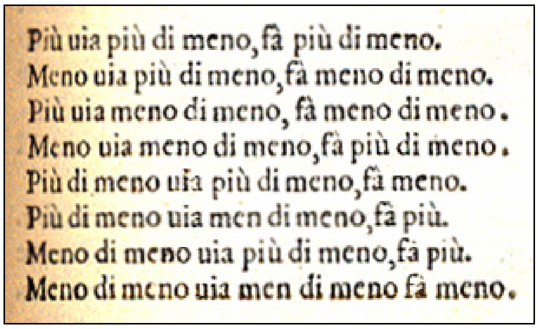

Più uia più di meno, fà più di meno.
Meno uia più di meno, fà meno di meno.
Più uia meno di meno, fà meno di meno.
Meno uia meno di meno, fà più di meno.
Più di meno uia più di meno, fà meno.
Più di meno uia men di meno, fà più.
Meno di meno uia più di meno, fà più.
Meno di meno uia men di meno fà meno.

Fig. 3

IMAGINARY NUMBERS FROM HISTORY TO DIDACTICS

It is worth noting that the introduction of imaginary numbers, historically, did not take place in the context of *quadratic* equations, as in $x^2 = -1$. It took place by the resolution of *cubic* equations, whose consideration can be advantageous. Their resolution, sometimes, does not take place entirely in the set of real numbers, but one of their results is always real. A substitution of $x = 4$ in the equation above ($4^3 = 15\cdot4+4$) is possible in the set of real numbers. In the quadratic equation, the role of *i* and of *–i* seems very important. As a matter of fact results themselves are not real, so their acceptance needs the knowledge of imaginary numbers.

Let us briefly summarize the results of an empirical research. In a first stage I examined 97 3rd and 4th year High School students (Italian *Liceo scientifico,* pupils aged 16–17 and 17–18 years, respectively). In all the classes, at the time of the test, pupils knew the resolution of quadratic and of biquadratic equations, but they did *not* know imaginary numbers. Responding to a question about the statement $x^2+1 = 0 \Rightarrow x = \pm i$ only 2% accepted the resolution (92% refused it; 6% did not answer). A subsequent question proposed the following as a resolution of the cubic equation $x^3-15x-4 = 0 \Rightarrow x = \sqrt[3]{2+11i} + \sqrt[3]{2-11i} \Rightarrow x = (2+i) + (2-i) = 4$. This resolution was accepted by 54% of the pupils (35% refused it; 11% did not answer).

So imaginary numbers *in the passages* of the resolution of an equation, but *not* in its result, are frequently accepted by pupils (the *didactical contract* ascribes great importance to the result). Under the same conditions, a similar test was then administered to 52 students of the same age group, where the equations were presented in the reverse order (Bagni, 2000): 41% accepted the solution of the cubic equation (25% rejected it and 34% did not answer). Immediately after that, the solution of the quadratic equation was accepted by 18% of the students, with only 66% rejecting it (16% did not answer).

These data suggest that teaching a subject using insights from its historical development may help students to acquire a better understanding of it.

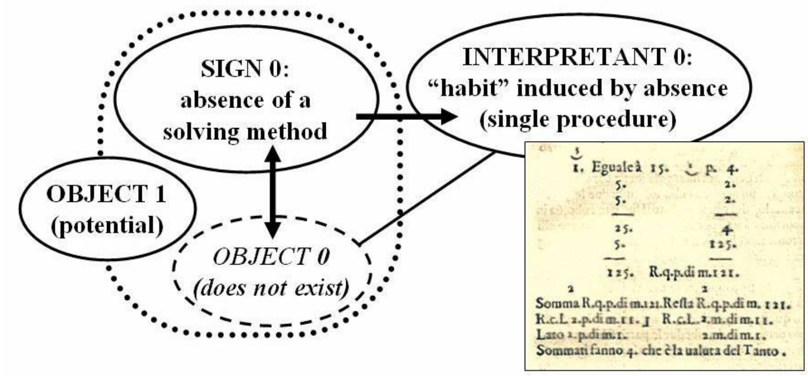
THE SEMIOSIC CHAIN

As previously noticed, this focus of this paper is not the detailed presentation of this experimental data (see, Bagni, 2000). Rather I shall consider some features of students' approach, making reference, in doing so, to Peirce's *unlimited semiosis*. As highlighted in section 2, every step of the interpretative process produces a new "interpretant n" that can be considered the "sign n+1" linked with the object (considered in the sense of an objectualized procedure, following Sfard, 1991, and Giusti, 1999, p. 26). However we must ask ourselves: what about the very first sign to be associated to our object?

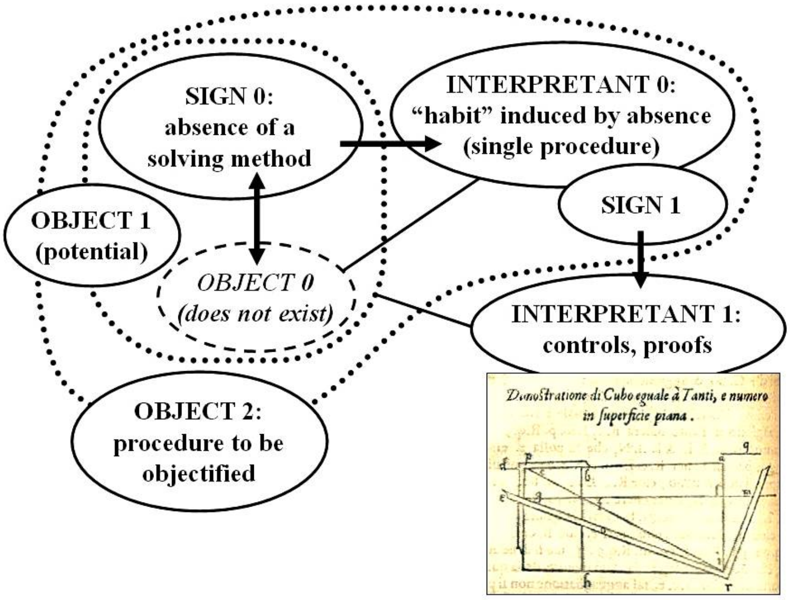
Our mathematical object (in this case, a procedure to solve an equation) would be represented by a first "sign". In fact, "absence" itself can be considered as a sign. Peirce (1931–1958, § 5.480) made reference to "a strong, but more or less vague, sense of need" leading to «the *first logical interpretants* of the phenomena that suggest them, and which, as suggesting them, are signs, of which they are the (really conjectural) interpretants». So I suppose that this kind of absence can be the starting point of the semiosic process.

From an educational viewpoint this is influenced by important elements, e.g. the theory in which we are working, the persons (students, teacher), the social and cultural context. Of course by that I do not mean that there is a unique historical trajectory for every "mathematical object". Nevertheless this starting point can be described as a complexus of "object–sign–interpretant" without a particular "chronological" order. It can be considered a *habit* linked to the absence of a procedure, or, better, *a procedure to be objectualized*. So the situation is characterized by some intuitive sensations, and by the influence of social, cultural, traditional elements. Later, with the emergence of formal aspects, our object will become more "rigorous" (making reference, of course, to the conception of rigor in an historical and cultural context – the rigor for Bombelli and the rigor for modern mathematicians are different). These stages are educationally important.

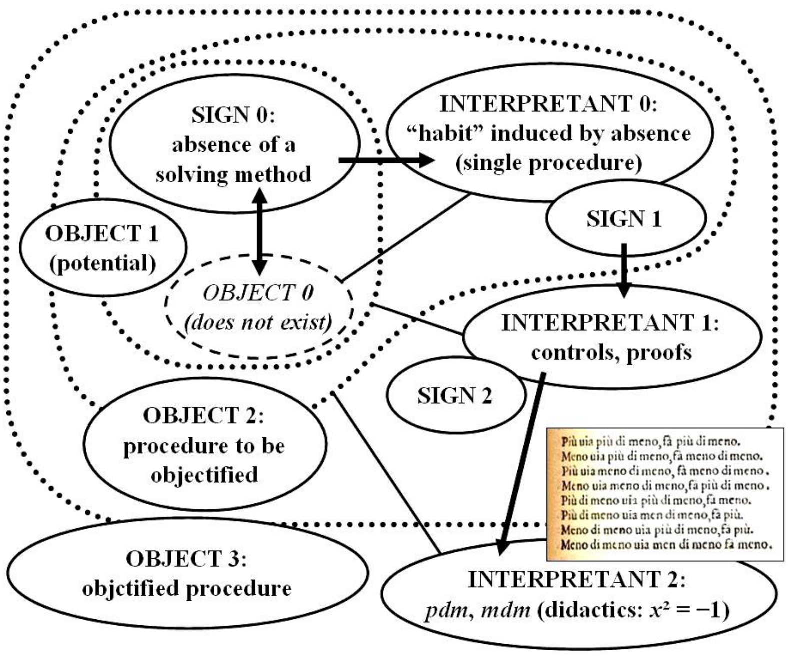
According to an ontosemiotic approach, knowledge is linked to the activity in which the subject is involved and it depends on the cultural institution and the context (Font, Godino & D'Amore, 2007, Radford, 1997). In the case considered, pupils have the perception of an absence, referred to the strategy to be followed, namely the procedure to be objectualized. Historical references gave them the opportunity to consider a situation, and the context is characterized by the "game to be played" (the resolution of an equation) at the very beginning of our experience. We cannot make reference to a semiotic function related to an object to represented. The "object" will be considered just later, on the basis of the solving strategy. A real strategy is actually absent, and only a "potential object" is connected to the possibility to find out an effective procedure in order to play the (single) game considered.



In Bombelli's work the iconicity has a major role, and this aspect can be relevant to students approach (further research can be devoted to this issue). Educationally speaking, in this stage the effectiveness of the procedure is fundamental. There is not a real mathematical object to be considered, nevertheless pupils have a "game to be played", and this can be considered as a sign (sign 1). Now controls and proofs are needed, and geometrical constructions can be considered as an interpretant (interpretant 1). So the possibility to provide a first "structure" to the strategy (e.g. the consideration of standard actions) makes it to become a procedure to be objectualized.



Both from the historical viewpoint (let us remember the aforementioned Bombelli’s geometrical constructions) and from an educational viewpoint (with reference to the substitution of the result, $x = 4$, in the given equation, $x^3–15x–4 = 0$ so $4^3–15·4–4 = 0$), a first objectualization can be pointed out. The experience considered do not allow to state that pupils reach a complete objectualization. In the following picture, the interpretant 2 is related to an objectualized procedure and it is referred to the “rules” listed by Bombelli (as noticed, only some students accepted them).



Later, the strategy will become an autonomous object and its transparency (in the sense of Meira, 1998) will be important from the educational point of view. It will not be linked to a single situation and it will be applied to different cases (Sfard, 1991). This stage can be characterized by the emergence of a *schema* of action (Rabardel, 1995).

According to Font, Godino and D'Amore (2007, p. 14), "what there is, is a complex system of practices in which each one of the different object/representation pairs (without segregation) permits a subset of practices of the set of practices that are considered as the meaning of the object". The starting point of the semiosic chain can hardly be considered in the sense of semiotic function. It can be considered as a first practice that will be followed by other practices in order to constitute the meaning of the object.

FINAL REFLECTIONS

In my opinion the importance of an ontosemiotic approach to representations can be highlighted by a Peircean (or post–Peircean) perspective giving sense to the starting point of the semiosic chain. The analysis of this stage of the semiosic chain can help us to comprehend both our pupils' modes of learning and the essence of mathematical objects themselves.

Nevertheless, from a cognitive viewpoint, the question is not only to show how a process becomes an object. The main problem is to understand how signs become meaningfully manipulated by the students, through social semiotic processes. It is also important to notice that Peircean semiotics seems not completely suited to account for the complexity of human processes in problem–solving procedures. In fact, we do not go always from sign to sign, but more properly from complexes of signs to complexes of signs (and usually they are signs of different sort: gestures, speech, written languages, diagrams, artifacts, and so on).

According to L. Radford and H. Empey, «mathematical objects are not pre–existing entities but rather conceptual objects generated in the course of human activity». It is worth noting that "that mathematics is much more than just a form of knowledge production – an exercise in theorization. If it is true that individuals create mathematics, it is no less true that, in turn, mathematics affects the way individuals are, live and think about themselves and others" (p. 250). As a matter of fact, a strategy to be objectualized can influence pupils' approaches both to mathematical tasks and to different (non–mathematical) activities: "within this line of thought, in the most general terms, mathematical objects are intellectual or cognitive tools that allow us to reflect upon and act in the world" (p. 250). These remarks lead us to reflect about the importance of "mathematical objects" and of their representations. They were conceived by mathematicians in the history, they are reprised and re–invented by our pupils today. So they affected – and, nowadays, affect – "all of society and not only those who practice it in a professional way" (p. 251).

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COGNITIVE CONFIGURATIONS OF PRE-SERVICE TEACHERS WHEN SOLVING AN ARITHMETIC-ALGEBRAIC PROBLEM

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The objective of this paper is to describe the cognitive configurations exhibited by the students when solving word problems which could be solved using arithmetic-algebraic methods. The configurations will be described in terms of theoretic elements provided by the onto-semiotic approach to mathematics knowledge and instruction.

Key words: elementary algebraic reasoning, cognitive configurations, primary teachers, didactic reflection.

INTRODUCTION

A number of researchers recommend the incorporation of elementary algebraic reasoning at different levels of primary education (e.g., Booth, 1988). Carraher and Schliemann (2007) state that algebra at the primary school is not simply a subset of the high school syllabus; rather, it is a rich sub-domain of mathematics education with its own approaches and problems to research.

The introduction of student primary teachers to elementary algebraic reasoning is a long and complex process (Van Dooren, Verschaffel and Onghema, 2003). It is considered that primary teachers should be able to recognize and to foster the algebraic reasoning manifested spontaneously by their students (Carraher and Schlieman, 2007). Therefore, research about fostering elementary algebraic reasoning in student teachers is of great relevance to initial teacher education (Borko et al, 2005).

On this research domain there are two questions posed by Carraher and Schliemann (2007, p.675): ‘can young students really deal with algebra?’ and, ‘can elementary school teachers teach algebra?’. Some researchers have tackled the second question. For example, Schmidt and Bernarz (1997) detail student teachers’ resistance and conflicts in the passage from arithmetic reasoning to algebraic reasoning. Similar findings are reported by Van Dooren et al. (2003).

Our purpose is to present the initial findings of a student teachers educational proposal on mathematics reasoning. The proposal offers opportunities to student teachers to develop didactic analysis knowledge (Godino, J. D., Rivas, M., Castro, W. F. y Konic, P, 2008) that could aid student teachers to recognize and to foster elementary algebraic reasoning in their pupils.

We focus the attention on the notion of cognitive configuration introduced by the “onto-semiotic approach”, OSA, (Godino, Batanero, and Roa, 2005; Godino, Batanero, and Font, 2007) to characterize the mathematic knowledge that is mobilized in order to solve an arithmetic-algebraic problem. We consider that this notion offers a wider view of the construct of strategy by considering the conceptual, propositional, argumentative, representational and situational aspects of knowledge alongside the traditional procedural approach.

INSTITUTIONAL CONTEXT AND METHODOLOGY

The research has been carried out with a sample of 94 primary student teachers enrolled in a mathematics method course at University of Granada, Spain. The course aims to develop mathematical knowledge as well as didactical reflection. It is to mention that algebra as such was not studied in the course. During the course several mathematical problems that could be solved using elementary algebraic reasoning were given to students. In this paper we analyze the students' solutions to one of these problems which were given during a test.

> *A ball is thrown from an unknown altitude; it bounces up to one fifth of the altitude it was thrown from. If after three rebounds the ball reaches an altitude of 6 cm, a) What is the altitude it fell from the first time?, b) Explains the resolution using algebraic notation.*

The problem belongs to a category of very well studied word problems. However, within the framework of this course, we are specifically interested in the arithmetic and algebraic solutions provided spontaneously by students.

EPISTEMIC ANALYSIS OF THE PROBLEM[4]

The OSA focuses on five dimensions in analysing the objects and meanings used in solving a mathematical problem: linguistic objects, concepts, properties, procedures and arguments. In what follows we analyse the problem using OSA[5]. This analysis has two purposes for the teacher educator: to explore the objects and meanings put into effect during the solution of the problem, and to identify eventual meaning conflicts and to foresee difficulties and errors that could emerge in students' solutions to similar problems.

The word problem is stated in terms of linguistic elements, which refer to quantities, magnitudes and relationships between them. These can be expressed in arithmetic or algebraic terms.

The statement "*A ball is thrown from an unknown altitude*" refers both to a real experience and to the unknown value of a quantity. Next it enounces a condition "*it bounces up to one fifth of the altitude it was thrown from*" that establishes the numeric relationship, invariant during the bouncing, between the altitude the ball falls from and the altitude to which it bounces, expressed by the fraction 1/5.

[4] To see an example of such analysis, we refer the readers to the work of Godino et al. (2008).

[5] A priori analysis of the solution to the problem done by an expert.

The statement “*If after three rebounds the ball reaches an altitude of 6 cm*” establishes that the numeric relationship is compounded three times with itself, fraction of fraction. Additionally it assigns a value to the last altitude.

Finally the statement, “*What is the altitude it fell from the first time?*” establishes the quantity that must be identified in terms of the given information in the problem wording.

The linguistic terms refer to mathematic concepts (e.g., fraction, equality, unknown, operation), whose meanings, properties and procedures are related argumentatively in a complex way and favors or inhibits the solution to the problem.

It is worth to mention that both the eventual arithmetic and algebraic solutions place the primary entities in different configurations. For instance, in an arithmetic solution, if it is assumed that 6 is the fifth of an unknown quantity, then we can find the unknown quantity by multiplying for five, inverting the fractioning operation used initially. However, in an algebraic solution, it is not necessary to use either this property or the associated concept. The unknown quantity is multiplied, three times, by 1/5 and this is equated to 6. Subsequently the unknown is isolated using a procedure that frames the solution in terms of multiplication/division.

COGNITIVE ANALYSIS OF THE STUDENTS’ SOLUTIONS

In what follows we will describe our typology of cognitive configurations evident in the solutions produced by the students. In each case, we identify the mathematical objects and meanings used by the students in representing their solutions.

Algebraic configurations[6]

Algebraic solutions are those where the use of unknowns is clearly manifested. The types of algebraic solutions are: use of unknown, assigning tags to equations, use of three unknowns, and additive relationships.

<u>ALC1[7]: Use of unknown.</u> On this type of procedure the unknown appears explicitly written and it is isolated. The students have attributed meaning to the linguistic objects “a bounce” and “*If after three rebounds*”, and have represented such linguistic elements in procedural objects, this can be deduced from the actions carried out on fractions, on relationships established and expressed by the equal sign and, finally, on isolating the unknown.

<u>ALC2: Assigning tags.</u> Students explicitly associate each rebound with an equation. They use a process made of three steps: initially identify the unknown “*altitude the ball fell from*” which is named *x*, later name the equation corresponding the first bounce as “*first rebound*”, and so two times more, up to the point where they write the equation that corresponds to the third bounce, and name it “*third rebound*”, equate to six and obtain the sought value.

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[6] See Godino et al. 2008.)
[7] The code ALC and ARC stands for algebraic and arithmetic configurations, respectively.

Every solution on this category is correct. It seems that students control the alleged difficulty that rises when dealing with unknowns by assigning a tag that lets them to isolate each rebound, represented linguistically, and at the same time allocated it in the problem context. On this type of solution the students have isolated the linguistic object *“it bounces up to one fifth of the altitude it was thrown from*”, and have identified it as an operative invariant in the whole process and have given it a procedural role expressed by multiplying by one fifth.

The procedural and linguistic objects are materialized argumentatively through the appropriate use of the equality in its relational meaning and by means of numerical operations and properties that are carried out on the equation with the purpose of isolating the unknown.

ALC3: Use of three unknowns. Students use three unknowns, each one of them associated to the unknown’s numerical values corresponding to each bounce. Then they propose an equation and they execute a nested replacement of variables, from the expression corresponding to the last one up to the expression corresponding to the first bounce, and they proceed to isolate the unknown.

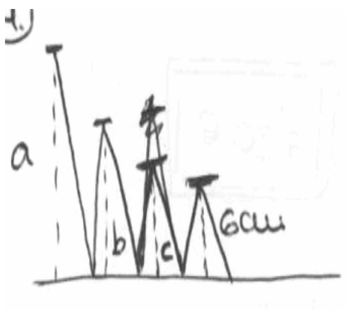
The problem is tackled by means of a procedure that breaks up it in three moments; the first and the second are represented by an equation with two unknowns, and the third, by an equation with one unknown. The mastering of linguistic elements that describe the relationships is predominant on this procedure.

The possible meaning conflicts on the description of the problem are overcome by assigning a semiotic function, whose antecedent corresponds to each and every bounce, and the consequent is a relationship, expressed as an equation.

On this procedure the students operate “with” and “on” the unknown (Tall, 20001) and spontaneously use the transitive property of equality (Filloy, Rojano and Solares, 2004).

It is observed, on this solution strategy, the use of procedures on two levels, the first that involves the “process” of dividing the problem in three parts, and the second, the use of properties and procedures, in the usual manner as mathematical procedures are used. This type of solution is illustrated on Figure 1.[8]

[8] A translation is provided



a is the initial height from which the ball is thrown. Each bounce *a, b,* 6 cm is 1/5 of the previous bounce. We isolated the first equation in order to substitute it in the others.

$$6=\frac{1}{5}of\ c=\frac{c}{5};\quad c=6.5=30;\ 30=\frac{1}{5}of\ b=\frac{b}{5};\quad b=30.5=150$$

$$150=\frac{1}{5}of\ a=\frac{a}{5};\quad a=150.5=750, the\ initial\ height\quad 750\,cm$$

Figure 1. Use of three unknowns (ALC3)

<u>CAL4: Additive relationships.</u> On this type of solution, the students use an unknown and produce expressions and equations that relate arithmetic data by means of additive expressions. Some students wrote expressions (not equations) to represent the problem. The operative invariant “one fifth” appears multiplying the unknown that is operated, additively with the numbers three and six but without establishing a relationship expressed by an equation. In some cases the fragility of knowledge about properties of rational numbers is manifested.

In some other solutions it can be seen that some relationships are proposed among the numerical values “three” and “six”, where “one fifth” multiplies the unknown, the students identify the presence of an unknown and recover the numbers out of the problem wording, however they do not related them in any way.

Arithmetic configurations

Arithmetic solutions were classified as those where only arithmetic operations are used without any reference to unknowns. The types of arithmetic solutions identified are: Reverse multiplication, multiplicative relationship, additive relationship, and rule of three.

<u>ARC1: Reverse multiplication.</u> The solution procedure consists of inverting the operation: it is known that the altitude to which the ball bounces is one fifth of the altitude it was thrown from, as 6 is the last altitude, therefore the previous altitude is 6x5 and the previous altitude to the last one is 6x5x5. Finally the altitude the ball was thrown from is: 6x5x5x5.

Students using ARC1 exhibit competence and fluency in the use of the multiplication operation in the context of known quantities. It is of note that this aspect of “operation sense” underlies algebraic thinking Slavit (1999, p.256).

On this category are located the right arithmetic answers given by the students. The only meaning conflict found on some answers is considering four bounces instead of three. Figure 2 illustrates this type of solution.

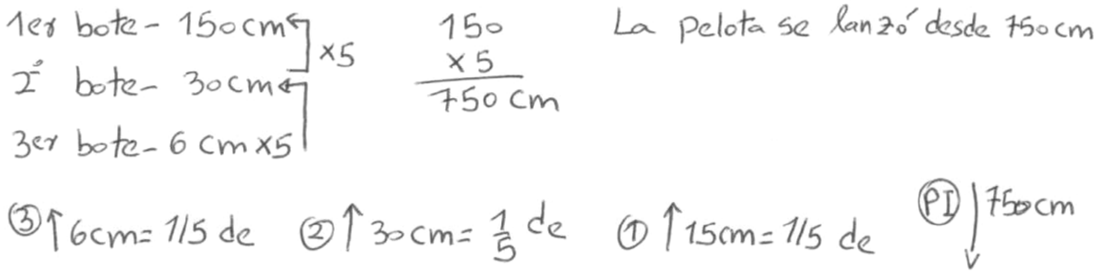


Figure 2. Reverse multiplication (ARC1)[9]

<u>ARC2: Arbitrary use of multiplication</u>. Students focus their attention simply on the numbers contained in the problem: 6, 3 and 5, and the solution they offer is an arbitrary combination of multiplicative operations among these three numbers. The students appear to construct their solution without paying any attention either to the conditions on numbers or to relationships among them. According to Garolafo (1992), these students do not exhibit a "numeric approach", because they do not display strategies neither to decide which operations to use nor to assess a plan to solve the problem.

It is deduced from the students´ solutions that they have not comprehended the meaning, in operative terms, of the linguistic objects "*first*", "*second*" and "*third*" bounce, nor in relational terms of "*If after three rebounds the ball reaches an altitude of 6 cm*". The students are incapable of expressing numerically the relationships present in the problem.

The two approaches to rational numbers duplicator/partition and stretcher/shrinker (Behr, et. al. 1997) are stressed on this strategy due to the fact that 6 cm is not identified as the last bounce, corresponding to one fifth of a quantity that can be found by multiplying for five, inverting the operation initially implemented, fractioning by five. The operative actions corresponding to adding up fractions are carried out correctly even though it seems to be a lack of meaning that students attach to the numbers and operations between them.

<u>ARC3: Arbitrary use of addition</u>. As with ARC2, the students only pay attention to numeric data, and simply add up the numbers, in some cases, without appearing to establish *any* relationship among them. It seems that students have assumed that the problem has an additive structure, where the length of the bounces are added up and the data 6cm, corresponds to the sum of the altitudes of the three bounces.

The meaning conflicts are located in the linguistic elements corresponding to "*first*", "*second*" and "*third*" bounce, as well as, to the statement "*one fifth of*", which is interpreted only in its numeric dimension. It seems that the relationships among the numbers and expressed linguistically in the problem wording are superfluous to students.

[9] The translation for the Spanish in the graph is: 1) Ball was thrown from 750 cm; 2) Bote stands for bounce

ARC4: Rule of three. The students establish a proportionality relation between the number three, that corresponds to the bounces and 6cm, then formulate the question: what is the altitude corresponding to one bounce? The meaning conflicts on this category are much more profound. It seems that students have associated the data format presentation and the problem wording to the archetypal format of proportionality problems that are solved through the so called “rule of three”.

On this type of solution the students carry out the change of type of register procedure that lets them to produce meaning in numerical terms but with no link to the problem. It seems that problem complexity compels students to veer towards more familiar grounds and to perform arithmetic operations (Herscovics & Linchevski, 1994).

A discussion of results

The last three types of arithmetic solutions (ARC2, ARC3 and ARC4) are characterized by a wrong meaning assignment to linguistic objects. Understanding the statement of a word problem requires recognition of the existence of dependence among meaning corresponding to elementary entities. Anghileri (1995) suggests that the close relationship between real settings and the procedures used to solve problems characterized the initial states in learning mathematics. The students have not succeeded in writing a numerical “argument” that links different objects appearing during the resolution process.

The difficulties in representing the problem arithmetically or algebraically are evident from the analogy between ALC4 and ARC3. Nonetheless the meanings and the ways they are related differ essentially. Along with each type of resolution it has been shown that the problem structure raises a number of interpretative challenges, and how the solutions correspond to particular configurations of primary entities, where these facilitate or hinder the arithmetic or algebraic problem representations. The mathematic objects invoked in the problem are the same but the meanings, the relationships among them and the meaning conflicts are diverse to students.

To Filloy, Rojano and Puig (2007), “the mode of thought- be arithmetic or algebraic- appears to be determined by the type of ‘ relational calculation’ that underlies the problem structure” (p.216). We consider that the relational calculation can be expressed and objectified in terms of primary entities, which could be useful for the teachers to recognize both the mathematic tasks complexity and the variety of mathematical reasoning leading to the solution.

RESULTS SUMMARY

Table 1 gives a detailed breakdown of the number and proportion of each type of algebraic and arithmetic solution.

Types of algebraic solutions Number of students	ALC1 25	ALC2 17	ALC3 5	ALC4 11	Correct/incorrect 37/58
Types of arithmetic solutions Number of Students	ARC1 4	ARC2 14	ARC3 3	ARC4 2	10/23
Do not answer	13				

Table 1: Type of configuration and number of students in each one

It can be seem that the number of algebraic solutions as the number of right solutions outnumbered the corresponding arithmetic solutions. The proportion between right solutions and solutions of each type is bigger for the case of algebraic solutions.

Even though students are asked to provide an algebraic solution in the second problem's item, they could have provided an arithmetic solution in the first problem item as well. Given that algebra was not studied during the course, it is worth noting the students' algebraic preference.

IMPLICATIONS FOR STUDENT TEACHER TRAINING

A finding of this research is that the algebraic methods used by the students to solve the problem outnumber in quantity and in effectivity the arithmetic strategies. Just a small number of students choose to solve the problem by means of a right arithmetic strategy in contrast to the findings reported by Nathan and Koedinger (2000). Another finding is the apparent disarticulation among the linguistic, conceptual and procedural elements in the cognitive configurations exhibited by the students, who do not manage to elaborate an "argument" leading to a problem solution.

We consider that teacher's activity not only concerns with planning mathematic tasks but also with the promotion and recognition of the meaning present in the students´ solutions, where the primary entities are articulated. Recognizing the entities involved students´ solutions could help teachers guide their didactic actions.

Therefore it is important to make teachers conscious of the network of objects, meanings and configurations that are put into effect during the mathematics problems solutions to help identifying the meaning conflicts manifested by pupils and therefore, to give answers to those conflicts in the classroom context. As a consequence, it is convenient to use the cognitive-epistemic analysis (Godino et al. 2008) in initial teacher training programs.

Some researchers have contended that teacher's competence to understand and to use the mathematic knowledge adapting it to students' achievements is important (Ball, 1990; Wilson, Shulman and Richert, 1987). More recently Hill, Rowan and Ball

(2005) found that content knowledge is related meaningfully to students' achievements.

We conclude with the observation about the arithmetic strategies that we have discussed above. Our study suggests that algebraic thinking underlies successful problem solutions. We believe that a focus on elementary algebraic reasoning can aid teachers in enabling their pupils to more fully understand the arithmetic domain.

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TRANSFORMATION RULES: A CROSS-DOMAIN DIFFICULTY

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The learning of a symbolic system such as algebra relies on the learning of the use of transformation rules. The implementation of rules in a CAS (Computer Algebra System) for students' modelling has pointed out some questions that are at the junction of three research fields: informatics, mathematics and didactics. Each of these communities has its own perception of algebraic objects, founded on models or practices. The implementation of objects that live in school has questioned object reliability. In this paper, a parallel is proposed between difficulties of informatics implementation of transformation rules and novices' difficulties.

Keywords: algebraic calculations, rules, informatics implementation, students' difficulties.

An important part of school algebra rests on algebraic calculations, what Kieran calls the "transformational activity", which she distinguishes from the generational and global activities (Kieran, 2001). This activity focuses on changing the form of an expression or an equation in order to *maintain equivalence*. This includes, for instance, collecting like terms, factoring and expanding expressions. These are algorithmic tasks like the transformation of $(5+x)x+10+2x$ into $(5+x)(x+2)$. The conservation of equivalence relies on *correct rules* that allow substituting expressions by others. These rules will be called "transformation rules" in this paper. They are supported by the laws of the polynomial ring –commutative law, distributive law and so on. Rules produce objects of a particularly interesting form. Their use is guided by what the desired expression has to look like: reduced polynomial expression or factored polynomial expression. Bellard et al. (2005) call them the constituent rules of mathematic theory: "these rules constitute the base of the [transformational] activity, govern the motion and predetermine the permitted actions". Such mathematical rules are supposed to be accurate and self-sufficient.

Nevertheless, Durand-Guerrier and Herault (2006) stress the fact that rules are objects the usage of which is not so obvious: "the rule is not only a way to learn but it is also an object which has to be learned". It is, in fact, impossible to present a rule alone to students. Rules have to be transposed, adapted and as such lose a part of their accuracy. The implicit notions of rules are compensated by a necessary didactical contract (Brousseau, 1997): "it is an illusion to believe that one can produce the meaning in the mind of someone by indirect ways through the rule and examples" (Wittgenstein, Ambrose, & Macdonald, 1979). Durand-Guerrier and Herault (op.cit.) also point out the illusion to think that the use of a rule is plain, such as "rails that would guide unfailingly and in advance the way to be followed". Actually, it is an interpretation that allows these implicit details between the rule and its application to be overcome. But what are exactly the notions underlying the learning of a transformation rule?

Our research is in line with the identification of systematic errors that students commit when solving transformational exercises. A library of correct and incorrect transformation rules has been built for that purpose and an automatic diagnosis mechanism has been implanted in order to associate a sequence of applied rules to student's transformation (Chaachoua, Croset, Bouhineau, Bittar, Nicaud, 2008). The implementation of these rules has raised questions about the kind of representation of a transformation rule. Automating the process forces the researcher to clarify some implicit mechanisms for the expert: how does a rule work? In which way does it work? How is it matched? It has led to three crucial points about implantation difficulties:

- The reading direction of a rule;
- The notion of sub-expression;
- The generic status of a rule.

Each of these points is discussed in the next sections. We propose, in addition, to link these three points to three classical difficulties which novices may experience when doing transformational activities: the difficulty of understanding the symmetric aspect of the equal sign (see e.g. (Kieran, 1981)); the difficulty of the structural aspect of an algebraic expression (see e.g. (Sfard, 1991)) and the difficulty of applying a general rule to a particular case (see e.g. (Durand-Guerrier, Herault, 2006, p. 144)).

The choices made to raise difficulties in programming may shed light an improvement of the teaching of algebraic rules and may overcome students' problems. Indeed, the reading direction of a rule is essential for a deductive reasoning, the notion of sub-expression allows matching correctly a rule and the generic status of a rule is the power of algebra.

1. REPRESENTATION, READING DIRECTIONS AND REASONING PROCESS

Transformation rules can be represented by two kinds of writings: equality or implication. Both present advantages and have good reasons to be used. Yet, we will see that rules as implication form are interesting in that it highlights the reasoning process in the transformation activity.

Rules as equality, used in school

The first representation –a rule as an equality– is the usual one used in school. Rules can be called by different names in the textbooks: proposition, property, identity, equality and sometimes even theorem (Bellard et al., 2005). Whatever their name, rules are often coming in the form of equality. For example, the distributive law is presented as:

$k(a+b) = ka + kb$, where k, a and b are real. (*Eq1*)

There is a double meaning of the equal sign: that of "identity" or that of "relation". In transformation rules, the equal sign is of course used as "identity", whereas in equations, the equal sign is used rather as "relation". This well-known duality is a real difficulty for students. Presenting rules as identity can, on the one hand, be interesting to get students used to and, on the other hand, provoke confusion.

Such representations are declarative rather than procedural: this form of identity has no explicit reading direction since the equal sign has a double way: from left to right and vice versa. A learning of the way to use such a rule has to be taught. Whereas the process-product has been many times denounced (Davis, 1975) and that special exercises are proposed to students in order to grasp the equivalence notion, here is a case where the equality has to be used in one of the two ways. In fact, textbooks sense that, most of the time, it is necessary to distinguish the two ways by proposing two identities: not only (*Eq1*), which is used to expand expressions but also "$ka+kb=k(a+b)$" to factor. This kind of presentation requires a specific work to become operative: associate a reading direction to the equality for application, according to the aim.

Rules as implication, used in informatics

The second representation –a rule as an implication– is the one used in informatics. One calls "implication" what Durand-Guerrier, Le Berre, Pontille, & Reynaud-Feurly (2000) call "formal implicative", representation used in geometry:

$\forall x \in \Re,\ P(x) \Rightarrow Q(x)$.

Implemented rules are represented as *oriented* mechanisms, also called "rewrite rules" (Dershowitz & Jouannaud, 1990): $A \to B$, where A is *rewritten* in B. The object A produces the object B and B can not produce A unless an other rule $B \to A$ is considered. For example, the rules:

$k(a+b) \to ka+kb$ (*R1*) is used to expand,

$ka+kb \to k(a+b)$ (*R2*) is used to factor.

It is rather a necessity in computing modeling to represent rules as oriented ones than a choice. Indeed, it is not really possible to implement rules as identity. If a single rule is implemented both for expanding and for factoring, there will be some loop and ending problems. For example, with the single rule (*Eq1*), the expression "$3(x+4)$" would be transformed into "$3x+12$", then into "$3(x+4)$" and so on.

Even if it is a necessity, this kind of representation is interesting because its reading direction is explicit: given a real or a polynomial expression under the form "$ab+ac$", where "a", "b" and "c" are reals or polynomials, it can be rewritten into "$a(b+c)$". One can suppose that the use of rules as implication is easier because of its procedural aspect. The kind of representation has an impact on its use easiness, as we will show in the next section.

Impact of the reasoning process

Although geometry is a special introduction field for proof, the latter is not a prerogative of geometry. The "deep structure" (Duval, 1995) of the transformational activities can be presented as a ternary organisation proposed by Duval. A premise (here, a certain expression), a proposition (a transformation rule) and a conclusion (an other expression), as shown in Figure 1, constitute a deduction step. These steps follow on, the conclusion of the current step becoming the premise of the next one. Using Duval's classification (Duval, 1990), the algebraic calculation is formed by *deductive reasoning of steps explicitly concatenated in reference to a transformation rule*. Thereof, this activity can be viewed as a process of demonstration:

> "Demonstration would be defined to be, a method of showing the agreement of remote ideas by a train of intermediate ideas, each agreeing with that next it; or, in other words, a method of tracing the connection between certain principles and a conclusion, by a series of intermediate and identical propositions, each proposition being converted into its next, by changing the combination of signs that represent it, into another shown to be equivalent to it" (Woodhouse, 1801)

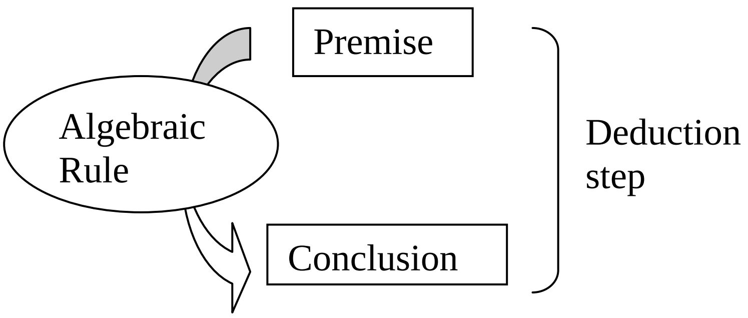

Figure 1: Deduction steps.

Representing rules as implications could allows the user to follow this reasoning process explicitly, as shown in Figure 2.

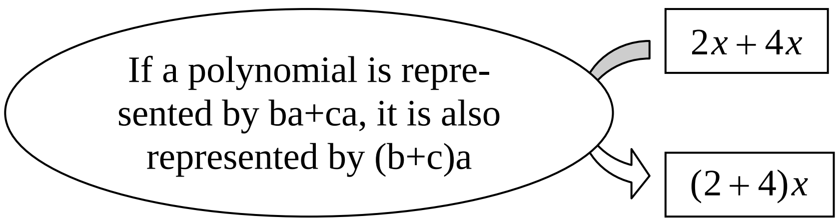

Figure 2: Example of the reasoning process in algebra. The level of making explicit a demonstration and the granularity of a deductive step evolves with the level of the student. Here, for example, we have omitted to explicit the commutative law. As Arsac notes: "any demonstration is shortened from another demonstration" (Arsac, 2004).

Splitting an identity into two implications conceals the fact that rules are equivalent but clarifies the *way of application* and, above all, it allows following the Duval's structure of a deduction step. This is the *modus ponens* mode: "*if p, then q*, now *p*, then *q*”. The representation form of a rule has an impact on its use easiness but it lets the difficulty to know to which object the rule can be applied.

2. MATCHING AND SUB-EXPRESSIONS NOTION

An unrefined syntactic unification between the premise of a rule and a part of an expression does not produce an algebraic behaviour. With an unrefined unification, a rule as $x+x \to 2x$ would transform “$5\boldsymbol{x}+\boldsymbol{x}$” into “$5\ \mathbf{2}\boldsymbol{x}$”, which has no sense (what is the operator between “5” and “2”?) nor the expected result. This is a well-known mistake committed by students: substitute an expression by another by working only on a syntax level and taking no account of semantics. Mastering substitution needs knowing the notion of what a *sub-expression of an expression* is.

The definition of an expression from the rewrite rule theory of Dershowitz (1990), in which rules are applied on *sub-objects*, underlines the notion of sub-expression, thanks to its recursive definition. Let us consider a set of symbols of terminal objects (e.g., integers), a set of symbols of variables (e.g., {*x*, *y*, *z*}), and a set of symbols of operators (e.g., +, –, ×, ^, sqrt, =, <, and, or, not). An *algebraic expression* is a finite construction obtained from the following recursive definition:

- a symbol of terminal object

- or a symbol of variable

- or a symbol of operator applied to arguments which:

    - are algebraic expressions,

    - are in the correct number (correct arity [1]),

    - and have correct types [2].

With this definition, matching a rule R to an expression E would consist of finding a sub-expression E1 of E, replacing E1 in E by the expression that produces R. For example, in “$5x+x$”, the algebraic (sub) expressions are “$5x$”, “x” (two times), “5” and “$5x+x$”. The expression “$x+x$” is not a sub-expression of “$5x+x$”. To deal with this problem, the internal representation of expressions in computer algebra systems (CAS) is a tree representation, in which the structure of the expression is explicit, as shown in Figure 3.

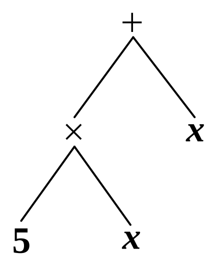

Figure 3: Tree representation of the expression $5x+x$**.**

The necessity of the tree representation appears also in school curricula. Although school approach of expressions is foremost syntactic -algebraic expressions are defined as "writings including one or more letters"-, new French curricula encourage making students work on tree representations. As they claim, tree representation allows pointing out the structural aspect of an expression as defined by Kieran:

> "The term *structural* refers, on the other hand, to a different set of operations that are carried out, not on numbers, but on algebraic expressions. […] the objects that are operated on are the algebraic expressions, not some numerical instantiations. The operations that are carried out are not computational. Furthermore, the results are yet algebraic expressions." (Kieran, 1991)

This structure notion is essential to deal with matching difficulties. It enables understanding why such rule like $k(a+b) \to ka+kb$ (*R1*) can be applied on sub-expressions of expressions such as $3+4(x+1)$. Nevertheless, is it sufficient to understand that this rule can be applied also on expressions such as $4x^2(x+1)$ or $4x^2(x+1+x^2)$? Either in informatics or at school, we will see that most of the time, one needs to precise as many rules as there are cases.

3. GENERIC STATUS OF RULES

The third idea which emerges of rules implementation turns on the generic status of a rule: how a rule such as (*Eq1*) or (*R1*) can be sufficient to apply to the expressions "$7(3+x)$", "$-7(3+x)$", or even "$7(3+x+x^2)$"? How to deal with the matching of "$a+b$" with "$3+x+x^2$"? It is, with no doubt, the principal difficulty for novice users of rules: the application of a general rule to a particular case. It is, in fact, the same in informatics. Although the two first points –reading direction & matching problems– have been easily resolved in informatics, it has not been the same for this third problem.

The entry by rewriting rules –and thus a syntactic presentation– leads to some new problems. Let us study again the case of (*R1*). For experts, it is not really this rule that is used but much more the single distributive law. With this last one, experts can expand any product of polynomials. In informatics, one needs rules to be implemented and so, the exact structure of an expression has to be specified. For (*R1*) implementation, "k", for example, has to be defined: is "k" a real, a product such as a monomial or a sum? It is not possible to just say "given a polynomial k". Indeed, to transform "k", its structure has to be specified. For example, if "k" is negative, the sign of the entire expression is changed. The main operator of the expression becomes "minus" and not "times": the entire internal tree representation is changed, as shown in Figure 4. The same difficulty is found when "$a+b$" is a sum of three terms: it can change the mechanism of the implementation of the rule. Without genericity, one needs to distinguish cases like "$k(a+b)$" from "$k(a+b+c)$". To deal with that, the concept of distributive law has been implemented. Let us consider two lists and an operator, the distributive rule can be written as:

$$(a_1,a_2,...,a_n)\Delta(b_1,b_2,...,b_n) \to (a_1\Delta b_1, a_1\Delta b_2,...,a_1\Delta b_n, a_2\Delta b_1, a_2\Delta b_2,...,a_n\Delta b_n).$$

We do not have to specify the length "n"of the lists.

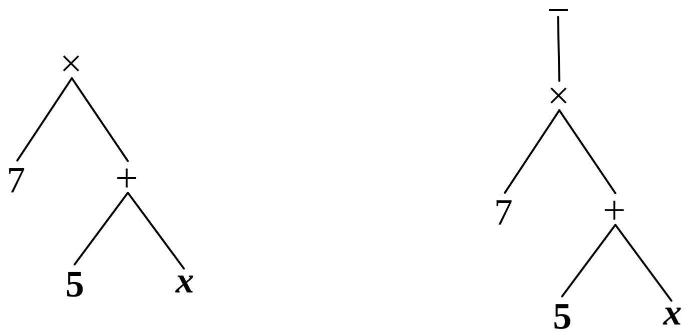

Figure 4: The single change of the real 7 into -7 changes the entire structure of the tree representation of the expression. On the left, the expression 7(5+x); on the right, the expression -7(5+x).

Another example is very representative of this problem: the rule of monomials addition, which can be written as $ax+bx \to (a \oplus b)x$, where $\oplus$ is the calculated sum operator. Such rule is not so easy to implement. If we ask the premise to be a sum of two products, this rule will not apply to expressions such as "$ax+x$" because "x" is a single argument and not a product: an automatic mechanism does not recognize "x" as the product of "1" and "x". To deal with this problem, some concepts have been implemented like the *monomial concept*. We have implemented the added fact that a monomial can be either a product of a real and a variable –of explicit degree or not– or a single variable –of explicit degree or not. Thus, expressions such as "$4x^5$", "$4x$", "x^1" or "x" are read as monomials, and the rule $ax+bx \to (a \oplus b)x$ can be easily implemented: one needs just to specify that the premise has to be a sum of two monomials.

The same problem occurs at school: the polynomial notion is not taught in France [3]. The variable "k" from the rule (*Eq1*) is then defined as a real, so are "a" and "b". Understanding that "a" can be itself a sum, or even a sum with variables, requires a real work. How do French textbooks deal with this problem?

To answer this question, we have used the concept of praxeologies from the Chevallard's anthropological theory of didactics. Let us remain that Chevallard proposes to describe any human activity by a quadruplet which enables an activity to be cut in types of task, which can be solved by techniques –a way of doing–, which can be explained by a rational discourse, "logos" (Chevallard, 2007) [4]. Our work in progress (Croset, 2009) shows that French textbooks distinguish three types of task for expanding expressions:

"$k(a+b)$", "$k(a-b)$" and "$(a+b)(c+d)$".

Cases like "$-(a+b)$" or "$+(a+b)$" are discussed in another part of textbooks ("how to remove brackets?"). Some textbooks propose even more distinctions: they discern also "$(a+b)k$" and "$(a-b)k$".

On the one hand, it seems that textbooks decide to specify many cases although all these tasks are explained by a single "logos": the distributive law. The fact that textbooks need to precise many cases points out the well-known difficulties of students to apply a general rule to particular cases. On the other hand, all possible cases cannot be specified. Textbooks do not specify types of tasks as "$k(a+b+c)$" or "$k(a+b)(c+d)$". Understanding the structure of the expression is supposed to be sufficient to deal with all these forms. Nevertheless, we have not found such work and reflection about the generality of rules. Only a few textbooks precise links between the three types of task described above. Explanations such as using $k(a+b) \to ka+kb$ to expand "$(a+b)(c+d)$" are not common. Neither are presented the iteration concept to expand "$k(a+b)(c+d)$" whereas our work (ibid.) shows that students' mistakes occur specially in this sub-type of task.

The second problematic example about monomials revealed by the computing implementation occurs also in students' difficulties: recognizing "x" as a monomial is not an easy task for a novice. A novice's common mistake is precisely to transform "$ax+x$" into "ax" because of the lack of the explicit coefficient "1" ahead of the "x": when "a" is added to "nothing", it remains "a" [5]. The concept of monomial is not taught currently in French curriculum. We speak about "like terms" but few textbooks precise that "x", "$1x$", and "x^1" are "like terms" which can be collected.

The force of algebra lies in the writings generic status. Its interest is lost if all cases are presented. To avoid that, a specific work on concepts such as distributive law or monomial could be proposed to novices, just like it has been done for the computing implementation.

4. CONCLUSION

The learning of the transformational activity cannot be restricted to memorizing rules. This requires a specific work about the application of rules. Our research focusing on automatic student modelling has brought to light three important difficulties concerning the application of transformation rules, which have been compared with similar novices' difficulties: knowing that a rule has a reading direction allows students to follow a reasoning process when they transform algebraic expressions; knowing the structure of an expression permits a correct matching; finally, having a good perception of the generic status of rules allows students to apply a general rule to a particular case. All these elements are necessary conditions for learning the algebraic symbolic system. Our paper has described the parallel between informatics implementation difficulties and the ones met by novices. One can wonder if the way to deal with the first ones could be used to deal with the second ones.

Regarding these three points, rules have been looked at from a technical point of view. Another point of view would be considering what experts' criteria are to control their transformations: substitute numerical values to equivalent expressions in order to verify the equivalence; in other words, being aware that equivalent expressions denote the same object. Similarly, another interesting point of view is to explore how to *choose* the appropriate rule. We have seen that a rule is general but the choice of a rule is crucial to obtain the form that one needs. The *raison d'être* of a rule, the strategic process and elements that guide an expert in choosing this particular rule, and not another one, have not been discussed here, despite the fact that informatics is also interested in such questions. We can expect that a parallel would be again possible between novices' strategic difficulties and the implementation ones.

NOTES

1. The arity of an operation is the number of arguments or operands that the operation takes. For example, addition is an operation of arity 2, sqrt is an operation of arity 1.

2. For example the expression "$\sqrt{5x=3}$" has not a correct type.

3. A recent study has compared the algebra learning in France and in Vietnam (Nguyen, 2006). It shows that algebraic expressions found in French textbooks rely on the notion of polynomial function whereas the ones that can be found in Vietnamese textbooks rely on the polynomial notion.

4. The reference (Chevallard, 2007) is not the best one for the notion of praxeology but it presents the advantage of being written in English. French reader can see also (Bosch & Chevallard, 1999).

5. Haspekian (2005) proposes another explanation to this mistake: the difficult notion of neutrality of the multiplicative law. We think that, in our context, the mistake is more relative to a visual lack.

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INTERRELATION BETWEEN ANTICIPATING THOUGHT AND INTERPRETATIVE ASPECTS IN THE USE OF ALGEBRAIC LANGUAGE FOR THE CONSTRUCTION OF PROOFS

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Abstract. *This work is part of a wide-ranging long-term project aimed at fostering students' acquisition of symbol sense (Arcavi, 1994) through teaching experiments on proof in elementary number theory (ENT). In this paper I present some excerpts of students discussions while working in small groups on activities of proof construction. My analysis of these transcripts is aimed at highlighting the incidence of anticipating thoughts and of the flexibility in the coordination between different conceptual frames and different registers of representation in the development of proof in ENT. In particular, I singled out four main sources of interpretative blocks, highlighting the strict interrelation between anticipating thought and students' difficulties in the interpretation of the algebraic expressions they produce.*

1. INTRODUCTION

Many research studies support an approach to algebraic language related to the development of reasoning. Arcavi (1994), for example, claims that, in addition to stimulating students' abilities in the manipulation of algebraic expressions, teachers should make them see the value of algebra as an instrument for understanding, introducing them to algebraic symbolism from the beginning of their studies through specific activities that encourage an appreciation of the value and power of symbols. A central aspect in Arcavi's approach to algebraic language is, in fact, the concept of symbol sense. The author chooses to characterize symbol sense highlighting, through meaningful examples, the attitudes to stimulate in students to promote an appropriate vision of algebra. Particular attitudes that he names include: the ability to know when to use symbols in the process of finding a solution to a problem and, conversely, when to abandon the use of symbols and to use alternative (better) tools; the ability to see symbols as sense holders (in particular to regard equivalent symbolic expressions not as mere results, but as possible sources of new meanings); the ability to appreciate the elegance, the conciseness, the communicability and the power of symbols to represent and prove relationships. Many researchers share a similar vision of the approach to the teaching of algebra. Among them, Bell (1996), states, in particular, that it is necessary to favour the use of algebraic language as a tool for representing relationships, and to explore aspects of these relationships by developing those manipulative abilities that could help in the transformation of symbolic expressions into different forms. This idea is strictly connected with Bell's description of "the essential algebraic cycle" as an alternation of three main typologies of algebraic activity: representing, manipulating and interpreting. Similar observations are also found in Wheeler (1996), who asserts the importance of ensuring that students acquire the fundamental awareness that algebraic tools "open the way" to the discovery and (sometimes) creation of new objects. Kieran (2004) also stresses the importance of devoting much more time to those activities for which algebra is used as a tool but which are not exclusively to algebra (global/meta-level activities according to Kieran's distinctions) because they could help students developing transformational skills in a natural way since meaning supports manipulations. Proof is certainly one of the main activities through which helping students develop a mature conception of algebra. I adopt Wheeler's idea that activities of proof construction through algebraic language could constitute "a counterbalance to all the automating and routinizing that tends to dominate the scene". I believe that activities of proof in ENT would both provide students with the opportunities they need to progress gradually from argumentation to proof (Selden and Selden, 2002)) and help them to appreciate the value of algebraic language as a tool for the codification and solving of situations that are difficult to manage through natural language only (Malara, 2002).

I agree with Zazkis, Campbell (2006) who state that "the idea of introducing learners to a formal proof via number theoretical statements awaits implementation and the pros and cons of such implementation await detailed investigations" (p.10). In order both to investigate these aspects and to foster the diffusion of activities of proof in ENT in school, aiming at making student appreciate the value and power of algebraic language, I am working with upper secondary school students (10^{th} grade) [1]. I planned and experimented a path for the introduction of proofs in ENT. The path was articulated through small-groups activities (some groups were audio-recorded), followed by collective discussions (audio-recorded) on the results of the small-group activities. In order to foster a widespread participation during group activities, I decided to work with homogeneous (according to competencies and motivations) small groups. In this work I will dwell on a central moment in the path: the small-groups' work aimed at constructing the proof of some conjectures they produced starting from numerical explorations. In particular I will present the main results of the analysis of group discussions when students were trying to prove one of the conjectures.

2. THEORETICAL FRAMEWORK WHICH SUPPORT MY ANALYSIS OF STUDENTS' DISCUSSIONS

Many different competencies are required of a student who has to face proof problems in ENT. In particular, he/she has to: (a) know the meaning of the mathematical terms in the problem text and interpret them correctly by reference to it; (b) translate correctly from verbal to algebraic language; (c) be able to interpret the results of the transformations operated on the algebraic expressions in relation to the examined situation; and (d) control the consequences of his/her assumptions. I identified a set of theoretical references that are both appropriate to the analysis of the transcripts of group discussions dealing with proofs and in tune with the view of algebra that I am trying to promote. The main reference in my research is the work by Arzarello, Bazzini and Chiappini (2001). The authors propose a model for teaching algebra as a *game of interpretation* and highlight the need for the promotion of algebra as an efficient *tool for thinking*. An awareness of the power of the algebraic language can be developed only once the student has mastered the handling of some key-aspects that arise in the development of algebraic reasoning. In particular, the authors highlight the use of *conceptual frames* defined as an "organized set of notions, which suggests how to reason, manipulate formulas, anticipate results while coping with a problem", and *changes from a frame to another* and from a knowledge domain to another as fundamental steps in the activation of the interpretative processes. According to the authors, a good command in symbolic manipulation is related to the quality and the quantity of anticipating thoughts which the subject is able to carry out in relation to the effects produced by a certain syntactic transformation on the initial form of the expression. Boero (2001) also argues that *anticipation* is a key-element in producing thought through processes of transformation. The author defines anticipating as "imagining the consequences of some choices operated on algebraic expressions and/or on the variables, and/or through the formalization process". In order to operate an efficient transformation, the subject needs to be able to foresee some aspects of the final shape of the object to be transformed in relation to the target. Arzarello *et Al.* stress that the ability to produce anticipations strictly depends on changes in the frame considered in order to interpret the shape of the expression.

Another theoretical reference that I take as fundamental for analyzing students' management of meaning in algebra is the concept of *representation register* proposed by Duval (2006). The author defines representation registers those semiotic systems "that permit a transformation of representations". Among them, he includes both natural and algebraic language. Duval asserts that a critical aspect in the development of learning in mathematics is the ability to change from one representation register to another because such a change both allows for the modification of transformations that can be applied to the object's representation, and makes other properties of the object more explicit. According to the author, real comprehension in mathematics occurs only through the coordination of at least two different representation registers. He analyzes the functions performed by different possible typologies of transformations, distinguishing between *treatments* ("transformations of representations that happen within the same register") and *conversions* ("transformations of representation that consist of changing a register without changing the objects being denoted") and highlighting both the fundamental role of each of these typologies of transformations and the intertwining between them.

In order to clarify how this set of theoretical references could help in analysing the role played by algebraic language in the construction of proofs (or attempts of proof) in ENT, the next paragraph will be devoted to the *a priori* analysis of the problem on which the working group activities, examined in this paper, were focused.

3. A PROBLEM AND ITS *A PRIORI* ANALYSIS

The problem, on which this paper is centred, is the following: "*Write down a two digit number. Write down the number that you get when you invert the digits. Write* *down the difference between the two numbers (the greater minus the lesser). Repeat this procedure with other two digit numbers. What kind of regularity can you observe? Try to prove what you state".*

The regularity to be observed is that the difference between the two numbers is always a multiple of 9; precisely it is the product between 9 and the difference between the digits of the chosen number. The proof requires the polynomial representation of each number: since a number of two digits m and n can be written as $10m+n$, where $m>n$, the difference can be represented as $10m+n-(10n+m)$. Through simple syntactical transformations it is possible to turn the initial expression into a form that makes the required property explicit: $10m+n-(10n+m)=9m-9n=9(m-n)$. The initial conceptual frames to which the statement of the problem refers are 'difference between numbers' and 'two digits numbers'. It can be assumed, therefore, that the student will not automatically choose the 'polynomial notation' frame to represent the problem (some students might apply the 'positional representation of a number' frame and then get stuck). The reference to the 'divisibility' frame, which allows them to foresee the desired final shape of the expression after correct treatments (i.e. $9{\cdot}k$, where k is a natural number), seems to be less problematic but possible blocks in the treatments to perform on the initially constructed polynomial expression can be ascribed to interpretative difficulties, which are strictly related to students' inability to correctly anticipate the final shape of the considered expression (it is necessary to recognize the transformation that leads to an expression that can be easily interpreted in the final frame 'divisibility'). Finally, some observations about possible students' behaviours could be proposed. Many students could end their numerical explorations after having observed that the difference between the two numbers is always a multiple of 9, without recognizing the relationship that exists between the two digits of the first number and the difference between the two numbers (i.e. the considered difference is the product between 9 and the difference between the digits of the chosen number). Consequently, the analysis of the final expression could provide another index of students' interpretative abilities, in that access to the new meanings it embodies depends on those abilities.

4. RESEARCH HYPOTHESIS AND AIMS

My hypothesis is that the production of good proofs in ENT depends upon the management of three main components: (a) the appropriate application of frames and coordination between different frames; (b) the application of appropriate anticipating thoughts; and (c) the coordination between algebraic and verbal registers (on both translational and interpretative levels).

The aim of this paper is to investigate the role played, in students' proving processes, by the three components I singled out and the mutual relationships between them. In this work I will present a sample of prototype-productions [2] helpful to highlight that the lack or unsuccessfully application of one of these components leads to failure and/or blocks of various types. In particular, I will highlight the interrelation between anticipating thought and interpretative blocks.

5. RESEARCH METHODOLOGY

Theoretical models I used helped us identify some interpretative keys for the analysis of protocols of students' discussion while working in small groups. My analysis focused on the following: (1) The conceptual frames chosen to interpret and transform algebraic expressions and the coordination between different frames; (2) The application of anticipating thoughts; and (3) The conversions and treatments applied and the coordination between verbal and algebraic registers.

My choice of analyzing small groups' discussions is motivated by the conviction that only when students are involved in a communication it is really possible for us to produce an in-depth analysis of the coordination between verbal and algebraic register. Moreover I believe that the analysis of the sole written protocols is not enough to highlight students' actual interpretations of algebraic expressions they construct. The need to communicate their reasoning to others forces students not only to verbally make what they are writing explicit, but also to explain both the objectives of the transformations they carry out and their interpretation of results.

6. THE ANALYSIS OF PROTOTYPE-PRODUCTIONS

In this paragraph I will present two examples of prototype-protocols of discussions, chosen because they highlight how students' interaction allows to identify the reasons of erroneous conversions and the difficulties in the interpretation of expressions.

6.1 Example 1:

The following example is characterized by the application of an initial suitable frame, not associated to an adequate conversion and a correct interpretation of the produced expressions.

After having considered many numerical examples, students A, C and N conclude that the considered difference is always a multiple of 9. The following dialog represents the proving phase.

27 C: Let us do with letters.
28 N: It is more complicated.
29 C: It will be $10x$ … plus …
30 A: …plus y (*they write* $10x+y$) [3]
31 C: If we invert the digits, it will be $y+10x$
32 A: and now … we have to do the difference
33 C: (*She writes and reads*) $10x+y$ … minus ... (*she writes* $y+10x$) it becomes $10x+y-y-10x$
34 N: I think there is a mistake because the result is zero … they cancel each other out. We are not able to prove it.
35 C: We have $10x+y$ and it represents the number … Then we have to …
36 A: (*She reads*) 'when you invert the digits' …

37 C: It is the same of having 1 and … It is as if we take it on this side, so y should be take on the other side… however, if we take 10 on this side, it will be left a …

38 A-N-C: one!

39 C: So it is not $10x$. I think it is x … So it would become $10x+y-(y+x)$. The two y cancel each other out, so they will be left $10x-x$. Exactly $9x$! We were able to prove it! …

40 C: … (*C. is looking to the numerical examples*) But here I can see something more, I think. I can see that, practically, this is … Look what I noticed (*she is looking at the differences 86-68, 92-29, 76-67, 52-25*) … if you subtract the two tens, 8-6, you have only to consider the product between 9 and the difference between the two tens: 9 times 2 is 18; 7-6 is 1, 9 times 1 is 9; 5-2 is 3, 9 times 3 is 27.

41 A: We have to write it down. I would have never noticed it!

42 C: (*she dictates*) It is always a multiple of 9 and we can observe that the result of the subtraction … you have to subtract the two tens and to multiply the result by 9… Do you know how I thought of it? Because I saw $9x$ and I said “it is a multiple” because there is 9 times x. Then I said “but … what is x? x is the tens!”. Then I tried to do x minus x.

43 A+N: Good!

This protocol can be subdivided in three key-moments: (1) *Initial conversion and first treatments* (lines 27-33); (2) *Identification of a problem, modification of the conversion and new treatments* (lines 34-39); (3) *Attempt of interpretation of the obtained expression and refinement of the conjecture* (lines 40-43).

Initially C carries out a *first erroneous conversion* (line 31), translating this concept through the expression $y+10x$. While students correctly interpret the natural language term “invert” when they work on numerical examples in order to formulate the conjecture, when they have to carry out a conversion into algebraic register, the concept “exchanging the place” is translated through the pure exchange of the order of the monomials which constitute the polynomial $10x+y$, dispelling serious difficulties in coordinating the ‘positional notation’ and ‘polynomial notation’ frames and lack in the internalization of the last. The difference (zero) they obtain starting from this erroneous conversion lead them to detect the inaccuracy of their initial conversion and to look for a new correct one. They detect a mistake in having supposed that $10x$ should represent the units digit, so they decide to correct this mistake, substituting x instead of $10x$, but they do not consequently modify the representation of y as tens-digit. Therefore, writing the polynomial as $y+x$, they carry out again an incorrect conversion. Probably because of the prevailing of the anticipating thought they carry out (expecting a multiple of 9, they only concentrate on the factor 9 when they look at the expression $9x$), once they obtain $9x$ as the difference between the two numbers, they do not immediately subject the new result to a careful interpretation. Only afterwards C interpret x as the tens-digit of the initial number and decide to investigate the considered examples in order to refine their conjecture. C concentrates on the tens-digits of the two numbers (x and y in the correct representation) and observes, starting from examples, that the result is obtained multiplying 9 by the difference between those digits. This observation, however, does not help her in critically interpreting the expression $9x$. In her final intervention, she even tries to translate into algebraic language, through the expression x-x, the difference between the two tens, but she is not able to 'grasp' the gap between the algebraic representation she proposes and her verbal considerations.

6.2 Example 2

In the following transcripts we can highlight what kind of difficulties students meet when appropriate application of the initial conceptual frame and conversions are not supported by anticipating thoughts and by a semantic control.

The three students G, B and A decide to work separately on the conjecture: while A and G analyze numerical examples only, B works on the algebraic formalization of the difference to be considered. Without speaking with her friends, B is able to perform the correct conversion, representing the considered difference with $10x+y-(10y+x)$. *Afterwards she performs correct treatments on this expression, obtaining* $9(x-y)$, *and she decides to illustrate her result to A and G.*

19 B: I obtained this thing … Why 9? 9 is 9! 9 is odd! Is it possible that the result is always an odd number?
20 A: No. Consider 20! The difference is 18!
21 G: I sincerely can't find a regularity …
22 B: I could only find that the result is 9 multiplied by x-y, but … why is 9 here? There is 9 only because there is 10!
23 G: Let's try with 28 … 82-28 … the result is 54! So … What have these numbers in common???
24 B: I found it!! I found it!! If I choose 65 and 56, the difference is 9. In the algebraic case the result is 9 multiplied by $(x$-$y)$!
25 G: Please, explain it!
26 B: Because, independently from the initial number, the difference is always 9.
27 G: No! Consider 82 and 28!
28 B: What a pity! I liked this observation! … Wait a moment … here (*she refers to the examples she chose*) we pass from a ten to the next ten. I found it! Only if we start from a number whose digits are consecutive, the difference is 9!!! 34 and 43 … All the numbers have consecutive digits!
29 G: It is true! 54 e 45!
30 B: 12, 23, … Do you understand? 1 and 2 are consecutive numbers.
32 A: 14 and 41? 15 and 51?
33 B: No! The two digits must be consecutive! When they are consecutive, the difference is always 9!
34 A: So … what does it happen?
35 B: I don't know … It happens that the difference between the numbers is 9. If you look at the algebraic case … Can you see that it is always 9 multiplied by something?
36 A: Only if the digits are consecutive the difference is 9?
37 B: I don't know why …
38 G: But … I think that the distance between the numbers is not the only reason …

(*silence*)
39 B: … It is always a multiple of 9!!!
40 A: In what sense?
41 B: Let's try! 52-25! The result is 27!
42 A: Also if we choose 15 and 51 …the result is 36!
43 B: They are all multiple of 9! Can you see that every case is the same?! Tell me other numerical examples!
44 A: 51-15 is 36
45 G: 52-25 is 27
46 B: 21-12 is 9, which is a multiple of 9!
47 G: So we can observe that the result is always a multiple of 9.

This excerpt could be subdivided in two key-moments: (1) *Attempt to interpret the expression produced during an 'algebraic exploration' of the problem situation* (lines 19-38); and (2) *Formulation of the conjecture* (lines 39-47).

Students' choice to proceed separately turns out to be not effective. In fact, while the analysis of numerical examples does not help A and G in formulating a conjecture, the total absence of anticipating thoughts about the objective of the algebraic manipulations B operates blocks her interpretation of the obtained expression *9(x-y)*. In fact, B initially tries to guess the correct interpretation of the expression as the representation of an odd number (line 19). When this interpretation is refuted by a counterexample proposed by A (line 20), B decides to refer to numerical examples in order to meaningfully look at the obtained expression. The choice of the numerical examples she considers (only numbers whose digits are consecutive) suggests her that the difference is always 9 (line 24). Now the presence of an anticipating thought (the difference is 9) negatively influences B's interpretation of the expression *9(x-y)*. When, again, G proposes a counterexample against B's conjecture (line 27), she does not try to re-interpret the expression and limits herself to look at numerical examples to understand what are the conditions under which the regularity she first observed (the difference is 9) is valid (lines 28 and 30). Although her correct observation about the interrelation between the digits of the initial number and the difference between the two numbers, again B is not able to correctly re-interpret the expression *9(x-y)*, focusing on the role assumed by the factor *(x-y)* (lines 35 and 37). B's troubled conquest of an only partial interpretation of the expression *9(x-y)* and her necessity to refer to numerical examples to understand what she obtained testify that, if algebraic manipulations are not guided by an objective, significant interpretations are blocked. An evidence of this problematical aspect is the fact that, paradoxically, the working group activity ends with the formulation of a conjecture.

7. CONCLUSIONS

The analysis I presented in the previous paragraph allows to offer some conclusions with respect to the role played by the three components I identified and the mutual relationships between them. The first protocol highlights the strict correlation between lack of flexibility in coordinating different frames, difficulties in carrying out conversions from verbal to algebraic register and lack of interpretative games in the analysis of the expressions produced. Moreover, it testifies how such correlation causes failures in the production of proofs in ENT. In fact, the three students display rigidity in their use of frames and an incapability of simultaneously manage different frames. Such rigidity makes them produce partial or incomplete interpretations of the constructed expressions, so they are not alerted about the non-acceptability of their proof. The second protocol testifies the strict interrelation between anticipating thoughts, the activation of conceptual frames and the subsequent interpretations of the produced expressions: since the conversion and the treatments operated by B are not oriented by an anticipating thought, the activation of a proper conceptual frame and a correct interpretation of the final expression are blocked. Moreover, this protocol represents a good example of results produced by the strict interrelation between *blind manipulations* (i.e. produced without an objective) and blocks in the interpretative processes. The rigidities highlighted in the analyzed protocols are shared by other protocols (not presented here because of space limitations), to which different problems could be add, such as: (a) blocks related to the activation of an incorrect initial frame of reference; (b) blocks in the treatments and in the interpretative processes due to an inability to foresee the expression to be attained by the activation of the correct final frame; (c) difficulties in the choice of the treatments to be operated caused by the absence of anticipating thoughts.

These observations helped us in singling out an initial classification of interpretative blocks in relation to causes that have produced them. Summarizing, I identified *interpretative blocks associated to*: a) *difficulties in simultaneously managing different frames* (example 1, line 42); b) *total absence of anticipating thoughts* (example 2, line 19); c) *activation of erroneous anticipating thoughts* (example 2, lines 24-26); d) *activation of a predominant (partial) anticipating thought* (example 1, line 39; example 2, lines 39-43). This classification let us highlight, in particular, the fundamental role played by anticipating thoughts during these kind of activities, thanks to the strict interrelation between them and students' difficulties in the interpretation of the algebraic expressions they produce.

In conclusion, my analysis of students' discussions during small group activities turned out to be an effective methodological instrument to verify my hypothesis on the importance of the key-components I singled out for the analysis of proof productions in ENT.

The results of this analysis will be a starting point for the next step of my research. I am convinced that the only way to make this approach to algebraic language really effective is to help teachers act as fundamental models in guiding their students toward the acquisition of the essential competencies that can help them overcoming difficulties and blocks identified in this work and developing awareness of the central role played by algebraic language as a reasoning tool. Therefore I will focus my research on the role played by the teacher during class activities in order to highlight the attitudes of an aware teacher, the choices he makes and the effects of his/her approach on students, from the point of view of both awareness shown and competencies acquired.

NOTES

1. The study was conducted in some classes (10th grade) of a *Liceo Socio-Psico-Pedagogico*, which is an upper secondary school originally aimed at educating future primary school teachers.

2. The term "prototype-production" is here used with the meaning of "representative of a category of productions of the same kind".

3. The difficulties I hypothesised in the identification of the initial frame are not highlighted by this protocol because students have faced the problem of the representation of two and three-digit numbers in a previous activity.

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EPISTEMOGRAPHY AND ALGEBRA

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We propose to address the problem of how to know students' knowledge in an entirely new approach called "epistemography" which is, roughly, an attempt to describe the structure of this knowledge. We claim that what is to be known is made of five tightly interrelated organised systems: the mathematical universe, the system of semio-linguistic representations, the instruments, the rules of the mathematical game, and the identifiers.

Keywords: epistemography, algebra, semiotics, language, subparadigm.

One of the most commonly shared principle of didactics of mathematics is that teaching must ground on students' previous knowledge. Therefore we researchers (and teachers too!) need to know what students know and what they are supposed to know.

But the point is that knowing what students are supposed to know is less easy to do than it appears at a first glance, particularly when they shift from primary studies to secondary studies and when there are frequent curricular changes in the primary studies. In this case, secondary teachers cannot rely on remembering their primary school time; reading curricular documents is not very helpful, neither discussing with primary teachers. The problem is the lack of a common language, or better said, that the common language is not accurate enough. Saying that "students know the sense of operations" or that they are able to solve "simple word problems" is far too fuzzy and superficial.

We propose to address this problem (how to know students' knowledge) in an entirely new approach called "epistemography" which is, roughly, an attempt to describe the structure of this knowledge.

Epistemography is based on an attempt to generalise and conceptualise findings about knowledge we made mainly during previous researches on algebraic thinking. According with many authors we found that semiotic and linguistic knowledge plays a central role in Algebraic Thinking. And we faced the following question: to what extent is this knowledge, *mathematical*? Letters and symbols are not mathematical objects in the same way that numbers or sets or functions are[11]; but on the other hand they are equally necessary to do mathematics.

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11 More precisely, digits, letters, symbols and expressions made with them form a "language". Languages are mathematically described by the "Language Theory" (a part of Mathematical Logic, shared with computer science).

Epistemography is a description of the structure of what the subjects have to know in order to actually do mathematics (and not just to pretend to do mathematics!). We chose to call this theory "epistemography" because it is about knowledge ("epistemo-") but, unlike epistemology, not in a historical perspective: rather, epistemography is a kind of geography of knowledge.

We claim that what is to be known is made of five tightly interrelated organised systems: the mathematical universe, the system of semio-linguistic representations, the instruments, the rules of the mathematical game, and the identifiers. We will now present in detail these five knowledge systems. Due to the lack of space this presentation is a quite schematic and abstract one; a much more detailed and discussed presentation of epistemography is to be written.

THE MATHEMATICAL UNIVERSE

To solve some algebraic problems, you must know that the product of two negative numbers is positive. You can believe that negative numbers are real numbers, or just "imaginary" ones; whatever philosophical option you take, if you want to do mathematics, you need to have some knowledge about something. We call a "mathematical object" this "something", and the Mathematical Universe the system made up of these mathematical objects (e.g. numbers), their relations (e. g. rational numbers are real numbers) and properties (e. g. the product of two negative numbers is positive). Usually, objects of the mathematical universe may be described as individuals (like the number 20) or classes (the even numbers).

SEMIO-LINGUISTIC REPRESENTATIONS SYSTEM

How to avoid, however, considering as belonging to the mathematical universe, objets or properties whose nature is totally different? We must, actually, distinguish carefully (mathematical) objects (like the number 20) from their (semiolinguistic[12]) representations (like the string of characters "20" made of a "2" and a "0", but also "XX" made of two "X" or "::::: :::::" made of twenty dots). This distinction –and its consequences– is essential and has been stressed by many authors (Drouhard & Teppo, 2004, Duval, 1995, 2000, 2006, Ernest, 2006, Kirshner, 1989, Radford, 2006, Bagni, 2007 amongst many others). Misunderstanding or neglecting this distinction may lead to quite severe consequences on mathematics learning and teaching studies. Hence our claim is that, besides knowledge about objects of mathematical universe, students must have some (at least practical) knowledge of the very complex and heterogeneous, and often hidden, system of semio-linguistic representations.

But, how can we decide if a given property is mathematical or semio-linguistic? There is a practical criterion: mathematic properties may be called "representation-free": they remain true whatever representation system is used. For example, the irra-

12 "semio-" means "related to signs" and "linguistic", "related to language"; see further.

tionality of √2 does not depend on how integers, square roots or fractions are written. Actually the Greeks' notations of the first proof had nothing in common with ours (in particular they did not use any symbolic writing). Semiotic properties, on the contrary, rely on representational conventions. The property that in order to write 1/3 you need an infinite number of (decimal) digits is true – in base ten only; it is false in base three ("<u>0</u>,<u>1</u>": zero unit and one third) or, as in the Babylonian system, in base sixty: : two times ten sixtieths.

Mathematical language

What are the characteristics of the semio-linguistic system? First of all, the "mathematical language" (in a loose sense) is a written one[13]. Mathematical semio-linguistic *units* are written texts. Following and extending Laborde's ideas (1990), written mathematical texts are heterogeneous, made of natural language sentences, symbolic writings, diagrams and tables, graphs and illustrations. Their organisation follows what we call the fruit cake analogy, the natural language being the dough and the symbolic writings, diagrams, graphs and illustrations being the fruit pieces. To describe rigorously such a complex structure is far from easy.

Linguistic system

Students' ability to understand natural language mathematical texts (the "dough") is linguistic by nature. Mathematical natural language (we call it the "mathematicians jargon") is mostly the natural language itself; but Laborde (1982) showed there are some differences (unusual syntactic constructions like "Let *x* be a number...") between the jargon and the mother-tongue, difficult to interpret by students.

Symbolic writings (like "b^2 - 4ac > 0") make up a language, too (Brown & Drouhard, 2004, Drouhard et al, 2006), which is far more complex and different from mother-tongue than it appears at first sight; detailed and accurate descriptions of this language can be found in Kirshner (1987) and Drouhard (1992). Students must learn this language and its syntax[14] – which allows symbolic manipulation (Bell, 1996): the actual mathematic language, ruled by a rigid syntax, permits to perform operations on the symbolic expressions rather than on (mental or graphic) representations.

The present mathematical language is also characterised by a complex but precise semantics. Semantics (the science of the meaning) is the set of rules and procedures which allows interpreting expressions, in other words which allows relating expressions to mathematical objects.

The most accurate description of this semantics (how symbolic writings refer to mathematical objects and properties) is based on G. Frege's ideas (Drouhard, 1995).

13 which puts upside down the usual relationship between oral speech and written texts

14 the syntax is the part of the grammar which deals with the rules that relate one to another the elements of a language. (Syntax says that a parenthesis must be close once opened...

G. Frege's key concepts are "denotation" (which can be a numerical value (in the case of "20"), a numerical function (in the case of "x+1"), a truth value (in the case of "1 > 20") or a boolean function ("x+1> 20"), according to the type of symbolic writing)[15], and "sense " (the way denotation is given). The linguistic nature of students ' difficulties with symbolic writings is often underestimated, or confused with conceptual difficulties.

Semiotic system

Let's give an example of a semiotic problem in algebra. How to represent an infinite series of decimals? Imagine I ask you what the properties of the number 0,666… are. When multiplied by 3 it gives 2? No. Actually I had in mind the number 1999/3000. And yes, I cheated: I broke the representational rule of decimals, which is a semiotic rule (on how to interpret elements like "…") about linguistic objects (the numeric expressions).

There are more than one approach to mathematics semiotics, which were fully presented in the special issue N° 134 (2003) of *Educational Studies in Mathematics*. Duval dedicated his lifelong work to an extensive and coherent theory of semiotics of mathematics education. Three key concepts are the semiotic representation registers, the treatments (within a register) and the conversions (between different registers). Other researchers (see amongst others Otte, 2006) are investigating how to interpret mathematics education using the terms of the founder of semiotics, Charles S. Peirce (1991): the three types of signs –index, icon, symbol– and, maybe more interesting, the three types of inferences –induction, abduction, deduction).

An entire communication paper would not suffice to present even a small part of the outcomes of semiotics for the study of algebraic thinking. Hence we called "semio-linguistic" the mathematics representation system. Therefore students must handle both aspects of this representation system, the linguistic as well as the semiotic one, and the complex interaction between them.

INSTRUMENTS

Up to now we have seen that to do mathematics, students must not only know objects and how to represent them: now we will see that they need also to know how to use instruments (Rabardel & Vérillon, 1995) to operate on the representations of objects.

However, unlike object/representation opposition, instruments are not characterised by their nature (mathematical objects can also be tools, as noted by Douady, 1986) but instead by their use. Students, then, must learn what these instruments are and how to use them. Given that instruments are only characterised by their use, it is possible to propose a typology, based on their nature: material instruments (like rulers or

15 The AlNuSet software, developed by Giampaolo Chiappini allows (in a totally original way) a dynamic view of the denotation of algebraic expressions.

compasses, see Bagni, 2007), conceptual instruments (mathematical properties, like theorems), semiotic instruments (manipulations on semiotic representations) – this idea appears in L. S. Vygotsky, 1986); eventually one may consider "meta" instruments like strategies and, more generally, meta-rules.

THE RULES OF THE MATHEMATICAL GAME

We have seen that students must know what mathematical objects are and their properties, how to represent them and how to use instruments. Is this sufficient to do mathematics? Not at all: using a given instrument to operate on a given representation may be, or not, legitimate (even if done properly). For instance, to solve some numerical problems, some procedures are arithmetical (and are not legitimate in algebra) and other are algebraic (and are not legitimate in arithmetics).

Therefore algebra is not just a question of objects, representations and tools, but also of rules, which are saying what the actions are that we *may* or *may not* do amongst the actions we *can* do. Algebra is not a game in the same sense that chess is a game, but, like chess, algebra does have rules. These rules, moreover, are changing with passing times: the present way of doing differs from, say, the Renaissance Italian way of doing algebra. L. Wittgenstein (the "second Wittgenstein", the author of the *Philosophical Remarks*, or *On Certainty*, 1986) is an invaluable guide to clarify the extremely complex relationship between objects, signs, practices and rules. (Ernest, 1994, Bagni, 2006).

SUBPARADIGMS

Some rules (in particular logic) are universal for all mathematics. But other rules are related to a certain domain of mathematics. A square number is always positive, except when studying complex numbers. We call these domains "subparadigms", which are analogous Kuhn's paradigms, but less vast, and commensurable between them). This notion of subparadigm allows us to understand the shift from arithmetics to algebra. Semantics (and instrumental value) of the "=" sign change, thus objects (the equalities, the expression with letters) also change. The semiotic systems, although looking quite the same ("2+1 = 3" and "2+x = 3"), are different in fact.

IDENTIFYING KNOWLEDGE

A last type of knowledge allows us to identify (or recognise) if what we do is mathematical or not, and to identify to what domain of mathematics it belongs. When a student writes something that superficially looks like algebra but actually is wrong or meaningless, the teacher might say: "This is *not* algebra"; and if later the student succeeds in writing a meaningful and correct algebraic text, the teacher might comment: "*This* is algebra". With these statements, the teacher speaks about the student's text but also about algebra; he is actually teaching the student *what* is algebra – and what is not[16] (Sackur et al., 2005). We call this Identifying Knowledge; it is also that which allows us to recognise whether a mathematical problem is arithmetical or algebraic, and to choose the appropriate instruments to solve it (without certainty: this kind of knowledge is more abductive that deductive, see Panizza, 2005).

THE LAYERED DESCRIPTION

As said above, epistemography is not the theory of everything (or, better said, of every kind of knowledge)! Firstly, we only consider here the part of knowledge which is specific to mathematics; this leaves aside nonspecific knowledge, related with the use of (oral and written) natural language or with general reasoning capabilities. “Mathematical activities”, however, remains too vague to allow a precise description. Then, by analogy with the Internet reference model, which is a layered abstract description for the very complex communications and computer network protocol design, we propose a layered description of students’ mathematical activities.

The five descriptive layers of students’ mathematical activities are:

1. the School Layer (what are the students’ rights and duties, why and how to work in the classrooms and at home, what kind of participation is expected by the teacher etc.). This is what french-speaking researchers like Sirota (1993) or Perrenoud (1994) call ““being a student” as a job”[17]. A great number of students’ difficulties may be analysed in terms of the school layer: when they don’t want to learn, or don’t know how to, for instance.

2. The Maths Classroom Layer (how to do maths in the classrooms and at home, what kind of participation is expected by the maths teacher and what is the math teacher supposed to do, etc.). This part of the students’ activities is ruled by what Brousseau (1997) calls the *didactical contract* (see also Sarrazy, 1995, for an extensive survey of this notion). Many students’ difficulties can be analysed in terms of didactical contract, as it was brilliantly done by Brousseau (ibid) and followers.

3. The Modelling Layer, which is the description of, for instance, how students change a word problem into a matematical problem, or even how they change a mathematical problem (i. e. expressed in mathematical terms) into an other problem which they can solve with their mathematical tools. A whole field of mathematics education is devoted to the modelling part of the students’ mathematical activities (see for instance Lesh and Doerr, 2002).

16 which would be almost impossible to do with an explicit discourse within this context: definition or characterization of mathematics are epistemological statements, not mathematical statements

17 unfortunately, according to Dessus (2004) this concept is almost non existent in English-speaking sociology of education studies.

4. The Discursive Layer, which is the description of students' reasoning on mathematical objects. This reasoning may be expressed by a discourse (like "if x is greater than -3 then x+3 is positive and therefore...”), hence the name of this description layer[18]. In France, Duval (2006) is a main contributor in this domain, which is closely related to researches on argumentation (see for instance Yackel and Cobb, 1997) and on proofs (see for instance Gila Hanna, 2000).
5. The deepest, Symbolic Manipulation Layer, describes how students operate on symbolic forms to yield other symbolic forms which represent the solutions of the problem. In the case of algebraic thinking, not too many authors (see for instance Bell, 1996 or Brown & Drouhard, 2003) stress on that – mainly because on the contrary it is often overemphasized by textbooks and teachers.

It is important to notice that what is layered is the description, not the student's activity. It is very similar to what happens in linguistics: the language's description is split in phonetics, syntax, semantics, pragmatics etc. but the subject's act of speech, on the contrary, is of a whole.

CONCLUSION

A way to cope with the problem of identifying students' mathematical knowledge has long been to focus on students' solving abilities and this can explain the prominent role which has been given to assessment throughout the world. However, many mathematics educators remain reluctant to reduce assessment criteria to solving abilities. Our point is that solving abilities are not so relevant clues on what students know and what they are supposed to know. On the one hand, the student's failure in achieving a task does not give much information on what his or her deficiencies or misconceptions are. On the other hand, the student's success may just show his or her technical abilities, but we cannot be sure that s/he understood conceptually.

Then, how can we determine what students know and are supposed to know? We claim that epistemography can provide accurate answers to this question.

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18 It is not called "reasoning layer" since that could lead to the erroneous idea that there is no reasoning outside this level.

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SÁMI CULTURE AND ALGEBRA IN THE CURRICULUM

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Abstract: The Sámi culture's richness of patterns and structures give rise to the question whether an implementation of Sámi culture in the teaching of algebra might improve this teaching for the Sámi pupils. The Sámi have their curriculum but Sámi culture does not seem to be implemented in its algebra syllabus. Mathematical archaeology with respect to metonymy upon the Sámi cultural elements duodji and joik indicate possibilities for the teaching of algebra. But a remaining question is the Sámi mathematics teachers' view of the situation and of the suggested possibilities. The paper aims to prepare for empirical studies which focus on the Sámi mathematics teachers' mathematical archaeology upon their own cultural elements, as a basis for the teaching of algebra.

Key words: algebra; curriculum; mathematical archaeology; patterns; Sámi

BACKGROUND AND RESEARCH QUESTIONS

The Sámi are an indigenous people of the arctic who live in the northern part of Norway, Sweden and Finland, and in the Kola Peninsula of Russia (Kuhmunen, 2006). In 1990 Norway ratified the ILO Convention No. 169 concerning indigenous and tribal peoples in independent countries, and after this the Sámi in Norway got their curriculum (KUF, 1997). In the three latest national curricula, the Norwegian Ministry of Education has worked out special Sámi syllabuses for several subjects, but not for mathematics. One quite common interpretation of the curriculum is that the teaching of algebra should be the same for pupils in the Sámi core area in Northern Norway as for any pupil in our capital Oslo in the south. A quite different interpretation is that the Sámi should have their syllabus in mathematics.

This paper constitutes parts of a basis for a project which intends to research the possibilities of a Sámi algebra syllabus. The idea is that one researcher and one group of Sámi mathematics teachers together design and develop a teaching of algebra based upon Sámi cultural expressions. One lower secondary school in the Sámi core area wants to join a meeting where this project is introduced. The aim of this paper is to obtain important basis material for this important meeting. The basis material includes a) an analysis of the present situation regarding the teaching of algebra for Sámi pupils, and b) an analysis of some Sámi cultural expressions with respect to possibilities for a teaching of algebra. This leads to the two research questions of this study: 1: How is Sámi culture implemented in the algebra part of the national mathematics syllabus for lower secondary school? 2: If there are any (algebraic) structures to be found in Sámi cultural expressions, then how may these structures emerge?

THEORETICAL FRAMEWORK

Algebra

According to Lakoff & Núñez' (2000, p. 110), "Algebra is the study of mathematical form or "structure"". According to the latest TIMSS framework (Mullis et. al., 2007) algebra consists of patterns, algebraic expressions, equations/formulas and functions. Barton (1999) describes mathematics as a system of quantities, relations and space. His term "relations" is interpreted to be wider than just algebra. Fyhn (2000) uses the metaphor "pattern" similar to Lakoff & Núñez' (2000) "structure". Lakoff & Núñez (ibid.) focus on the terms "essence" and "structure" in their approach to algebra,

> Algebra is about essence. It makes use of the same metaphor for essence that Plato did – namely, Essence is form. …Algebra is the study of mathematical form or "structure". Since form (as the Greek philosophers assumed) is taken to be abstract, algebra is about abstract structure. (ibid., p. 110)

The analyses in this paper use the term algebra as by Lakoff & Núñez (ibid.).

Aesthetical Expressions as Basis for the Teaching of Algebra

Fyhn (2000) searched for and analysed relations between pupils' participation in different leisure time activities and their score in some TIMSS mathematics tasks from 1995 and 1998. The pupils were categorised according to their participation in different leisure time activities, activities which they performed at least once a week. The results pointed out some common features for the categories "creative-crafts-girls", girls who participate in activities that concern drawing or handicraft, and the "musicians", pupils who play an instrument. The creative-crafts-girls' mean test score was below the mean score, while the musicians scored high above the mean. Geometry was expected to be a domain where the creative-crafts-girls had their highest score, but their score in geometry turned out to be rather low. Actually these girls' highest scores were on tasks which tested the pupils' understanding of patterns. The musicians turned out to have a test score profile that to a large extent was parallel to the creative-crafts-girls' (ibid.). This gave raise to the idea of a teaching of algebra that is based on the pupils' understanding of patterns.

Symmetry is an important part of the two latest Norwegian mathematics syllabuses for primary school (KUF, 1996; KD, 2006b). But the approach to symmetry is limited to be via geometry. Norway give less priority to algebra in school, and algebra is the domain where the Norwegian pupils have their lowest score in the TIMSS (Trends in International Mathematics and Science Studies) (Grønmo, Bergem, Nyléhn & Onstad, 2008). This opens for new ways of teaching of algebra. Due to the Sámi culture's apparently richness of patterns and structures, a good implementation of Sámi culture in the mathematics subject syllabus might lead to an improved teaching of algebra for Sámi pupils. Before any approaches can be done towards the design of new approaches to school algebra, there is a need for investigating how and to what extent structures and patterns from Sámi culture are integrated in the mathematics syllabus.

Parts of this investigation will take place in cooperation with the teachers; the rest will take place in this paper. In addition the apparently richness of structures and patterns in Sámi cultural expressions need to be confirmed and described before they can be treated as a basis for the teaching of algebra.

Mathematical Archaeology

Mathematics can be integrated into an activity to such a degree that it disappears for both the pupils and the teachers. According to Skovsmose (1994, p. 94) "Mathematics has to be recognised and named, that is the task of a mathematical archaeology." It makes a difference whether the teaching is built upon situations that contain possibilities for application of mathematics or just for descriptive purposes. Many sorts of descriptive uses of mathematics can be possible as well as appropriate through mathematical archaeology; mathematics can be treated as an emerging subject (ibid., p. 90). It is important to a project, which contains mathematics as an implicit element, to spend some time on mathematical archaeology. The reason is: "If it is important to draw attention to the fact that mathematics is part of our daily life, then it also becomes important to provide children with a means for identifying and expressing this phenomenon" (ibid., p. 95). If there exists any algebra in the Sámi culture, it has to be implicit and hidden. A result of a mathematical archaeology may be that such algebra is recognised, named and described. A description of such algebraic structures may lead to an increased consciousness about possibilities for the teaching of algebra.

METHOD

The first research question will be answered by a) a survey of the development of the Sámi Curriculum in general and analyses of the treatment of algebra in it, b) a survey of the mathematics textbooks for Sámi pupils and analyses of their treatment of algebra, and c) analyses of the treatment of algebra in the national tests for Sámi pupils in mathematics and Sámi language. The second research question concerns the emergence of mathematics from elements in the Sámi culture. The research question will be enlightened by performing mathematical archaeology (Skovsmose, 1994) upon duodji and joik. Duodji is the name of Sámi craft, handicraft and art (KD, 2006a), while joik is the old Sámi folk music (Graff, 2001). The emergence of mathematics is categorised into three different levels; 1: recognition, 2: naming and 3: description.

ANALYSES

The Sámi Curriculum

The Sámi's right to take care of and develop their language and culture has not always been accepted in Norway. The *norwegianisation* (assimilation) of the Sámi has been extensive and long-lasting (Minde, 2005). The norwegianisation also has led to a disparagement of Sámi culture, and this gives reasons to believe that there are few tracks of Sámi culture to be found in the Norwegian curriculum. In 1989 the Ministry of Education published the Sámi syllabuses (KUD, 1989) as a special supplementary booklet to the national curriculum for the compulsory school. The intention was to adapt the traditional syllabuses to the Sámi culture and the Sámi surroundings. Some subjects got their own syllabus, but mathematics did not. The 1997 national curriculum (KUF, 1996a; KUF, 1996b) included a special Sámi curriculum (KUF, 1997). The mathematics syllabus was identical with the Norwegian one except for the illustrations.

The national curriculum of 2007 (KD, 2006a) includes a special Sámi syllabus for seven subjects, but not for mathematics. Reasons for a particular Sámi mathematics syllabus are that the Sámi and the Norwegian numerals are structured differently (Nickel, 1994), and that the traditional Sámi measuring units are based on body measures and not on the SI-system (Jannok-Nutti, 2007). For the pupils who learn Sámi as their first language, Sámi units of measurement and mathematical methods are treated as Basic Skills in mathematics, as an integrated part of the subject Sámi language (KD, 2006c). And “skills in mathematics require understanding of form, system and composition” (ibid., p. 3). According to Lakoff & Núñez (2000), this is algebra. But according to the curriculum, this is part of the syllabus in Sámi language. For the pupils who learn Sámi as their second or third language, basic skills in mathematics mean general concept development, reasoning and problem solving as well as the understanding of quantities, amounts, calculations and measurements (KD, 2006a). For these pupils the syllabus has no aims regarding their understanding of form, system and composition.

In the Norwegian national curriculum (KD, 2006b), the subject area “numbers and algebra” for the lower secondary school is presented this way

> The main subject area numbers and algebra focuses on developing an understanding of numbers and insight into how numbers and processing numbers are part of systems and patterns… Algebra in school generalises calculation with numbers by representing numbers with letters or other symbols. This makes it possible to describe and analyse patterns and relationships. Algebra is also used in connection with the main subject areas geometry and functions. (ibid., p. 2)

As for the pupils who learn Sámi as their first language at school, the understanding of form, system and composition may be integrated with descriptions and analyses of patterns and relationships in the mathematics lessons. But this message is only implicit in the curriculum. Thus an interesting question is whether the Sámi culture is integrated in the teaching of algebra for the Sámi pupils.

Textbooks

The Sámi mathematics textbooks are Norwegian textbooks translated into Sámi language. The Norwegian Directorate for Education and Training (Udir, 2004) presents two mathematics textbooks in Sámi language for lower secondary school; one of them is approved for the curriculum of 1997, and the other one is Finnish. For economic reasons Norway offer the lower secondary school pupils no Sámi mathematics textbooks which are approved for our latest curriculum. However, these pupils have their right to get appropriate books: The United Nations' Declaration on the Rights of Indigenous Peoples Article 14 claims that "Indigenous peoples have the right to establish and control their educational systems and institutions providing education in their own languages, in a manner appropriate to their cultural methods of teaching and learning" (UN, 2007, p. 6). The Sámi parliament, the Sámediggi, is an elected representative assembly for the Sámi in Norway (Kuhmunen, 2006). The Sámediggi's Youth Committee underlines the importance of getting Sámi textbooks. They sent an open letter to the Norwegian Minister of Education where they demand that the Ministry carry out necessary actions in order to improve the school-days for Sámi children (Nystø Ráhka, 2008). The textbook situation for Sámi pupils is far from satisfactory. Thus it is not any surprise that no attention is paid towards including Sámi culture in the algebra paragraphs in the existing textbooks.

National Tests

From 2003 Norwegian pupils have taken part in national tests as part of a national system for quality assessment (KD, 2003). From 2007 the mathematics tests were replaced by tests in mathematics as a basic skill in every subject. One result of this is that algebra is no longer part of the tests. The tests are translated from Norwegian to Sámi language; the Sámi pupils are offered no special tasks. The Ministry of Education and Research have decided that pupils who have Sámi as their first or second language will be tested in mathematics as a basic skill in this subject (KD, 2007c; KD, 2008). The Norwegian Directorate for Education and Training will carry out the translations of the mathematics tests into the three Sámi languages (ibid.). The national tests in mathematics as a basic skill do not reflect the pupils' achievement of any goals which are particular for the Sámi curriculum, and the algebra goals for the Norwegian pupils are neither reflected in these tests.

Duodji and Joik

The ornamentations in Sámi handicraft, *duodji*, and the Sámi folk music, *joik*, are both rich on structures and patterns. This claim is based upon the doctoral dissertations of Dunfjeld (2001) and Graff (2001). According to Dunfjeld (2001) the Sámi people's understanding of their own ornamentation differs from the pure formal understanding of ornamentation that we find in Western Europe. Thus she introduced the term "Tjaalehtjimmie" which has a meaning beyond pure decoration; "it is composed by signs, ornamentals and symbols which together may give meaning "(ibid., p. 102, my translation). For example may the meaning of the triangular engraving be decided from its localisation and orientation related to other symbols in a composition like in figure 1.

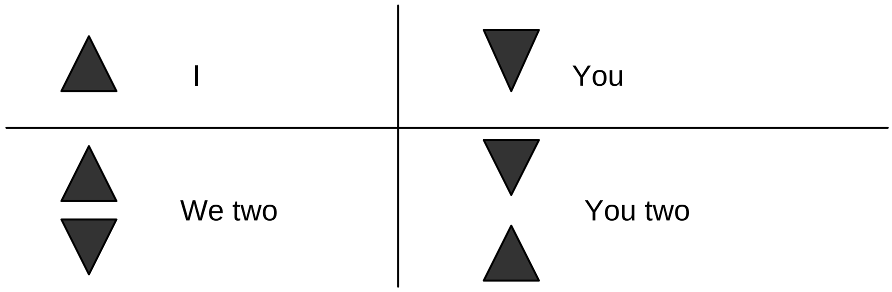

Figure 1. How the triangular engraving may symbolise personal pronouns in first and second person, and in singular and dual (Parts of figure from Dunfjeld, 2001, p. 109)

In duodji there are several more or less advanced plaited patterns. Fyhn (2006) describes hair plaiting by first splitting the hair into three equal parts. Plaiting can be further described by numerous repetitions of “take the right part and cross it over the mid-part. Then take the left part and cross it over the mid part”. The right part, whichever it is, can refer to all of the three parts of the hair, and so is for the mid part and the left part as well. This is what we understand with conceptual metonymy (Lakoff & Núñez, 2000), and this exists outside mathematics.

> This everyday conceptual metonymy …plays a major role in mathematical thinking: It allows us to go from concrete (case by case) arithmetic to general algebraic thinking… This everyday cognitive mechanism allows us to state general laws like “x + y = y + x”, which says that adding a number y to another number x yields the same result as adding x to y. It is this metonymic mechanism that makes the discipline of algebra possible, by allowing us to reason about numbers or other entities without knowing which particular entities we are talking about. (ibid, p. 74-75)

According to the curriculum the Sámi ornamentations are geometry (KD, 2006a). Dunfjeld (2001) denotes these structures as geometry, too, and she refers definite to figures as triangles, rhomboids, squares and rectangles. Her mathematical archaeology is at level two; naming. When she refers to the organisation of the geometrical figures and the patterns they shape, she does not denote it as mathematics anymore. Fyhn’s (2006) description of ornamentation as metonymy is mathematical archaeology at level three, description.

Graff (2001) claims that researchers have focused on joik from different perspectives: as text, as melodies and rhythms, and as communication. To “joik” a person means to perform a particular joik which is dedicated to this person; the joik is an expression with a meaning (ibid.). The pitch constitutes an analogy for conceptual metonymy in music, when two or more people sing together. The structure of the song is given on beforehand; independent of what particular pitch to use. Graff (ibid.) points out, among other things, that the melodic motive in joik is based upon melodic patterns which in turn might have different shapes. The structuring of the joiks which he investigated, show that a rhythmic motive might be repeated throughout the complete joik (ibid.). Algebra is the study of mathematical form or “structure” (Lakoff & Núñez, 2000), and joik is just a way of expression that like other music is built up by given structures. According to the curriculum, the understanding of how different patterns and structures influence artistic and musical expressions is part of mathematical skills in the subject music (KD, 2006a). Graff's (2001) term "rhythmic motive" is the name of a structure and could be denoted as mathematical archaeology at level two. He gives thorough descriptions of the structures as well, and he uses words like "ascending –descending melody line (inverted U-form)" (ibid., p. 210, my translation) and "transposition" (ibid., p. 214, my translation). But there is no mathematics connected to the names and the descriptions of these structures. The structures that constitute a basis for duodji ornamentations and for joiks may be identified and described by mathematical terms. The process in which algebra is emerging from these aesthetic expressions can be carried out as mathematical archaeology (Skovsmose, 1994) at three levels. But because joik as well as duodji express more than just aesthetics and structure, the meaning aspect need to be focused and enlightened.

CONCLUSION

The Sámi curriculum (KD, 2006a) offers a special Sámi syllabus for several subjects, but not for mathematics. "The understanding of form, system and composition" is part of the syllabus for Sámi as first language. Together with "descriptions and analyses of patterns and relationships" from the algebra syllabus, this opens for an integration of elements from the Sámi culture in the mathematics lessons. But that depends on whether the Sámi mathematics teachers are aware of and agree to these possibilities, and how the Sámi language teachers approach "form, system and composition" in their lessons. Due to the norwegianisation (Minde, 2005) there are reasons to believe that the teachers are not aware of the possibilities of integrating elements from their culture in the teaching of algebra.

The United Nations' Declaration on the Rights of Indigenous Peoples (UN, 2007), states that the Sámi lower secondary school pupils have their right to get appropriate mathematics textbooks in their own language. There are Sámi versions of Norwegian textbooks for primary school and for some of the grades in lower secondary school, but many of these books are based on a lapsed curriculum. And no special attention is paid towards including Sámi culture in the algebra parts of these textbooks. The lack of Sámi mathematics textbooks results in extra work for the teachers. Sámi pupils are offered translated versions of the Norwegian national tests in mathematics as a basic skill in every subject. The fact that these tests do not concern any algebra is an example of how Norway gives less priority to algebra in school. The national tests neither reflect the pupils' achievement of any goal in the Sámi curriculum. Aesthetic expressions may become a resource in the teaching of algebra: According to the Sámi curriculum (KD, 2006a) the relations between aesthetics and geometry are elements in the work with duodji decorations, while the music syllabus focus on the understanding of different patterns and structures. No connection between aesthetics and algebra is found in the Sámi curriculum.

One question for the further research is whether and to what extent the Sámi mathematics teachers find the project relevant and worthwhile taking part in. One more question is how metonymies might function in bridging the gap between Sámi cultural discourses and the algebra teaching discourse. The term "discourse", is here used as by Foucault (2004, p. 53), "…discursive practice is a place in which… objects is formed and deformed." These questions are closely interwoven and the further development of the project depends on the meeting between the researcher and the teachers. Maybe the teachers really want to join the project. But one other outcome is that the teachers dislike the ideas of creating an algebra teaching based upon Sámi cultural expressions. Another outcome might be that the teachers give priority to other parts of mathematics than algebra at the moment. A third possible outcome is that the teachers want to take part in the project, but that metonymies turn out to be less useful than they seem at the moment.

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PROBLEM SOLVING WITHOUT NUMBERS AN EARLY APPROACH TO ALGEBRA

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Abstract: *This paper reports a research project that aims at finding a good approach to school algebra using magnitudes and measurement. Thereby we not only focus on the way algebra can be taught effectively but also at on when in student's mathematical education a geometric and measuring approach can be successful. For this purpose we provide a theoretical framework and modify an early algebra program developed for first-graders to implement it in different age-levels.*

Key Words: *Algebraic Symbolizing, Early Algebra, Cognitive Gap, Measurement*

INTRODUCTION

In Germany, as in many other countries, algebra is taught as generalized arithmetic (see e.g. Lins & Kaput, 2004) after a long term arithmetical education. Reasons can be found on the one hand in the historical development of algebra as a medium for solving advanced arithmetical problems, on the other hand in the Piagetian stages of cognitive development. According to Piaget's theory children achieve the formal operational stage – and therewith the capability for abstract reasoning - not before the age of eleven (Piaget & Inhelder, 1972). It is however not self-evident that all aspects of algebraic thinking require achievement of the full formal operational stage.

Linchevski (2001) talks about a "cognitive gap", which characterizes "these steps in the pupil's learning experience where without a teaching intervention [...] he or she would not make a certain step" (Linchevski, 2001, p. 144), and this is independent of the Piagetian stages.

If one reinterprets the cognitive gap in terms of Wygotski's zone of proximal development (Wygotski, 1987), the cognitive gap marks not only the difference between what a learner can achieve without help and what a learner *cannot* achieve *without* help, but what a learner *can* achieve *with* help: in this case developing algebraic skills.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

Early Algebra

The idea of teaching algebra in earlier grades beyond a preparatory pre-algebraic way is most welcome as one can see in several early algebra projects (see Carraher & Schliemann, 2007). A reason for the popularity of early algebra is that the problems that students have with school algebra is likely to be based mostly on long experience of arithmetic classes without algebraic contents (see McNeil, 2004). This leads us to a first question:

1. Are there coherences between students' arithmetical skills and their effective approach to algebra?

From Carraher and Schliemann's (2007) review of the seven most common difficulties middle and high school students have with algebra (Carraher & Schliemann, 2007, p. 670) we can extract at least two main ideas that are demanded in arithmetic but are no longer desired while dealing with algebra. These are on the one hand the belief that the equal sign only represents an unidirectional operator that produces an output on the right side from the input on the left side, and on the other hand a focus on finding particular answers.

Algebraic symbolizing

Regardless of whether it is taught as regular school algebra in grade 7 or as early algebra in an earlier grade, if algebra is to be taught at school we have to think about what school algebra is meant to be. School algebra is taught as dealing with algebraic symbols, terms and equations, but often without context. This is accompanied by the problem, that students do not see the point in algebraic symbolizing.

"The lesson from history has implications for teaching in the sense that the potential of dominating algebraic syntax will not be appreciated by students until they have experienced the limits of the scope of their previous knowledge and skills and start using the basic elements of algebraic syntax." (Rojano, 1996, p. 62)

Van Amerom proposes that "algebra learning and teaching should be based on problem situations leading to symbolizing instead of starting with a ready-made symbolic language." (van Amerom, 2002, p. 10)

An alternative to conventional algebraic symbolizing is to allow the students to develop their own sign system when solving algebraic problems. But the algebraic syntax, as we know it and the way it is used worldwide, is a sophisticated tool for communicating about algebraic problems, and thus the understanding of and the ability to use and manipulate conventional algebraic symbolism is an important goal of algebra education (see Dörfler, 2008).

Summarizing, on one hand there is a negative correlation between students' advanced arithmetical skills and their effective approach to algebra. On the other hand there is the need to teach algebraic syntax in an environment that brings students to the limit of their mathematical abilities. This leads us to the conclusion that if algebra and algebraic syntax can in fact be taught in early grades successfully then it should indeed be taught in these early grades for the following reasons.

First of all, an earlier approach to algebra offers a lot more mathematical exercises that children can understand but cannot solve with the mathematical knowledge they've achieved up to then. At the same time the emphasis on arithmetic is reduced, which may decrease a habituation effect to arithmetic. Apart from that, lower achievers in arithmetic may profit from an early approach to algebra and algebraic syntax can support their algebraic thinking strategies.

The MeasureUp- Program

An unconventional way of teaching school algebra is taken by the MeasureUp-Program (Dougherty & Slovin, 2004) which combines early algebra with a fast introduction to common algebraic symbolization, at an early stage in primary school even before numbers are introduced. MeasureUp is based on a teaching experiment from the 60s implemented by Davydov (1975), a Wygotskian student. Within this teaching experiment the students develop abstract algebraic thinking by comparing magnitudes, like length, area, volume, etc. of concrete objects. The comparison of magnitudes is written down firstly with the help of signs of different sizes and finally with letter inequations and equations. The teaching of numbers follows only when the students can handle the algebraic syntax of elementary linear equations properly.

Our main concern is with the idea of introducing the abstract use and manipulation of the algebraic symbol system by concrete comparison of the magnitudes excluding numbers. We want to find out if this concept, which we will call the MeasureUp-Concept, will work for primary school children even though they have already have been introduced to numbers and arithmetic. This leads us to the following question.

2. Does the MeasureUp-Concept give German primary school-children a “good” approach to algebra and algebraic symbolism?

To answer this question we concentrate on two basic ideas of algebra, expressing magnitudes and their relations in letters and detaching the thinking from the concrete context.

The various aspects of letter variables range from letters as specific unknown over letters as generalized numbers to letters as changing quantity (see e.g. Küchemann, 1978). In our very first approach we have not seen it as important which of these aspects the children were working with. We are primarily interested in the question of whether the children are really seeing the letters as numbers and not developing the misconception of seeing letters as objects. As it is not intended to focus the children on magnitudes as numbers we have to differentiate the two categories letter as magnitude and letter as object. Bertalan (2008) claims, that a geometric approach supports the (mis)conception of letters as objects.

Within the intervention the children are working with concrete objects whose different magnitudes are compared. We want to know if the children are able to detach their thinking from the concrete material and if they are able to deal with word problems that do not refer to concrete material.

When to teach algebra and algebraic syntax?

Our focus of interest lies in the Measure Up-Concept, the introduction of abstract use and manipulation of the algebraic symbol system by concrete comparison of the magnitudes excluding numbers, which is only a small but important part of the MeasureUp-Program. Because the MeasureUp-Program starts with the first grade it is reasonable to arrange our first observations at this age-level.

However, there are several widespread reasons, why algebraic syntax without numbers should *not* be taught in primary school, including curricular issues and the argument that this is too far away from a primary school students' everyday use of mathematics and thus should not be subject of mathematic lessons. With these reasons in mind, we come to another question of interest:

3. Does the Measure Up-Concept work in high school grades lower than grade 7 in the sense that none of the difficulties named above appear.

METHODOLOGY

Our research is based on the paradigm of design based research (DBR), which "blends empirical educational research with the theory-driven design of learning environment" (The Design-Based Research Collective, 2003, p. 1). It contains two main goals which have to be well-connected. These are on the one hand designing learning environments, on the other hand developing theories of learning. DBR happens in multiple cycles of design, implementation, analysis and redesign. The following investigation marks the first completed cycle of design, implementation and analysis. Later we will state conclusions for redesign.

As the starting point for the intervention we chose the MeasureUp-Program which we modified for our purpose. As variables are not part of primary school curricula, we have been looking for a school that enables us to teach the MeasureUp-Concept. We found that a Montessori primary school class with mixed age-groups would fit best for our first investigation. The self directed activity of children in a Montessori class allows us a flexible intervention alongside the regular class.

Implementing the MeasureUp-concept in a Montessori class made it necessary to develop material that children can work with on their own. So we developed exercise books which contain the introduction and comparison of magnitudes not only of Montessori but also other concrete materials, the setting up of equations and inequations, the so-called statements, and transforming inequations in equations, including transitivity and commutativity.

Example 1:

Compare

1. Take boxes I, II and III

2. Name the volumes of the boxes.

3. Compare the volumes of boxes I and II, write a line-segment and a statement.

4. Compare the volumes of boxes II and III, write a line-segment and a

statement.

5. Which statement can you write down for the volumes of boxes I and III without comparing the volumes?

The last exercise book contains word problems that do not refer to concrete material and word problems that contain numbers.

Example 2:

Word problems

A street has length A. Julia has already walked length B. How far does she still have to go?

A street has length L. Tim has already walked 200 m. How far does he still have to go?

A street has length 845 m. Hans has already walked 220 m. How far does he still have to go?

To address the question of whether there are coherences between students' arithmetical skills and their effective approach to algebra, we had to collect data about the arithmetical knowledge of the children. Thus every student attended the half-standardized interview ElementarMathematisches BasisInterview (EMBI, basis interview on elementary mathematics,) before the intervention (Peter-Koop et al, 2007). Thus we are able to compare high achievers with low achievers.

Then we introduced the exercise books to the children and allowed them to work with them during their free activity time. With some students or student groups we made appointments which gave us the opportunity to videotape the students while they were working with their exercise books and explaining their work to an interviewer. This happened within the principles of the Montessori school which means: students join voluntarily, the intervention will take part in an individual atmosphere and mistakes are not to be corrected. The work will consider the individual stage of development and, if required, the exercises will be extended or modified. So the interviewer held a double role as interviewer and teacher. Then we transcribed the videos and conducted a series of qualitative content analyses. To answer our first question

1. Are there coherences between students' arithmetical skills and their effective approach to algebra?

we have been coding in regard to the following topics:

- The students' possible belief that the equal sign only represents a unidirectional operator that produces an output on the right side from the input on the left side.
- The students' focus on finding particular (i.e. numerical) answers.

These we used as categories for our content analysis. Then we compared the findings of a, according to the EMBI, lower achiever with findings of a higher achiever.

To answer our second question

2. Does the MeasureUp-Concept give German primary school-children a "good" approach to algebra and algebraic symbolism?

we concentrated on the ideas of expressing magnitudes in letters and detaching the thinking from the concrete context. We did a qualitative content analysis with the two categories letter as number and letter as object. Also we did a qualitative content analysis on the children's work with concrete material and also on the situations where children are solving word problem which does not refer on material (Example 2). For the latter we did not use pre-set categories, but generated them inductively.

For answering the third question,

3. Does the MeasureUp-Concept work in lower high school grades than grade 7 in the sense that none of the difficulties named above appear.

we are planning further cycles of design, implementation, analysis and redesign in a 5th grade of a German high school.

OBSERVATIONS ON STUDENTS' ACTIVITIES

The design of the study only allowed us a focus on a small number of students. So our following interpretations are based on two case studies, Jay and Elli, which have been chosen for following reasons. Both students, a boy and a girl, are 3rd graders and will leave the class in the following year to join grade 4-6.

As showed by the EMBI, Jay is good at counting and handles interpreting and sorting of numbers beyond 1000 easily. He shows multiple strategies in addition, subtraction and multiplication and is able to solve division problems in an abstract way. Elli is also good at counting, but not as good as Jay and she is able to interpret and sort three-digit numbers. She is solving addition and multiplication problems through counting and needs proper material for solving multiplication and division problems. So we can call Jay a higher achiever and Elli a lower achiever. This is important for our first question, whether success in algebra class depends on arithmetic skills.

The analyses of both the transcripts and the exercise books showed that there is no dominance of the belief that the equal sign only represents a unidirectional operator that produces an output on the right side from the input on the left side. Jay and Elli both wrote and completed several equations of the form D+B=A and D=A-B, without accounting for the direction of the equation. The transcripts also did not show any sign of preference or confusion about writing the equations the one or other way.

We had a different result when analyzing the focus on particular answers. We take a look at how Elli and Jay dealt with Example 2 (see above).

Jay: …how far does she still have to go?

Teacher: Right, you have said...
Jay: D. *J wants to write down D, but the teacher stops him.*
Teacher: Wait, can you write an equation?
Jay: What's that?
Teacher: A statement, with equal signs and plus and minus.
Jay: Err, D plus B equals A.
Teacher: Yes, right, you can write that down. *J writes it down*.
Jay: Yes, but first of all I can write down D. *J writes down D and underlines it.*

Here we can see that Jay is looking for a particular answer. He names the length that still has to be travelled with D and wants to write it down as answer. The intervention of the teacher reminds him, that he can find a statement that shows how he can get length D with length A and B. Certainly, because of the early intervention of the teacher, we do not know if Jay would have written a statement without prompting. As we can see, he has no difficulties in finding the equation D+B=A and later on he will have no problems with transforming the equation into D=A-B. But for him, both equations do not belong to the solution. In his exercise book we can find both equations in a subsidiary position. By contrast he insists in writing down and underlining D “first of all” right behind the word problem. The underlining is an indicator that for Jay D is the particular answer of the word problem but the equations are not.

Elli handles the word problem differently. At first she has problems with understanding the question and after the encouragement of the teacher she draws the street and attaches the given information. Then she suggests different statements that are however not solution-orientated. With some help by the teacher she finally writes down the statement S=A-B.

The following transcript shows that generally Elli feels comfortable with using letters.

Elli: A street has length 845 meters.
Teacher: Hm.
Elli: Is the length M. Hans already walked 200 meters. How far does he still have to go?
Teacher: Hm.
Elli: I want to do that with letters.
Teacher: You want to do that with letters? Ok. Which letters do you want?
Elli: N and M.

By contrast Jay again is eager to calculate the solution and notes “that’s easier”.

If we interpret the observed situation, while keeping the research question in our mind, we explicitly have to differentiate algebraic thinking from using algebraic syntax. Elli’s difficulty with the last word problem that prompts the wish to use letters is a sign of her low achievement in arithmetic. We can also see her difficulties with algebraic thinking and algebraic syntax, but nevertheless Elli is expecting benefit from using algebraic syntax. Jay on the contrary has no difficulties with solving the word problems because he realizes their algebraic structure. He does not use the algebraic syntax, but this is not because he cannot use it. We have seen that he can easily find a proper statement and is able to manipulate the equation. We conjecture he does not use algebraic syntax because the word problems are easy for him and he is focussing upon an answer where the approach is a minor matter.

We do not suspect that lower achievers in arithmetic will be likely to have fewer difficulties with algebraic thinking and using and manipulating algebraic syntax than higher achievers. But they may be more open for the use of algebraic syntax while working on word problems, because they expect a benefit for solving word problems and therewith are more accessible for the use of algebraic syntax.

As we have seen students at that age-level can work easily with letters as denotation.

For a "good" approach to algebra we need to know whether they name the object or the magnitude. By viewing the transcripts we found evidence for both letter as object and letter as magnitude. But we also observed a third category as is seen in the following transcript.

Teacher: Which letter stands for example for this length? *The teacher shows a grey stick.*
Jay: Err, the lowest, the lowest letter of all, which...ah...which is the lowest one? *Jay is sorting the letter-cards*
Jay: So we call the small grey ones U. This is an U.
Teacher: So, then you can name all.
Jay: A is always the biggest one.

Jay is naming "the small grey ones". Thus he is naming not only one object, but a class of objects with the same attributes. But he is naming the objects and not the magnitudes. Although the letter U names an object, the size of the object is still contained in the letter, because it is "the lowest" letter and the grey sticks have the lowest length. There is no lower letter than U because the letters V - Z are not available on letter-cards. Furthermore we can see that there is also a highest letter, the letter A which names "always the biggest one". Elli shows a different but similar performance when she has to compare the width of two stripes which have same width but different length.

Elli: Do you have an U?
Teacher: I do.
Elli: Like Urs? And a D like Donatella?
Teacher: A D like Donatella? Ok.
Elli: My mother. An U and a D like my mum.
Teacher: There's the D, look. So, you can already write that down. Here is.... which has the width U?
Elli: Dad is bigger.

Like Jay, Elli includes the size of the object in the letter. For that purpose she refers to the size of family members. But Elli is focusing on what differentiates the objects and not on what is being compared. So she is choosing the letters while focusing on the length and not the width. Therefore she picks two letters that refer to two family members which different length, U for her "bigger" dad and D for her smaller mum.

Beside the categories letter as object and letter as magnitude we can summarize the above observation under the category letter as object with a certain size. This leads to new questions of interest. Does a geometric approach to algebra support the idea of a letter as object with a certain size instead of letter as object and letter as magnitude? And if so, is it to be seen as positive or negative for a "good" approach to algebraic thinking and/or algebraic syntax?

Finally we take a look at the word problems of Example 2 again, to find out how Jay and Elli handle problems that do not refer to concrete materials but to imagined objects, in this case a street. Both were offered the opportunity to use paper strips or sticks to represent the street or to draw the street. For solving the second word problem, which mixes letters with numbers, Eli drew a street, while Jay used paper strips. The following observation was made as Elli was working on the word problem.

Teacher: So, a street has length N, Tim already walked 200 meters.
Elli: Then he still has to go 400 meters.

With Jay we can make a similar observation.

Jay: ...that is length L. *J displays a different paper strip.*
Teacher: That is length L? Ok.
Jay: 200 meters, how big is a man, that big, then, I think, these are about 200 meters.
Teacher: Ok.
Jay: And this small edge here, that goes here, are the remaining…?
Teacher: Meters. How do you call the remaining meters?
Jay: 50 meters?

Both understand the offered material not as aid for visualising the real street but as a scaled down model version of the street. They can't detach themselves from the concrete material thus they are not able to solve this word problem without assistance.

PERSPECTIVE

In regard to our questions the evaluation of the exercise books and the transcripts did not provide as conclusive results as we had hoped for. In particular looking at how the students perceive the letters brought up new questions. These questions have to be considered in our redesign. We also have to work more closely on the abilities of the children. We have seen that Jay did not use the algebraic syntax in some cases because he did not require it. As a main goal of the intervention is to adapt the use of algebraic syntax, we have to modify these particular exercises so that we can adapt them easily and flexibly at the abilities of the students. Furthermore we decided to move the question of the ability to detach the thinking from the concrete context to the projected intervention in grade 5. There we also will try to gain more clarity if a long term arithmetic education gets in the way of an effective approach to algebra.

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THE AMBIGUITY OF THE SIGN $\sqrt{\ }$ [19]

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In this paper an educational problem is discussed deriving from the ambiguity of the radical sign, $\sqrt{\ }$, produced by its shift in meaning when passing from arithmetic to algebra. This problem is concerned with understanding difficulties that are linked to a particular tradition of teaching in which the radical sign is introduced by means of the square root notion. As a conclusion it indicates that any teaching proposal should take into account the distinction between root and radical.

Key words: roots, radicals, meaning, textbook

INTRODUCTION: The problem under investigation

The ambiguity of the sign $\sqrt{\ }$ as a consequence of the change in its meaning when passing from arithmetic to algebra often goes unnoticed by teachers and textbook authors. This lack of perception may be the cause of certain cognitive conflicts experienced by teachers and students.

This work takes its cue from one of these conflicts. It is a conflict expressed by a Spanish secondary school mathematics teacher called Patricia, on attempting to understand the definition of equivalent radicals. She states that the equality $\sqrt[6]{3^2} = \sqrt[3]{3}$ cannot be true, since in the expression on the left the index of the root is even, so that it has two opposing roots, two solutions, whereas in the expression on the right the index is odd so it only has one root, which means that the two expressions have a different number of roots.

The conflict expressed by Patricia leads to the difficulties and controversies related to the values, properties and rules of radicals, which are the ultimate aim of this work.

Examples of that, are the students opinion about the statement $\sqrt{\boldsymbol{25}} = \pm\ 25$. (Roach, Gibson and Weber, 2004), the value of $(-8)^{1/3}$ =-2 (Even and Tirosh, 1995; Goel and Robillard, 1997; Tirosh and Even, 1997), and the rule for multiplying imaginary numbers (Martínez, 2007).

THEORETICAL FOUNDATIONS

To support the work carried out, a theoretical approach has been adopted that has three fundamental references.

1. One of these looks at the cognitive side, taking into account the need to reconceptualise signs that change meaning when passing from arithmetic to algebra (Kieran, 2006, p. 13).

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This happens with the sign $\sqrt{\ }$ which changes meaning, since it either indicates an operation, as happens in $\sqrt{\boldsymbol{4}}$, or indicates the main root of this operation, as happens in the solution to the equation $x^2 - a = 0 \rightarrow x = \pm\sqrt{\boldsymbol{a}}$.

There are examples of this double meaning to be found in the teaching tradition that appears in textbooks by such influential authors as Euler.

Euler considered that:

> 150. (…) the square root of any number always has two values, one positive and the other negative; that $\sqrt{4}$, for example, is both +2 y -2, and that, in general, we may take $-\sqrt{a}$ as well as $+\sqrt{a}$ for the square root of a (…). (Euler, 1770, p. 44)

In Euler's text the $\sqrt{\ }$ sign is used ambiguously. In $\sqrt{4}$ it is perceived as an indicated operation (finding the square root of 4) and it is associated to the set of two results, in this case +2 and - 2. In $+\sqrt{a}$ it is perceived as a result of the aforementioned process and designates one of the two roots of *a*.

This duality of meaning starts in arithmetic when introducing the $\sqrt{\ }$ sign in order to indicate an operation in an abbreviated way, *the fifth elementary operation*[20]. In arithmetic this number can be found and it is unique. Thus, for example, the square root of 4 is 2, which is written $\sqrt{4} = 2$.

Things change in algebra, since the square root of a $(a > 0)$ cannot be calculated, so that to indicate its value the expression $\sqrt{a}$ is introduced, which no longer represents an indicated operation but a result.

3. The second reference looks at the formal component. The mathematicians have decided to assign to the radical expression, $\sqrt[n]{\boldsymbol{x}}$, $x \geq 0$, only one value, one of the roots of x, the root no negative, the one that they name principal root. With this restriction, the right thing is to write $\sqrt{\boldsymbol{4}} = \boldsymbol{2}$, not $\sqrt{\boldsymbol{4}} = \pm\boldsymbol{2}$.

> We agree to denote by $\sqrt{\boldsymbol{a}}$ the positive square root and call it simply the square root of a. Thus $\sqrt{\boldsymbol{4}}$ is equal to and not -2, even thought $(-2)^2=4$ (Lang, 1974. p. 10).

With this decision, the mathematical problem of the ambiguity of the radical sign disappears, but no the didactic problem. Students do not learn only what they are told; much of students' learning occurs when they attempt to make sense of the mathematical situations that they encounter (Roach, et al. 2004). To help students to make sense of the formal definition there are several options:

A) To avoid contradictions. If $\sqrt{\boldsymbol{4}} = \pm\boldsymbol{2}$, then $\sqrt{\boldsymbol{4}} + \sqrt{\boldsymbol{4}} = (\pm\,\boldsymbol{2}) + (\pm\,\boldsymbol{2}) = \{-\boldsymbol{4},\boldsymbol{0},+\boldsymbol{4}\}$; $\sqrt{\boldsymbol{4}} - \sqrt{\boldsymbol{4}} = (\pm\,\boldsymbol{2}) - (\pm\,\boldsymbol{2}) = \{-\boldsymbol{4},\boldsymbol{0},+\boldsymbol{4}\}$ and $\sqrt{\boldsymbol{4}} + \sqrt{\boldsymbol{4}} = \sqrt{-\boldsymbol{4}} + \sqrt{-\boldsymbol{4}}$

[20] This consists of given a number, find another which when multiplied by itself gives the first.

B) To satisfy the requirements for the definition of operation of exponentiation to rational exponents. This definition should not depend on the representatives of numbers involved in the operation. We want $a^r = a^{\frac{m}{n}} = \sqrt[n]{a^m} = a^{\frac{km}{kn}} = \sqrt[kn]{a^{km}}$ (see Tirosh,& Even, 1997, p. 327). Nevertheless, if $\sqrt{4} = \pm 2$, then $\sqrt[6]{3^2} \neq \sqrt[3]{3}$. And, in general, $\sqrt[kn]{a^{km}} \neq \sqrt[n]{a^m}$, when *kn* is even and *n* is odd,

C) To satisfy the requirements for functions. The basic arithmetic operations addition and multiplication by a number different from zero establish bijective functions: $x \to x+a$, $x \to x \cdot a$, $a \neq 0$. These functions have unique inverse functions corresponding to the inverse operations. But, an operation like: $x \to x^2$ does not establish an injective function; because $x^2 = (-x)^2$. Consequently, the function $x \to x^2$ has to be confined to one of its branches to be inverted, $x \geq 0$. In the same way the inverse operation, $x \to \sqrt{x}$, has to be confined to positive domain, and range, in order to be unique.

2. The third reference takes on a psychological point of view, taking into account the dual operational/structural nature of mathematical conceptions and their role in the formation of concepts, indicated by Sfard (1991).

Sfard (1991) supports this theory with the fact that a mathematical entity can be seen as an object and a process. Treating a mathematical notion as an object leads to a type of conception called structural, whereas interpreting a notion as a process implies a conception called operational.

For Sfard, the ability to see a mathematical entity as an object and a process is indispensable for a deep understanding of mathematics, such that the “concept formation implies that certain mathematical notions should be regarded as fully developed only if they can be conceived both operationally and structurally” (p. 23).

It is worth pointing out that when referring to the role of operational and structural conceptions, Sfard conjectures that when a person gets acquainted with a new mathematical notion, the operational conception is usually the first to develop, whereas the structural conception follows a long and difficult process that needs external interventions (of a teacher, of a textbook), and may therefore be highly dependent on a kind of stimulus (of teaching method) which has been used (p. 17).

Pointing out that, the investigation on the conceptualization of the radical sign should be held in a revision of manuals and textbooks.

OBJECTIVES

Once the general problem to be studied has been pointed out, as well as the theoretical references, it is necessary to specify the general aims that are to guide the investigation's design and methodology:

1. To determine the characteristic aspects of teaching the radical sign, just as they are shown in textbooks today.

2. To diagnose mathematical knowledge with respect to the radical sign that some secondary school teachers have.
3. To explain teachers' possible conceptual and operational difficulties.

PATRICIA'S CONFLICT

The aims are linked to Patricia's conflict. Patricia is a high school mathematics teacher (in Spanish public education) and a student in a post-graduate programme. She presented the following conflict to her professor:

> In the textbook, the concept of equivalent radicals is defined as follows: "Two radicals are equivalent if they have the same roots" (and so I had learned). On the other hand, simplifying a radical by dividing the index of the radical and the exponent of the radicand by the same number, results (in theory) in a radical equivalent to the first. However, in a case like the sixth root of three squared, the cube root of three is obtained. As the index of the first radicand is an even number, two solutions exist (one being the opposite of the other) but in the second case, the index is an odd number and therefore there is a single root. Therefore, it cannot be said that these two radicals have strictly the same roots. So, are they equivalent?

Patricia says:

(A) Two radicals are equivalent if they have the same roots.

Also Patricia makes reference to the following equivalency:

(E) $\sqrt[nk]{a^k} = \sqrt[n]{a}, k, n \in N^*, n \geq 2, a \geq 0.$

Applying the equivalency (E), Patricia obtains than: $\sqrt[6]{3^2} = \sqrt[3]{3}$. However, to her the sixth root of three squared has two opposed roots, “two solutions”, as the index is an even number and the cube root of three has a single root as the index is an odd number, which means that the two expressions do not have the same number of roots and so according to (A) they would not be equivalent.

Hypothesis in relation to this conflict

In order to try to explain the causes of conceptual and operative difficulties that give rise to Patricia's conflict, the following hypothesis has been formulated:

(H_1) The lack of perception of the difference between the operational and structural conceptions of the radical sign that Patricia expresses is the cause of her conflict.
(H_2) This lack of perception is a product of a traditional teaching proposal, which does not pay attention to the need to re-conceptualise the √ sign when passing from arithmetic to algebra.
(H_3) In an alternative teaching proposal, where the meanings of root and radical are formulated, the conflict expressed by Patricia is not expected.

METHODOLOGY

To verify the solidity of the hypotheses an exploratory study was carried out, as a step prior to a more rigorous inquiry in terms of methodology, still to be carried out.

This exploration is based on a revision of current and representative textbooks of two alternative proposed ways of teaching: the Spanish one, which introduces the radical sign in arithmetic, and the Rumanian one, which introduces it in algebra. The revision of textbooks is has been complemented by a questionnaire followed by an interview with two representative individuals, Patricia (Spanish) and Iulian (Rumanian), two typical high school mathematics teachers.

With the revision of textbooks an attempt has been made to identify characteristic features in the teaching of roots and radicals in Spanish and Rumanian textbooks, and to identify comments that may favour the ambiguity of the $\sqrt{\ }$ sign, and Patricia's conflict.

The questionnaire

The questionnaire consists of a paper and pencil test which included four tasks. The first one is based on the teaching proposal given in the Spanish textbooks. In the task it is considered, as in Euler's text, that the square root of any positive number has two solutions, one positive and another negative. However, to represent this set of results the symbolic form $\sqrt{4} = \pm 2$ is used as well as the rhetorical form: "the solution is double, positive and negative". The intention of this task was to know if the difference is perceived between the structural and operational conception. The task is:

> In the class of 9th grade, after introducing the theme of the roots and radicals, the students were asked to calculate the square root of four.
>
> One student wrote $\sqrt{4} = \pm 2$, justifying thus:
>
> "As the radicand is positive and the root's index is even, then the solution is double, positive and negative".
>
> Is this correct?

Task 1

The interview's design took into account the answers produced by Patricia and Iulian to task 1. If the answer was "No", then the interviewee was asked to justify why and if it was "Yes", then they were given the second task with the aim of bringing in a cognitive conflict, in order to study the students' reaction

The second task is based on substituting $\sqrt{4}$ for ± 2 in a context of calculation. With this the aim was to put the affirmative answer to the task 1 into conflict.

> If $\sqrt{4} = \pm 2$ then complete:
>
> $\sqrt{4} + \sqrt{4} = (\pm 2) + (\pm 2) = \ldots$
>
> $\sqrt{4} - \sqrt{4} = (\pm 2) - (\pm 2) = \ldots$
>
> Explain the answer.

Task 2

A third task is based on the restriction of the property of radicals in the case where *k* is an even number and $a < 0$, which requires the intervention of the module.

$$(P) \quad \sqrt[nk]{a^k} = \begin{cases} \sqrt[n]{a}, & k,n \in N, n \geq 2, a \geq 0 \\ \sqrt[n]{|a|}, & k,n \in N, k - \text{even}, n \geq 2, a < 0 \\ \sqrt[n]{a}, & k,n \in N, k - \text{odd}, n \geq 2, n - \text{odd}, a < 0. \end{cases}$$

Here, the intention was to confirm that the interviewee was taking into account the radical's formal definition, in a traditional problematic case. The hypothetic situation that is present is the following:

> In a class of 10th grade, after introducing the radicals theme, the students were asked to simplify:
>
> $$\sqrt[6]{(-8)^2}$$
>
> One student wrote: $\sqrt[6]{(-8)^2} = \sqrt[2\cdot 3]{(-8)^2} = \sqrt[3]{-8} = -2$
>
> and said: "I have applied the following rule: $\sqrt[nk]{\boldsymbol{a}^{mk}} = \sqrt[n]{\boldsymbol{a}^m}$. Is this correct?

Task 3

If the answer to the task was "No", then the interviewee was asked to justify why and if it was "Yes", then the fourth task was given with the aim of introducing a cognitive conflict, in order to study the student's reaction.

Task 4 imposes the strategy for calculating $\sqrt[6]{(-8)^2}$ that leads to a different result from -2. With this, the intention was to put the affirmative answer given previously to the task 3 into conflict, in order to again study the reaction of the interviewee.

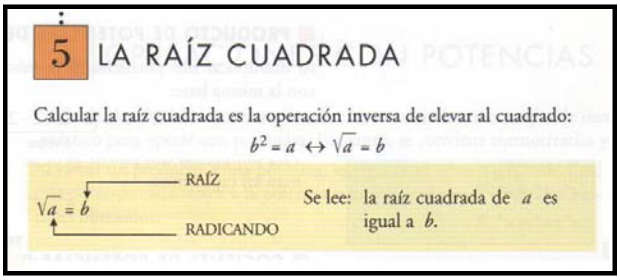
> If you consider:
>
> $\sqrt[6]{(-8)^2} = \sqrt[2\cdot 3]{(-8)^2} = \sqrt[3]{-8} = -2$
>
> then complete:
>
> $\sqrt[6]{(-8)^2} = \sqrt[6]{64} = ...$

Task 4

RESULTS OF TEXTBOOKS REVIEW

1. In the Spanish textbooks reviewed the sign $\sqrt{}$ is used to express the reverse operation of taking a number to the power of two (Figure 1):

> Calculating the square root is the reverse operation of calculating the power of a square: $b^2 = a \leftrightarrow \sqrt{a} = b$.


5 LA RAÍZ CUADRADA

Calcular la raíz cuadrada es la operación inversa de elevar al cuadrado:

$b^2 = a \leftrightarrow \sqrt{a} = b$

$\sqrt{a} = b$ — RAÍZ — RADICANDO

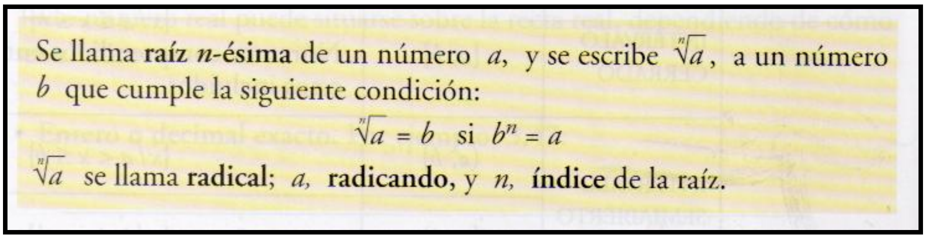
Se lee: la raíz cuadrada de a es igual a b.

Figure 1. 1º Secondary (7th grade), Anaya 2006, p. 52

The expression that has the $\sqrt{\ }$ sign is called a radical, that is to say the operation shown, and not the main root of said operation (Figure 2).

It is called the n[th] root of a number a, and is written $\sqrt[n]{a}$, where a number *b* meets the following condition: $\sqrt[n]{a} = b$ and $b^n = a$

$\sqrt[n]{a}$ is called radical; a, radicand, and n, the root's index.

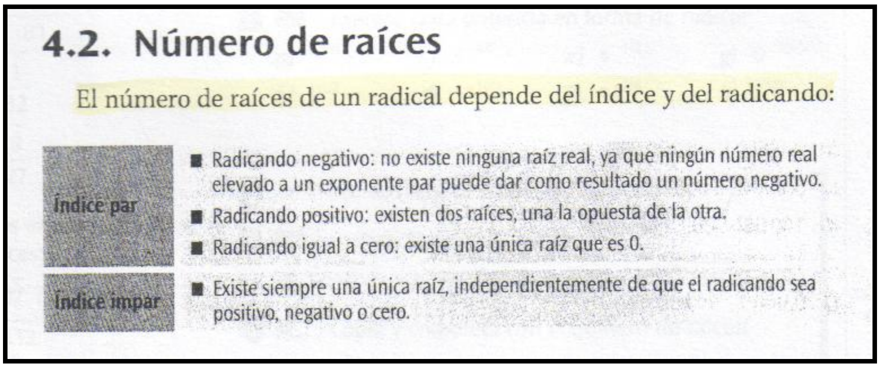

Se llama **raíz *n*-ésima** de un número a, y se escribe $\sqrt[n]{a}$, a un número b que cumple la siguiente condición:

$$\sqrt[n]{a} = b \quad \text{si} \quad b^n = a$$

$\sqrt[n]{a}$ se llama **radical**; a, **radicando**, y n, **índice** de la raíz.

Figure 2. 4º Secondary (10th grade), Anaya, 2006 b, p. 32

As a consequence it is considered that a radical has roots and that its number depends on the index of the radicand's sign (Figure 3).


4.2. Número de raíces

El número de raíces de un radical depende del índice y del radicando:

Índice par	■ Radicando negativo: no existe ninguna raíz real, ya que ningún número real elevado a un exponente par puede dar como resultado un número negativo. ■ Radicando positivo: existen dos raíces, una la opuesta de la otra. ■ Radicando igual a cero: existe una única raíz que es 0.
Índice impar	■ Existe siempre una única raíz, independientemente de que el radicando sea positivo, negativo o cero.

Figure 3. 3rd Secondary (9th grade), Oxford, U. P., 2007, p.32

So, equalities appear written as $\sqrt{36} = \pm 6$ (Figure 4).

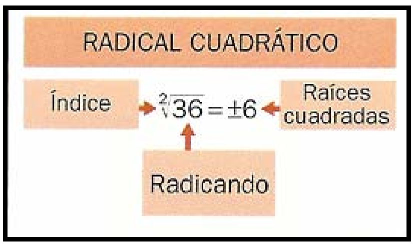


Figure 4. 3rd Secondary (9th grade), S. M., 2003, P. 36

The properties of the radicals are stated without mentioning their field of validity. So it is not taken into account that $\sqrt{a^2} = |a|, \forall a \in \mathbf{R}$ (Figure 5).

Propiedades de los radicales

- $\sqrt[np]{a^p} = \sqrt[n]{a}$
- $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
- $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$
- $(\sqrt[n]{a})^p = \sqrt[n]{a^p}$
- $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

Figure 5. 4º Secondary (10th grade), Anaya, 2006 b, p. 36

2. In the Rumanian textbooks reviewed, the sign $\sqrt{\ }$ is associated with the radical notion. The radical with an index two of a positive number a is defined as the positive solution of the equation $x^2 = a$ and is denoted by $\sqrt{a}$. (Figure 6)

4 Radicalul de ordinul 2 şi radicalul de ordinul 3. Proprietăţi

Radicalul de ordinul 2

Fie $a \geq 0$. Se poate arăta că există şi este unic $x \geq 0$ astfel încât $x^2 = a$. Privind acest număr ca fiind obţinut prin operaţia inversă ridicării la pătrat, acceptăm

Definiţia 4.1 *Numărul pozitiv x, cu proprietatea $x^2 = a$, se numeşte radicalul de ordinul 2 al lui a (rădăcina pătrată a lui a) şi se notează $\sqrt{a}$:*

$$\boxed{(\sqrt{a})^2 = a}, \quad a \geq 0.$$

1) $\sqrt{a} \geq 0$, pentru orice $a \geq 0$.
2) Nu se defineşte $\sqrt{a}$ pentru $a < 0$ întrucât $x^2 \geq 0, \forall x \in \mathbf{R}$. Astfel, $\sqrt{a}$ are sens numai pentru $a \geq 0$.
3) Deşi $2^2 = 4$ şi $(-2)^2 = 4$, avem $\sqrt{4} = 2$. Cu alte cuvinte, a scrie $\sqrt{(-2)^2} = -2$ este o greşeală!

Figure 6. 10th grade, Fair Parteners, 2005, p. 13

It is taken into account that $\sqrt{\boldsymbol{a}^2} = |\boldsymbol{a}|, \forall \boldsymbol{a} \in \mathbf{R},$ and the domain of validity of the radical's properties is specified. (Figure 7)

Proprietăţi ale radicalilor de ordin 2:

1) $\sqrt{a^2} = |a|, \quad \forall a \in \mathbf{R}$; 2) $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}, \quad a \geq 0, \ b \geq 0$;

3) $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}, \quad a \geq 0, \ b > 0$; 4) $\sqrt{a^n} = (\sqrt{a})^n, \quad a \geq 0, \ n \in \mathbf{N}^*$;

5) dacă $a \geq 0$ şi $b \geq 0$, atunci $a < b \Leftrightarrow \sqrt{a} < \sqrt{b}$ şi $a = b \Leftrightarrow \sqrt{a} = \sqrt{b}$;

6) $\frac{2}{\frac{1}{a} + \frac{1}{b}} \leq \sqrt{ab} \leq \frac{a+b}{2}, \quad \forall a, b \in \mathbf{R}, \ a > 0, \ b > 0.$[1]

Figure 7. 10th grade, Fair Parteners, 2005, P. 13

FINDINGS AND CONCLUSIONS

As for the first objective, the review of textbooks shows that there are substantial differences in dealing with the $\sqrt{\ }$ sign. Specifically, it can be said that in the Spanish textbooks studied, the conception associated with this sign is operational, whereas in Rumanian texts it is structural.

As regards the second objective, Patricia and Iulian's mathematical knowledge with respect to the radical sign shows significant differences.

In tasks 1 and 2, Patricia identifies $\sqrt{4}$ with the set of two solutions (2 and -2), and does not see the radical as the positive root when the index is even. In the interview, to emphasize this in task 2, she indicated that in reality there are not two solutions, but there are contexts in which it is replaced by +2 and others in which it is replaced by -2.

Iulian does not agree with $\sqrt{4} = \pm 2$, arguing that the radical of an even index of a positive number belongs to the interval $(0, \infty)$ and specifies that, in any context $\sqrt{4}$ represents a number, that is, the positive square root of 4.

In task 3 and 4, Patricia does not take into account that $\sqrt{a^2} = |a|, \forall a \in \mathbf{R}$. On the other hand Iulian correctly applies the restriction of the property of radicals and he realizes the error that the hypothetical student commits.

In conclusion, it can be said that Patricia has procedural knowledge of the $\sqrt{\ }$ sign, whereas Iulian has structural knowledge, and that these conceptions are consistent with what is shown in the textbooks studied.

As for the third objective, this part of the work was restricted to Patricia's conflict, the answers to the questionnaire and the interviews that provide indications suggesting the validity of the hypotheses.

(H_1), Patricia does not distinguish between operational and structural use of radical sign.

(H_2), the review of Spanish texts evidences that the teaching proposal reflects the ambiguity of the radical sign, used in the expression $\sqrt{4} = \pm 2$, and does not use the formal definition of radicals, so that it is plausible to think that they encourage the appearance of Patricia's conflict.

(H_3), in the revised Rumanian texts, the formal definition of the radical sign is observed, so that it is possible to think that they support Iulian's way of acting, which does not encounter the conflict that Patricia expresses.

Finally, the important educational implication that should be pointed out is that in any educational proposal that aims to avoid conflicts such as the one expressed by

Patricia, the formal definition of radical must be considered, and it must be guaranteed that students understand the reasons for this definition.

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BEHIND STUDENTS' SPREADSHEET COMPETENCIES: THEIR ACHIEVEMENT IN ALGEBRA? A CASE STUDY IN A FRENCH VOCATIONAL SCHOOL

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Research on the use of spreadsheet in mathematics education usually points out its potentialities in the learning of algebra. The link between spreadsheet and algebra is thus often seen in the direction "spreadsheet for algebra". This paper follows the opposite direction, i.e. "algebra for spreadsheet", by questioning the role of algebra in students' spreadsheet competencies. It reports a case study, based on computer tests, in the framework of a French research project studying students' spreadsheet uses and competencies. The results of the test show algebra raising out again, playing a role behind students' achievements and actions with spreadsheets. Algebra, Spreadsheet competences, Computer tests

INTRODUCTION

What role can technology play in mathematics education? Usually, didactic research approaches ICT questions through this direction, i.e. "technology for mathematics". This is the case for many studies on spreadsheets which consider this latter as a good tool to help pupils understanding algebraic concepts.

Here, we take the opposite direction: "what about algebra for spreadsheet?" by questioning the role algebra plays in students' mastery of spreadsheets. This issue comes from the analyses of experimentations in the context of *DidaTab*, a French research project studying students' spreadsheet competencies. To identify the basic competencies students have acquired, the *DidaTab* project realised tests of competencies in several classes. In the analyses of the results, the relation with algebra stands out again, raising issues on the relations between students' achievements and actions with this kind of software and their mastery of algebra.

In the first part, we give a quick description of the *DidaTab* project. The second part focuses on relationships between spreadsheets and mathematics learning. Then, to get a more concrete view of spreadsheet mastery problems, we detail the results of a computer test administrated to 17 y.o. students in a vocational marketing school. The results of this test put in perspective students achievements, actions, and software interaction understanding, with their knowledge (or their lack of) in algebra.

THE DIDATAB PROJECT

According to educational authorities of many countries, ICT has to be used in classrooms. In the case of secondary education, all countries have established detailed recommendations (Eurydice, 2004, p. 24). In general, using ICT to enhance subject knowledge or learning correct use of a word processor or a spreadsheet are part of the objectives at lower secondary level. But, if ICT seems to be included in prescribed curricula, we only have very few data about effective practices in classrooms and ICT competencies of students. Some data from PISA 2003 (Eurydice, 2005) provides interesting results (for example, that less than half of students are familiar with using a spreadsheet to plot a graph) but rely on declarative statements. We don't know whether students under or over estimate their competencies. To get a more comprehensive picture, we considered that it was not fruitful to take into account ICT as a whole, and decided to focus on specific software. Spreadsheets, prescribed in French curricula for ten years now, were a good indicator of ICT mastery. What do students learn about spreadsheets? Which basic competencies do they have acquired at the end of their schooling?

DidaTab (didactics of spreadsheet[1]) was a three year project (2005-2007) founded by the French ministry of research and dedicated to study personal and classroom uses of spreadsheets in French context. The methodology combined questionnaires, interviews (students and teachers), classroom observations, computer tests, content analysis of official curriculum texts, websites and resources, and some comparative studies with other countries (Belgium, Greece, Italy) have been made. As results (Blondel & Bruillard, 2006), we have an almost complete cartography of spreadsheets uses in the French secondary education, including an overview of personal uses, and we began to describe kinds of genealogy of uses, according to subject matters (e.g. mathematics, technology, social sciences, experimental sciences…). But we have not yet built a theoretical framework to explain spreadsheets uses and competencies of students. Some of these competencies relate to knowledge of mathematical nature, especially algebraic one. In a next part, we discuss this particular relation between spreadsheets and mathematics.

SPREADSHEET AND MATHEMATICS COMPLEX RELATIONSHIPS

In the title of this section, we play on the word "mathematics" to relate two points: mathematics as a school subject, this questions the place of spreadsheet within syllabus, or mathematics as knowledge that spreadsheets may bring into play, this questions the place of mathematics within the spreadsheet objects.

Spreadsheets within mathematics syllabus

Spreadsheets have been introduced at many different teaching levels and courses of the French Educational system. Part of the mathematics syllabus since 1997, first in middle school (grade 6 to 9) then in high school (grade 10 to 12), their place varies according to the school streams, as mathematics education appears under different aims. Two main tendencies can be distinguished, each of them promoting a different use of spreadsheets.

In the scientific streams, mathematics is a very theoretical discipline also used to select students. In this "abstract" approach of mathematics, spreadsheets appear as a

[1] In French, spreadsheet is "tableur". See http://www.stef.ens-cachan.fr/didatab/en/index.html for other information and results in English about *DidaTab* project

tool to serve the learning of mathematical concepts. Then spreadsheets' role is to support and enhance learning.

In some other streams, as vocational or literary, mathematics is considered as a more experimental subject oriented towards everyday life problems. This objective favours the use of different kinds of software such as spreadsheets, which allow a more concrete approach of mathematics opened on its everyday applications.

First vision leads to a very small place for working spreadsheet competencies. Moreover, as we will elaborate in the next section, using spreadsheet to enhance mathematical learning is "double-faced" as far as spreadsheet is not neutral on mathematical concepts. Second vision opens a larger place for building some spreadsheet competencies. In these streams, a hypothesis would be that students' difficulties in mathematics could be counterbalanced by some instrumental abilities and some mastery of this software[2]. But the situation is not as simple because of the specific relationships existing between spreadsheet and mathematics: spreadsheet mastery requiring mathematical knowledge.

Mathematics within spreadsheet objects

ICT use in mathematics education is a question among the more general problematic of technology use in human activity, studied in the field of cognitive ergonomics. A theory of instrumentation (Vérillon & Rabardel, 1995); developed in this field, provides a frame to tackle the problematic of the learning in complex technological environments. In this frame, an instrument is not *given* but *built* by the subject (Vérillon and Rabardel, 1995) through a progressive individual *instrumental genesis*. This genesis, is not *neutral*, instruments have impact on the *conceptualisation*. This idea of non neutral «mediation» between subject and tools provides a way to report on the strong imbrications that exist, and have always existed, between mathematics and the instruments of the mathematical work. It led to an instrumental approach in didactics that has been used in several researches on symbolic calculators (CAS) in mathematics education (Artigue 2001, Lagrange 1999, Drijvers 2000, Guin, Ruthven & Trouche 2005). What about spreadsheets?

Some "computer" objects, characteristics of spreadsheets, do not strictly correspond to mathematical knowledge transposed in a computer environment, even not to a computer transposition of school knowledge, but are *however linked with* mathematics. The basic principle of spreadsheet, which consists in connecting cells between themselves by "formula", gives an example of these objects, linking spreadsheet to the domain of algebra. Such a particular relation with mathematics is precisely the reason why many research in didactics from different countries (Ainley (1999); Arzarello et al. (2001); Capponi (2000); Dettori et al. (1995) or Rojano and Sutherland (1997)) give spreadsheets a role in the learning of algebra.at elementary stages, iden-

[2] For instance, in the literary stream, a place is given to concrete aspects of mathematics and this is precisely a stream where spreadsheets take an important part in the mathematics syllabus.

tifying them as tools of arithmetic-algebraic nature. Haspekian (2005a), having adopted an instrumental approach, showed that in spite of an apparent simplicity of use, it is not so evident for teachers to take benefit from these characteristics. The tool generates some complexity: spreadsheets transform the objects of learning and the strategies of resolution by creating new action modalities, new objects, and by modifying the usual ones (as variable, unknown, formula or equation…). Here are some examples.

In a paper and pencil environment, variables in formulae are written by means of symbols (a letter generally for the school levels concerned here). This variable 'letter' relates to a set of possible values (numerical here) and exists in reference to this set. In spreadsheet, let us take for example the formula for square numbers. The Fig.1 shows a cell argument A2 and a cell B2 where the formula was edited, referring to this cell argument.

	A	B
1		
2	5	=A2^2

Figure 1 A2 is the cell argument; B2 calculates the square of the value in A2

Here again the variable is written with symbols (those of the spreadsheet language) and exists, as with paper and pencil, in reference to a set of possible values. But this referent set (abstract or materialised by a particular value, e.g. 5 in Fig.1) appears here through an intermediary, the cell argument A2, which is both:

- an abstract, general reference: it represents the variable (indeed, the formula does refer to it, making it play the role of variable);
- a particular concrete reference: here, it is a number (in case nothing is edited, some spreadsheets attribute the value 0);
- a geographic reference (it is a spatial address on the sheet);
- a material reference (as a compartment of the grid, it can be seen as a box)

So, where in paper and pencil environment, we stick a set of values, a cell argument overlaps here, embarking with it, besides the abstract/ general representation, three other representations without any equivalent in paper and pencil (Fig.2).

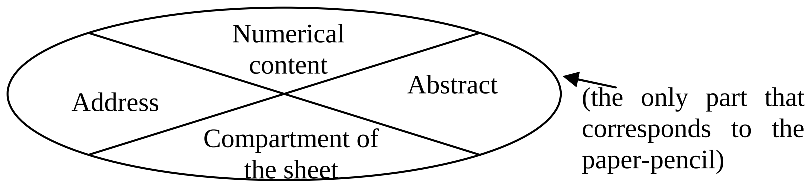

Figure 2: The "cell variable"

Other examples of the changes due to spreadsheets are given in Haspekian 2005a.

From an institutional point of view, these changes have different impact following the different way chosen to introduce algebra. As the recent ICMI study showed (Stacey et al., 2004), different aspects of algebra can be focused on: as a tool of generalisation, a tool of modelling, or a tool to solve arithmetical, geometrical or everyday life problems through what is called since Descartes, the « analytical method ». Following the case, different mathematics is brought forward: variables, formulae and functions on one hand, unknowns, equations and inequations on the other hand. In the French school culture, it is traditionally the analytic way that is chosen, the resolution, though equation solving, of various problems appears as emblematic of pupils introduction to algebra. Table 3 gives a quick insight of the distance between the algebraic culture in the French secondary education and the algebraic world carried out by spreadsheets.

"Values" of algebra	In paper pencil environment	In spreadsheet environment
Objects	unknowns, equations	variable, formulae
Pragmatic potential	tool of resolution of problems (sometimes tool of proof)	tool of generalization
Process of resolution	"algorithmic" process, application of algebraic rules	arithmetical process of trial and refinement
Nature of solutions	exact solutions	exact or approached solutions

Table 3: distance between different "algebraic worlds"

More generally, the mathematical culture sustained by spreadsheets is an « experimental » one: approximations, conjectures, graphical and numerical resolutions, implementing everyday life/ concrete problems, statistics… Thus, this vision fits with the aim of mathematics in particular streams of the French Education, especially where students not very good at mathematics are supposed to use spreadsheet with stronger objectives. It is thus interesting to investigate with students of non scientific streams and test them at the last year of their schooling (grade 12).

As we will see next, the computer test confirms the complex relation between spreadsheet and mathematics. Algebraic aspects; especially the use of cell-variables in formulae, stand out again as one of students' main difficulties with the tool.

STUDENTS' SPREADSHEET COMPETENCES: A CASE STUDY

We report here the example of a one hour computer test administrated in 2005 in a class of 13 students of vocational school[3] (17 year old) preparing a marketing diploma. After presenting the objectives and a brief description of the test, we first give an overview of the general results and then an analysis on the algebraic aspects that these results lead to focus on.

Objectives and description of the test

For this part of the *DidaTab* project, the objective was to assess students' spreadsheet competences in a computer test. In order to design such tests, a first step consisted in the identification of basic spreadsheet skills, that have been actually organised in five categories (see below), then the definition of some general and simple tasks corre-

[3] This school is identified as rather difficult in the sense that students have behavioural difficulties and social problems.

sponding to each ability, and finally the construction of a database of skills, questions and tasks (for more details on this step of the project, and especially on the design of the tests, see Tort and Blondel, 2007).

From the database we selected 24 exercises relevant to the school year of the students and covering all categories of skills. Then, the students' mathematics teacher chose 11 exercises from this list according to the competences she assumed that her students have. With regard to the classification, the 11 exercises are divided in the following way:

1. "Cells and Sheets Editing" (3 were selected)
2. "Writing of formulae" (4 were selected)
3. "Translating data into graphs" (1 was selected)
4. "Managing data tables" (2 were selected)
5. "Modelling" (1 was selected)

The tasks were proposed in the computer test with increasing order of difficulty, in a spreadsheet file. Students had to answer directly within the tool and record their work at the end of the test. The collected data are constituted by these file records, observation and the complete recording of the actions for one of the students.

An overview of the results

Among the five categories of skills, clear differences between basic skills linked to superficial manipulations (not requiring knowledge of the contents) and abilities requiring deeper knowledge appear.

The best rates of success for the 13 students, concern cell formatting: italic (10), bold (11), date format (9). The results decrease then as the tasks require more understanding of spreadsheets objects. Some tasks requiring deeper knowledge of spreadsheet functionalities have been moderately achieved: recopying a format (6), sorting out data (6), or representing data with a graph by choosing the best type of representation (4). Finally, more specific knowledge as the conditional format (0) or specific displays either of numerical data (fractional format: 1) either of graphics (displaying labels on the X axis: 2) seem rather unknown from these students.

All exercises of the formulae category are part of the competences that have been failed in. Actually, the success rates for the four tasks of this category are the lowest of the test, varying from 0 to 2 good answers for each item: Writing a formula to calculate the AVERAGE of a line of data in adjacent cells (2), Writing a formula calculating a subtraction (0), a product (0), a division (0), Writing and copying down a formula using relative and absolute references (0, only 1 student answered: he gave a number…), Writing a conditional formula (using the IF function) (0).

Three main issues can be raised from these observations:

1) The inadequacies between the skills we thought students have and their actual level of competence. Students' abilities were clearly lower than expected.
2) The teacher tended as well to overestimate the skills of her students. The exercises she has chosen were globally too difficult.

3) The very bad results concerning the formulae category raise the question of spreadsheet's relation to algebra. Obviously, the formulae, the copying of formulae, the use of relative/absolute references as variables in formulae and the conditional formulae appear in students' results, as the less achieved competences.
In the next section we analyse this last point in more details.

Algebraic aspects in students' achievements

Competences just mentioned are all linked with algebraic knowledge of students, their understanding of the concepts of variable and formula. These results join other research in didactics of mathematics (Capponi, 2000, or Dettori & al. 2001). For Capponi, benefiting from spreadsheet potentialities requires from the user the understanding of some algebraic knowledge such as the notion of formula, and students' difficulties with spreadsheets show their needs in this domain: the work remains at the numeric level (data tables, numbers, operations) without reaching the level of an algebraic treatment (dynamic sheet of calculations, formulae).

About formulae

Looking further the tasks of the formulae category, we note that sometimes, not only the correct formula had not been found, but not even wrong formulae have been tried. Some students edit, instead of formulae, the corresponding arithmetic operations, some others edit directly the results they calculate by hand, but most of them do not answer anything. Another surprising point concerns the calculus of the average: we did not find any formula such as "(A5+B5 +… +N5)/ 14" or equivalents and only 2 students achieved the calculus of this average.

Observation during the test brings out some more elements. One of the students who succeeded in the average used the AVERAGE functionality (and seemed yet surprised to have directly the response). This can seem paradoxical, but to calculate the average of the given numbers, he directly used the function "AVERAGE" provided by spreadsheet; the references to the adequate cells are then automatically made. The student has to calculate an average, he has an "average" function (as a key of calculator), and he uses it without controlling more what this feature produces. The use of "AVERAGE" can thus mask its lack of understanding of what is really a formula in spreadsheet and the way it can be used. We have the same observation for the other student who used the average function. Finally, in the whole test, we did not find any other formula at all except these automatic formulae as average or sum. And the very surprising result that is coming to light with these analyses is that no student used a single relative reference in the entire test! According to us, this is precisely linked to the problem of the cell variable. Very few students used formulae which send back automatically the cell references[4] (such as SUM or AVERAGE) and not even a sin-

[4] The spreadsheet used in this experiment is Microsoft Excel. The interface provides buttons that you can directly activate and obtain the writing of a formula including cell references

gle student was able to write a formula which requires finding and entering the cell variable.

About the cell variable

The use of cells as variable in a formula seems more difficult than the use of formulae itself. In tasks which require a formula which does not automatically send back the cells references, either students do not find any formula or they use again an “automatic” formulae (AVERAGE or SUM) even when these functions have nothing to do with the task! For example, a correct answer to a task was a formula with a multiplication and one student has written the following formula: “=SUM (C12*10)”. SUM is here totally useless and used in a non standard way. The student invoked the function and turned the usual argument automatically written by the software (“=SUM (C12)” in this case) into a multiplication. By using this automatic formula, he did not enter himself the cell reference in the formula.

We have exactly the same phenomenon in another task: 2 students used SUM in both columns although the answer has nothing to do with a sum. One of them, after using the function SUM transformed the separating sign ":" in the syntax of this function into a subtraction (and the result is then correct)! All the others had not answered or had put either an operation or directly the numerical result instead of a formula. Once again, the use of cells as variables in a formula seems to be problematic, the type of functions as SUM or AVERAGE being apparently the only type of formula those students manage.

The problem of the cell-variable is also revealed by the use of the recopy. Here again, a deeper analyse of the answers of the whole class shows that it is not so much the fill handle that raises problem than copying downwards formulae. The recopy becomes problematic when it puts at stake some cells references which have to be incremented. This principle of the spreadsheet functioning, which is one of its most basic interesting feature, but which has an algebraic nature (the recopied cell playing the role of variable in the formula and the spreadsheet keeping the structure of the formula during the recopy), seems not to be understood by students. Results concerning recopy are quite different whether the recopy does concern cell-variables (copying a formulae with references: 0) or not (as copying down a date: 6).

In conclusion, it seems clear that these students do not master the ability of self editing a cell-variable in a formula or the way the recopy affects the cell references.

DISCUSSION AND PERSPECTIVES

The computer test reveals difficulties of grade 12 students, not so much in surface manipulation skills, but in their lack of understanding of algebraic concepts. Using a formula in a spreadsheet requires having understood the concept of "variable" in the spreadsheet (the cell argument in the formula). Using a recopy of a formula requires seeing the increment of the references produced by the recopy as a means for the spreadsheet to preserve the algebraic structure of the formula along the copy. The syntactic writing varies in every line but the algebraic structure is preserved. These types of knowledge were analyzed as algebraic competences which constitute a difficulty for students at the pre-algebraic level (Capponi, 2000, Haspekian 2005b).

In an exploratory study with younger students (grade 7), which consisted of a first approach to algebra through the use of spreadsheet, Haspekian (2005) found similar results. The students were asked to write, interpret or transform formulae. The observations have shown that the technique of using a formula and copying it down was the competence the longest to acquire and created most difficulties to the students. The difficulties were the following ones:

- comprehension of formulae (some remained in a use of arithmetical level of the spreadsheet);
- use of the fill handle, in particular at the beginning. But even afterward, when they experienced it several times, they had difficulty in appropriating it and its use was not systematic.

The experiment in vocational high school shows that students of grade 12 have the same difficulties as regard to these algebraic concepts embarked in the tool. It would be interesting to make paper-pencil test on their level in algebra to validate this hypothesis.

Another interesting point is the question of the modalities of spreadsheet learning. In the experiment of Haspekian (2005b), half of the students had followed a training course about spreadsheet (hands-on work) some months before the experiment. In particular they had seen formulae and recopy of formulae, and the teacher of this course had asserted that these students would have no difficulty with the tasks of the experiment. The results showed that they had the same difficulties and took the same time to answer the exercises that the other half of the students, those who had never used spreadsheet previously. Our computer test points out the same difficulties.

It is also interesting to compare with students of other professional fields or students of general fields. In *DidaTab*, another computer test has been administrated in a class of literary stream. Results show that students have less difficulty with recopy and formulae but have much more difficulties with manipulation skills. Yet in France, this stream is the general stream where spreadsheets use is the most strongly prescribed by curriculum... Certainly, as mentioned in part II, spreadsheets change too much the traditional mathematics that live in the general streams, teachers do not seriously enough take into account spreadsheet learning (not enough time devoted to spreadsheet learning, lack of structured training sessions, etc.) in these general streams, and many students are not able to manage important spreadsheet features. This result is confirmed by many interviews of students in the *DidaTab* project. Thus, our small experiment with 12th graders gives a rather different picture from general discourses about students great competencies. It seems that intrinsic difficulties of spreadsheet concepts are not sufficiently taken into account in mathematics education, even in the school streams where mathematics objectives and views are connected to every day life. In conclusion, students of professional fields who are mostly supposed to use spreadsheet due to their school profile are unfortunately those who are precisely blocked by their difficulties in algebra, and students of general streams with a better level in mathematics are those who will not "meet" spreadsheets enough because of the specificity of their stream…

To go further, it would be interesting to deepen the research with more computer tests in different levels and settings, and try to define thus kinds of students trajectories of uses.

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DEVELOPING KATY'S ALGEBRAIC STRUCTURE SENSE

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In this paper we follow one student through a sequence of tasks and describe our observations of how her algebraic structure sense develops.

Key words: algebraic structure sense, high school algebra

INTRODUCTION

In this paper we take a close look at how one Israeli 11th grade high school student (age 16) performed during a series of teaching interviews designed to develop algebraic *structure sense*.

The term *structure sense* was coined by Linchevski and Livneh (1999). Subsequently the idea was developed and refined by Hoch and Dreyfus (2006) who arrived at the following definition.

> Students are said to display structure sense for high school algebra if they can:
>
> - Recognise a familiar structure in its simplest form.
> - Deal with a compound term as a single entity, and through an appropriate substitution recognise a familiar structure in a more complex form.
> - Choose appropriate manipulations to make best use of a structure.

See Hoch (2007) for a full definition and examples.

In an earlier paper (Hoch & Dreyfus, 2007) we showed how, through a simple intervention, students acquired the ability to recognise and exploit the properties of algebraic expressions possessing the structure $a^2 - b^2$. We described what is structural about $a^2 - b^2$, and showed how a student can learn to recognise structure. Hoch (2003) discussed and analysed structure in high school algebra, considering grammatical form (Esty, 1992), analogies to numerical structure (Linchevski & Livneh, 1999) and hierarchies (Sfard & Linchevski, 1994), culminating in a description of algebraic structure in terms of shape and order. In this research we took a similar approach, relating to any algebraic expression or equation as possessing structure, which has external and internal components. External components include shape and appearance. Internal components are determined by relationships and connections between quantities, operations, and other structures.

We designed a series of tasks with the aim of facilitating the improvement of structure sense. The tasks were deliberately devoid of any context other than the structural and technical, because the students had shown themselves unable to use certain algebraic techniques in different contexts, a phenomenon also noted by Wenger (1987). If a meaningful context had been chosen, then the issue of whether the students were familiar with the context and how well they understood it would have had to be considered.

The tasks were based on five structures that Israeli students meet in high school: $a^2 – b^2$; $a^2 + 2ab + b^2$; $ab + ac + ad$; $ax + b = 0$; and $ax^2 + bx + c = 0$. Hoch and Dreyfus (2006) identified students' difficulties with these structures. The creation of the tasks was based on the first author's analysis of structure sense and supported by her teaching experience. She placed emphasis on verbalising about mathematical concepts. In order to speak about a mathematical concept (or object), students must be able to deal with the result of some process without having to think about the process itself. The process is performed on a familiar object and then the result becomes another object (Sfard, 1991; Sfard & Linchevski, 1994). For example, in exercise 3 below the term 3xy is the result of the process of multiplying three elements. The student is required to relate to this result as an entity, in order to find its value.

In one task, the aim is to familiarise the student with equations that could be considered to have linear or quadratic structure when a product is related to as the variable. The student is presented with the following exercises in sequence:

1. Find xy: $8xy + 15 = 0$.
2. Find xy: $8x^2y^2 + 6xy – 9 = 0$.
3. Find 3xy: $17xy – 25 = 13 + xy$.
4. Find 2xy: $34xy – 4x^2y^2 = 10xy – 13$.
5. Find x: $17x^2 – 45 = 0$.

The student is asked to say which structure each equation possesses, to make up similar equations, and in some cases to devise efficient ways of solving them. The fifth equation is obviously quadratic, but the student is asked whether it could be considered to have a different structure if the instruction was "Find x^2".

In another task the student is required to describe each of the five structures listed above in words, and make up expressions or equations similar to those shown. The idea here is that the need to explain a structure in words causes the student to think more carefully about it. Gray, Pinto, Pitta, and Tall (1999) considered the use of language a powerful method of dealing with complexity. The student is asked to create expressions or equations that might be difficult for a friend to recognise. The rationale for this is that the act of creating more examples deepens the personal relationship with the structure. Rissland (1991) and others (e.g., Bills et al., 2006) said that generating examples is an important cognitive activity and that the ability to generate examples as needed is a cognitive tool of experts, often lacking in novices.

TEACHING INTERVIEWS

A series of three teaching interviews was designed, comprising tasks including the ones described above, with the purpose of improving students' structure sense. A pre-test measuring structure sense was administered to two 11^{th} grade classes of intermediate to advanced students. Ten students who performed badly on the pre-test were chosen to participate in individual sessions of approximately 45 minutes each, over a period of up to two weeks. Throughout the sessions the researcher encouraged the students to verbalise about what they were thinking and doing, with emphasis placed on the correct naming of each algebraic entity and structure. A post-test was administered individually in a separate session a fe w days after the third session, and several months later a delayed post-test was administered.

All ten students displayed considerable improvements in structure sense, as measured by the immediate post-test. These improvements were maintained over time, to varying extents. We chose to report on Katy because she displayed the highest level of retention of learned abilities, and also because she was enthusiastic and highly verbal. On the pre-test Katy displayed technical skills such as opening parentheses, collecting like terms, and factoring trinomials. However her structure sense was poor—she was unable to factor an expression without first converting it into an equation and could not recognise a common factor. We will present here some excerpts from Katy's interviews. The excerpts are presented in chronological order: excerpt 1 is from the first session, excerpts 2 and 3 are from the second session, and excerpts 4 and 5 are from the third session.

EXCERPT 1: DIFFERENCE OF SQUARES

Katy displayed difficulties in factoring $49 - y^2$ as $(7 - y)(7 + y)$, and only reluctantly agreed that the expressions $x^2 - 16$ and $49 - y^2$ belong in the same structure group. When asked to give a general formula for the expressions in this group, she first suggested the formula $a^2 - b$. She observed that $49 - y^2$ confused her, "because for me the 'squared' is always plus". With a little help she arrived at the formula $a^2 - b^2$. However she was confused when asked to give a name to the structure represented by $a^2 - b^2$. The following extract is typical of students' difficulties when trying to explain mathematical concepts in words. (K = Katy; I = interviewer)

K The expression is made up of …

I How did you decide that these belong together? [Points to $x^2 - 16$ and $49 - y^2$]. What characterises them?

K That squared minus that squared. Of the first degree.

This is an example of careless use of terminology. Earlier Katy had described linear equations as being of the first degree, yet here she assigns this name also to a quadratic expression, despite the fact that she first mentioned the squared terms.

I You called them $a^2 - b^2$.

K Ah. So … eh … how to give it a name?

I Um , a description.
K Can I call it $a^2 - b^2$?
I Yes.
K Is that a name?
I No, that's a formula. You have a number squared minus a number squared. What do we call the result of a number minus a number?
K A ratio?
I No, that's a number divided by a number.
K Difference?
I That's right. So we can call this the difference of two squares.
K Ah, I understand, the difference of two squares.

Many of the students were unable to name the result of subtraction without heavy prompting.

EXCERPT 2: COMMON FACTOR

In the pre-test Katy failed to answer any of the questions that required extracting a common factor. In the first session different types of factoring were mentioned, though not practised, including extracting a common factor. Subsequently, in the second session Katy had no problem factoring the expression 36axy – 16aby. She was able to relate to the common factor 4ay as a single entity. However the expression $16x + 40xy + 50x^2$ presented her with more of a challenge. She rewrote it as $50x^2 + 40xy + 16x = 0$, and extracted a common factor to get $x(50x + 40y + 16) = 0$.

I Why did you write "equals zero"? I don't see an equation.
K [Scores out "equals zero".] I can't do anything else.
I You extracted a common factor. I don't think you extracted the greatest common factor.
K Ah. Two. [Writes: $2x(25x + 20y + 8)$.]
I Fine, but why did you change the order?
K It's just simpler for me to have the x squared at the beginning.

The above extract illustrates Katy's diffidence about what she can "do" with an expression, although she knows what to do with an equation. It mirrors her performance on the pre-test. She does not, probably cannot, justify her preference for having "the x squared at the beginning" other than that she feels it is simpler. This preference was shared by other students, and perhaps reflects the manner in which textbooks and teachers present quadratic expressions. Although Katy succeeded in factoring the expression, she did not relate to 2x as an entity—she extracted first x, then 2.

EXCERPT 3: EQUATIONS

When it came to equations, Katy was overconfident, making some instant decisions that were not always correct. She was asked to copy each equation under its structure (quadratic or linear). Here is her response to $(2x^2 – x)^2 + 2(2x^2 – x) – 35 = 0$.

K Wow. This also doesn't belong here (pointing to $ax^2 + bx + c = 0$) but
I If it doesn't belong, don't write it there.
K No, it does belong, if we use t, where t is 2x squared
I Why?
K Because there will be x to the third.
I Yes, I agree you need to use a substitution, what will your t be?
K 2x squared.

Here followed a brief discussion about the viability of such a substitution.

K [Thinks] Then I'll get an equation with t equals x and tx squared and t squared. x to the third can be t squared.
I How would you solve such an equation?
K Eh …
I I don't know either. Can you think of a different way?
K [Thinks]
I Continue with the idea of t.
K Oh I didn't look. 2x squared minus x is t.

Substituting t in place of a compound variable in an equation is taught in 10th grade and using it without regard for the appropriateness of the substitution is typical of many students. The fact that Katy said "I didn't look" rather than "I didn't see" suggests that she is self-reflecting and aware of what she should have done.

Katy very quickly classified $(x^2 + 3x)^2 = 2x^2 + 6x + 15$ as having structure $ax^2 + bx + c = 0$. The interviewer asked her to write down the appropriate quadratic equation.

K The quadratic equation? The equation …
I Let's see. What will t be?
K Eh. [Writes $(x^2 + 3x)^2 = 2x^2 + 6x + 15$] To open and solve?
I How would you solve it?
K [Writes $x^4 + 6x^3$]

Eventually Katy was led to make the appropriate substitution. It seems that her original perception of the equation's structure was based on a guess, probably provoked by the fact that the term in parentheses is squared, or perhaps by looking only at the right hand side of the equation.

EXCERPT 4: NAMING A STRUCTURE

After Katy factored $(x + 3)^4 – (x – 3)^4$ correctly the interviewer pointed out that most students found that extremely difficult, and asked Katy why she thought that might be.

K Because of the fourth power? They didn't identify …
I Uhm.
K They didn't see the structure.
I But there was this expression x to the fourth minus y to the fourth that nearly everyone succeeded in factoring. [Writes $x^4 – y^4$].
K Because, in my eyes, it's different. Simply, that's clear [points to $x^4 – y^4$] and that's not [points to $(x + 3)^4 – (x – 3)^4$].
I And now, with new eyes?
K That's also clear [points to $(x + 3)^4 – (x – 3)^4$].
I Are they different?
K Yes, because of the words.
I What?
K Because in my head I see "difference of squares".

This extract clearly shows that being able to think about structure and give it a name helped Katy identify it.

EXCERPT 5: EXEMPLIFYING

Table 1 shows Katy's responses when asked to describe each structure in words and create more examples. Katy only managed to give the name of each structure (note that she said common denominator instead of common factor, a mistake made by many students) rather than a more wordy explanation. This, too, was typical of all the students. She displayed enthusiasm over the task of creating new examples, and made an effort to produce something out of the ordinary.

Table 1 Verbalising and exemplifying

Structure	Explanations	New examples
$a^2 + 2ab + b^2$	It's sum squared	1. $(3 + 2x)^2 + 6(3 + 2x) + 9$
		2. $(4x^2 + 12x + 9)^2 + 6(3 + 2x) + 9$
$a^2 – b^2$	Difference of squares	3. $z^2x^2 – 9$
		4. $x^2(3x + 2)^2 – 64$
ab + ac + ad	Common denominator	5. $(x + 2)y + (x^2 + 5x + 6) +(x + 2)(x + 5)$
ax + b = 0	Eh … linear equation	6. $2(2x + 4)^2 – 9 = (4x^2 + 16 + 16x) + 5$ Find $(2x + 4)^2$
$ax^2 + bx + c = 0$	Quadratic equation	7. $9x^2y^2 + 6xy + 2 = 0$
		8. $9x^2y^2 + 6xy + 4 = 0$
		9. Solve for $(x^2 + 2x)^2$ $(x^2 + 2x)^4 + (3x^2 + 6x)^2 + 9 = 0$
		10. $(x^2 + 2x)^4 + 3(x^2 + 2x)^2 + 9 = 0$
		11. $(3x + 2)^6 + 9 = (3x + 2)^3$

Katy wrote example 1 and, when asked to write another one even more difficult, adapted it to get example 2, commenting, "I would never be able to solve that". The interviewer asked her why she thought these examples might be difficult for other students.

K Because when you come to an exercise, you don't look at the general structure, unless it is really obvious to the eye.
I Uhuh, okay.
K And because … I wouldn't get it. I would have to figure out how the 9 got there, in order to extract 3 plus 2x.

It seems that here Katy was talking about how she behaved before the teaching interviews.

In between writing examples 3 and 4 Katy said, "Just a minute, something more complicated? Now this was the one I really didn't understand the most, now it seems the simplest, it's impossible to make it more difficult." We consider this a testimony to her structure sense development.

Katy changed example 7 into example 8 because she thought that the former had no solution while the latter had a solution. She seemed surprised to be informed that it was perfectly permissible to write a quadratic equation with no real solution. "Oh," she laughed, "I didn't know." In fact she should have known, since in class she had learned to analyse quadratic equations, and in fact mentioned this kind of analysis at the end of the first session. This is an example of how Katy has compartmentalised her knowledge.

Katy corrected example 9 to example 10. She stated, "I meant this. Like x squared plus 3x plus 9".

At the end of the session the interviewer commented on how well Katy had done, and asked her if she had been practising.

K [Laughs] The penny dropped.
I How did the penny drop? Do you think you could tell me?
K I don't know. But at least three times in class I found myself using this.
I Yes? I am very pleased.
K I said to myself, here are connections, suddenly I recognised a structure.

Katy's self-reflection and enthusiasm were a foreshadowing of her performance in the post-tests.

POST-TESTS

In the immediate post-test, Katy answered all the items correctly. After the test she commented that she felt it had taken her too long because of, "The common factor. I don't think about that. I will have to think about the common factor." (Note that this time she said factor, not denominator.) When asked to account for her excellent performance:

K Do you know what helped me the most? It's the order; three different things. Everything I see I categorize. And in addition it helps – how it sounds, subtraction of squares, that's … like … Now that we're doing trigo, that appears a lot, a lot, a lot a lot, in identities.

I And you think of the …?

K Today, there were three exercises, like, I work ahead with two boys, and I see that I'm three exercises ahead of them, and I stop to look what they've got stuck on, and I see that they're stuck on the subtraction of squares, and I said, but it's obvious what to do.

In the delayed post-test, several months later, Katy answered almost all the items correctly. Overall, Katy's structure sense improved considerably, and this improvement was sustained over time. Although the improvements in structure sense of the other participating students were less than that of Katy, their improvements also stood the test of time, providing evidence for the efficacy of the teaching interviews.

DISCUSSION

A close look at Katy's transcripts reveals that she displayed much typical behaviour: confusion between expression and equation, denominator and factor, ratio and difference; tendency to change the formulation of quadratic expressions; difficulty with verbalizing. She showed a clear improvement in structure sense from session to session, yet there is no instance that pinpoints the actual learning process. However, naming a structure helped her to use it, and she actually said that she succeeded "because of the words" that she sees in her head. Naming the structure is an important part of learning it – the name is part of the definition. One of the roles of a definition is to introduce a concept and convey its characterising properties. Another is to create a uniformity that allows easier communication of mathematical ideas (Borasi, 1992; Zaslavsky & Shir, 2005). A known concept or object can be given a definition by describing a few characteristic properties (De Villiers, 1998; Shir & Zaslavsky, 2001).

In conclusion, there is evidence that learning has taken place. Since there is no way of pointing to any one incident of knowledge acquirement, it can be surmised that the learning occurred as a process over time.

After the first post-test Katy said, "I think you should tell the teachers to do this with all the students. It would help them so much. Really." Of course, one-on-one intervention is not possible in a classroom situation, so the tasks would have to be adapted to make them suitable for group work, and yet enable the teacher to intervene when necessary. These tasks were designed as a form of remediation, to be used with 11th grade students who were assumed to be familiar with the algebraic structures. This raises the question whether it would be more effective if students' attention were drawn to structure at a much earlier stage, perhaps even before they practised using the formulae. Answering this question requires further research.

Further research is also required to answer other questions arising when attempting to develop students' structure sense. For example, can the teaching interviews be adapted for whole class activities? At what stage in the learning of algebra would this kind of intervention be most appropriate? Could the improved structure sense manifest itself in other subject areas, with other structures? The improvements in structure sense were maintained over a period of a few months. What would a longitudinal study show?

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CHILDREN'S UNDERSTANDINGS OF ALGEBRA 30 YEARS ON: WHAT HAS CHANGED?

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In this paper, we outline the design and method of the research project Increasing Student Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS). Phase 1 consists of a large-scale survey of attainment in algebra and multiplicative reasoning, using test items developed during the 1970s for the Concepts in Secondary Mathematics and Science (CSMS) study (Hart, 1981). This will enable a comparison of the current attainment of students aged 11-14 with that of 30 years ago. Phase 2 consists of a collaborative research study with 8 teachers extending the investigation to classroom / group settings and examining how formative assessment can be used to improve attainment. Although the focus of this paper is on reporting the research design, some early analysis of data from the initial survey data from 2008 (n = 2400) is reported.

INTRODUCTION

Over the past 30 years, there has been a great deal of work directed at, first, understanding children's difficulties in mathematics and, second, examining ways of tackling these difficulties. Yet, there is no clear evidence that that this work has had a significant effect in terms of improving either attainment or engagement in mathematics. Indeed, children continue to have considerable difficulties with algebra and multiplicative reasoning in particular (e.g., Brown, Brown & Bibby, 2008; Wiliam et al., 1999). In this paper, we describe the project Increasing Student Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS), a research study designed to address these problems.

ICCAMS is a 4-year research project involving a research team from King's College London and Durham University together with eight teacher-researchers from four schools. The project consists of a large-scale survey of 11-14 years olds' understandings of algebra and multiplicative reasoning in England followed by a collaborative research study with the teacher-researchers extending the investigation to classroom / group settings and examining how formative assessment can be used to improve attainment and attitudes. Although the project is in its early stages, we report some initial tentative results later in this paper. These initial results compare children's current understandings with a similar survey, the Concepts in Secondary Mathematics and Science (CSMS) study (Hart, 1981), which was conducted 30 years ago. When completed, the full results will enable us to examine what gains, if any, have been made over the intervening period. The Phase 2 findings will extend the results to children's understandings in group and classroom settings.

BACKGROUND

Mathematics education in the UK[21] is facing a crisis; insufficient students are choosing to continue studying mathematics post-16, whilst university teachers and others point to falling standards in the subject (CBI, 2006; Smith, 2004). There is considerable research in the UK addressing reasons for non-participation in mathematics - students stop studying mathematics because they experience it as difficult, abstract, boring and irrelevant (e.g., Osborne et al., 1997). The most recent findings relating to 16 year-olds (Brown, Brown & Bibby, 2008) suggest that students' attainment and attitudes are strongly inter-related. A major factor is that even relatively successful students perceive that they have failed at the subject and lack confidence in their ability to cope with it at more advanced levels, especially in comparison to the perceived 'clever core' of fellow-students. When pressed about the reasons for their feelings of failure, students suggest that they do not understand parts of what they have been taught; this commonly relates to algebra and to aspects of multiplicative reasoning (e.g. percentages, and ratio) and its applications (e.g. in trigonometry). Students' negative attitudes commonly relate to the predominance of routine and formal work on algebra and multiplicative reasoning. Performance in these topics has been shown to be particularly weak in England relative to other countries (e.g. Mullis et al., 2004). Yet algebra and multiplicative reasoning are both essential for further study in mathematics, in science & engineering (as well as health and medicine, economics, etc.) and for mathematical literacy in the workplace and elsewhere (e.g., CBI, 2006).

The original CSMS study was conducted 30 years ago. The study made a very significant empirical and theoretical contribution to the documentation of children's understandings and misconceptions in school mathematics (e.g., Booth, 1984; Hart, 1981). In the intervening period, there have been various large-scale national initiatives directed at improving mathematics teaching and raising attainment: e.g., the National Curriculum, National Testing at age 7, 11 and 14, the National Numeracy Strategy and the Secondary Strategy[22]. Many of these initiatives have drawn directly on the CSMS study. During this period examination results have shown steady and substantial rises in attainment: e.g., the proportion of students achieving level 5 or above in Key Stage 3 (KS3)[23] tests has risen from 56% in 1996 to 76% in 2006 and the proportion of students achieving grade C or above at GCSE has risen from 45% in 1992 to 54% in 2006. However, independent measures of attainment suggest that that these rises may be due more to "teaching to the test" rather than to increases in genu-

[21] This crisis in mathematics education is not confined to the UK. It is also a concern in the US and elsewhere in Europe.

[22] These initiatives are particular to England. However, similar initiatives relating to testing (and accountability) and to national curricular are evident elsewhere in the world.

[23] In England, compulsory secondary school consists of two Key Stages: KS3 (11-14 years) and KS4 (14-16 years). In 2008, and for more than a decade previously, 14 year olds took a 'high stakes' test at the end of KS3, although this assessment has been abandoned for 2009 and future KS3 assessment arrangements are currently under review. GCSE (General Certificate in Secondary Education) is the examination taken at age 16, the end of compulsory schooling. Almost all 16 year olds in England take GCSE mathematics.

ine mathematical understanding. Replication results from the science strand of the CSMS study (using a test on volume and density) suggest that students' understanding of some mathematical ideas as well as the related science concepts has declined (Shayer et al., 2007). Studies at the primary level indicate that any increases in attainment due to the introduction of the National Numeracy Strategy have been at best modest (Brown, Askew, Hodgen et al., 2003; Tymms, 2004). Results from the Leverhulme Numeracy Research Programme suggest that any increase in attainment at Year 6 is followed by a reduction in attainment at Year 7 (Hodgen & Brown, 2007) Further, Williams et al. (2007) find that, following this dip at Year 7, there is a plateau in attainment across Key Stage 3.

AN ALTERNATIVE APPROACH: FORMATIVE ASSESSMENT?

National initiatives in mathematics education in England have largely focused on specifying what mathematics should be taught (e.g., the National Curriculum), how mathematics should be taught (e.g., the Secondary Strategy) and summatively assessing what mathematics has been learnt (e.g., National Tests). However, research suggests that a much more effective approach to increasing attainment and engagement would be formative and diagnostic assessment: the tailoring of teaching to students' learning needs (Black & Wiliam, 1998). In an extensive meta-analysis study Hattie (1999) found that interventions involving feedback are more effective than any other educational intervention, with an effect size of 1.13. Further, Wiliam (2007) calculates that, for the achieved effect size, the cost of formative assessment is lower than for other comparative educational interventions. Yet, whilst there has been a great deal of activity nationally and internationally in formative assessment, there is also considerable evidence that teachers have substantial difficulties implementing these ideas (Bell, 1993). These difficulties in implementation relate to three issues. First, formative assessment has largely been described generically rather than in subject-specific terms (Watson, 2006). Second, formative assessment has been poorly described theoretically and pedagogically (Black & Wiliam, 2006). Third, teachers' ability to use formative assessment in mathematics is limited by their knowledge about key ideas, and the likely patterns of progression in student learning. Thus if teachers focus on teaching mathematical procedures they may find it difficult to see what is causing problems for students in mastering and applying these, and though aware of the importance of questioning, they may not know what questions they should ask (Hodgen, 2007).

THE NEED FOR A COLLABORATIVE APPROACH TO DISSEMINATION

Much of both the research and the implementation of initiatives in these areas of mathematics have been "done to" teachers, which may in part explain the limited influence in schools. Leach et al. (2006) found that research evidence cannot simply be presented to teachers; research findings need to be "re-worked" as teaching materials. However this process of re-working, or recontextualisation, is not straightforward

(Ruthven, 2002). We hypothesise that in order for change to occur teachers must have greater insight into the problems of student understandings and attitudes, a profound understanding of fundamental mathematics (Ma, 1999), and understanding of how available resources relate to student understandings and underlying mathematical ideas. These approaches have been tried before in e.g. diagnostic teaching experiments – also based on the CSMS research - and have proven success (Bell, 1993; Swan 2006). Existing experience of collaborative research methods (e.g., Black & Wiliam, 2003) suggests that disseminating these approaches more widely and implementing them in ordinary classrooms is more likely to be successful if these approaches have been grounded in teachers' practices.

THE RESEARCH STUDY

ICCAMS is investigating engagement and achievement by focusing on the two topics at KS3 that are central to the current mathematics curriculum: algebra, and multiplicative reasoning. These topics are also fundamental to further study in mathematics and other numerate disciplines (e.g., science, engineering, economics[24], etc.) The study will focus on KS3, because this is where students first meet algebra and more abstract multiplicative reasoning, and where attitudes begin to deteriorate (Mullis et al., 2004). There is also evidence of a plateau in student achievement at KS3 (Williams et al., 2007).

Phase 1: The large-scale survey of algebra and multiplicative reasoning 11-14

In Phase 1, we are conducting a large-scale survey of attainment in algebra and multiplicative reasoning and attitude to mathematics, involving both cross-sectional and longitudinal elements. This will use test items first developed during the 1970s as part of the CSMS study (Hart, 1981). Based on a representative sample of schools and students in England, the survey will provide a comprehensive and detailed analysis of current student attainment in algebra and multiplicative reasoning. It will provide up-to-date information on student understandings of basic ideas in the areas of algebra and multiplicative reasoning enabling us to plot where changes have occurred since the original study. It will extend the CSMS study by linking understanding of concepts and student progression to student attitudes, to teaching, and to demographic factors. Analysis is being conducted using a variety of techniques, extending those used in the original CSMS study with Rasch and other techniques.

The full survey will consist of both cross-sectional (n=6000) and longitudinal (n=600) samples identified using the MidYIS database (Tymms & Coe, 2003). Three original CSMS tests (Ratio, Algebra, Decimals) will be administered with some additional items relating to fractions (drawn from the CSMS Fractions test) and spread-

[24] ICCAMS is funded by the Economic and Social Research Council in the UK as part of a wider initiative aimed at identifying ways to participation in Science, Technology, Engineering and Mathematics disciplines.

sheet items. Piloting indicated that only minor updating of language and contexts was required.

The test items range from very basic to sophisticated, allowing broad stages of attainment in each topic to be reported, but also each item, or linked group of items, is diagnostic in order to inform teachers about one aspect of student understanding.

Phase 2: The collaborative research study investigating formative assessment

In Phase 2, we are conducting a collaborative research study with teachers, which will indicate how they can best use a formative assessment focus within these curriculum areas to improve student confidence and competence, and thus participation, engagement and attainment. In this phase, we adopt a design research methodology (Cobb et al., 2003). Central to our approach will be the analysis of children's difficulties from both teaching and research perspectives.

Initially teachers will be supported in interpreting and acting upon the survey results of their students; later they will use classroom-based formative assessment based on the frameworks for learning provided by the tests, and assessment for learning approaches. They will also draw on research-informed approaches to the teaching of these curriculum areas. This study will, first, examine how teachers can make use of existing resources and initiatives to respond to students' learning needs, and, second, develop and evaluate an intervention designed to enable a wider group of teachers with much less support to do this. In the final year of the study, the approach will be implemented and evaluated with a further group of teachers and classes.

The Phase 1 findings will provide up-to-date information on student understandings of basic ideas in the areas of algebra and multiplicative reasoning to inform the teachers and teacher-researchers in Phase 2 both about their own students and about where they lie relative to the general population.

A central question for Phase 2 is how the generic approach of formative assessment can be adapted to the particular needs of mathematics teaching and learning. This will be done in several ways. First, the diagnostic results for individual students assessed against the learning and progression framework developed by CSMS will guide teachers in planning appropriate work for students and in further formative assessment. The CSMS tests were carefully designed over the 5-year project starting with diagnostic interviews in order to focus on student progression in understanding of key concepts such as variable and rational number. (See below for a fuller description of the Algebra test.) Second, we will identify and link existing teaching resources into the developmental and diagnostic learning structure provided by CSMS, building on and extending our existing work in this area which is underpinned by a combination of Piagetian and Vygotskian theories (Adhami, et al., 1995; Brown, 1992). There is extensive research evidence relating to the teaching and learning of both algebra and multiplicative reasoning that can inform this intervention (e.g. Bednarz et al., 1996; Sutherland et al., 2000; Ainley et al., 2006), but these research findings and resources have only made a limited impact on teaching practices in classrooms. The solution lies not in designing yet another resource for the teaching of algebra and multiplicative reasoning, but in supporting the judicious use and interpretation of existing resources by teachers (Askew, 1996). Third, we will develop our existing work in this area (Hodgen & Wiliam, 2006).

THE WORK TO DATE AND EARLY ANALYSIS

In June 2008, tests were administered to a sample of around 3000 students in each of Years 7, 8 and 9[25]. Approximately 2000 of these students took the Algebra test. The full cross-sectional sample will be completed in Summer 2009 when a further subsample of around 2000 students will be tested. We report here on the early analysis of this data. We note that these early results should be treated with caution. In particular, the current sample of students appears to be slightly higher attaining than the general population in England. This early analysis suggests that student attainment in algebra at age 14 is broadly similar to that of 30 years ago, although the patterns across the attainment range and in earlier years are more complex.

Students' understandings of letters

We now focus on just four linked items due to space constraints: 9a-d, illustrated in Figure 1. These items have been chosen to give a flavour of the test.

The CSMS algebra test was carefully designed over the 5-year project starting with diagnostic interviews. The original test consisted of 51 items. Of these 51 items, 30 were found to perform consistently across the sample and were reported in the form of a hierarchy (Booth, 1981; Küchemann, 1981). The test items range from the basic to the sophisticated allowing broad stages of attainment to be reported, but also each item, or linked group of items, is diagnostic in order to inform teachers about one aspect of student understanding. The focus of the test was on generalised arithmetic, and in particular it looked at different ways in which pronumerals can be interpreted (Collis, 1975). Items were devised to bring out these six categories (Küchemann, 1981):

> Letter evaluated, Letter not used, Letter as object, Letter as specific unknown, Letter as generalised number, and Letter as variable.

The four items, 9a-d, were amongst the consistently performing items that formed part of the original hierarchy. Item 9a, at Level 1 in the hierarchy, and items, 9b and c, at Level 2, can be solved without having to operate on the letters as unknowns; the letters can be treated as objects (i.e., the name of the various sides of the figures). Items 9b and c additionally require the explicit use of some mathematical syntax. Item 9d, at Level 3, was designed to test whether students would readily 'accept the lack of closure' (Collis, 1972) of the expression 2*n*, where the given letter, *n*, has to be treated as at least a specific unknown. The proportions of 14 year old students an-

[25] Key Stage 3 is made up of three academic years: Y7 (age 11-12), Y8 (age 12-13) and Y9 (age 13-14).

swering these items correctly in 1976 reflect this variation in difficulty: 94% for 9a; 68% for 9b; 64% for 9c; 38% for 9d.

The item facilities for 1976 and 2008 are presented graphically in Figure 1. This suggests that the *pattern* of progression is similar in 1976 and 2008: an initial relatively steep rise is followed by a much smaller rise subsequently. However, although the initial steep rise now appears to take place a year earlier, this initial advantage is not sustained and by age 14 students' attainment appears similar in 1976 and 2008. The results for item 9a are more of an anomaly: this relatively easy item appears to be more difficult now than in 1976.

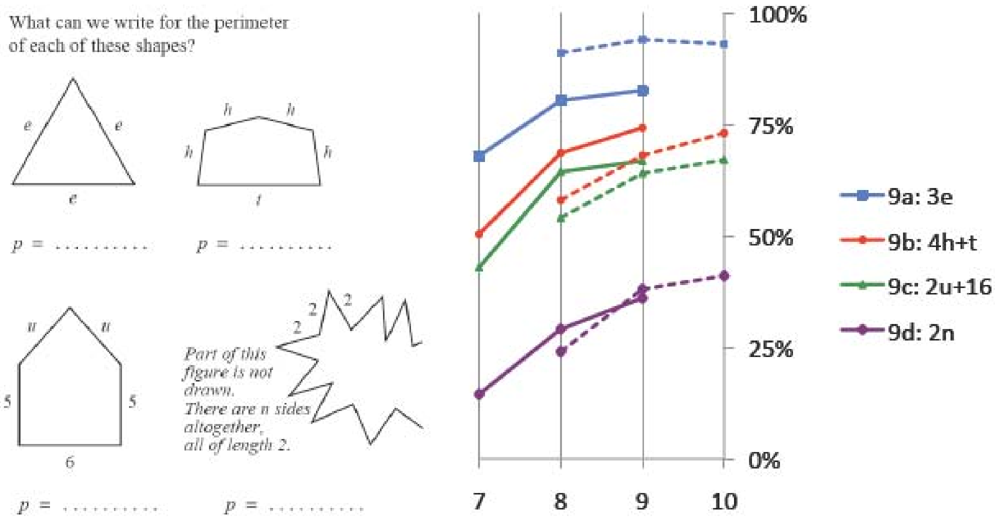

Figure 1: Items 9a-d. Facilities for items in both 2008 [continuous] and 1976 [dotted] for Year 7 to Year 10 (ages 11-14). In 2008 data were not collected for Year 10; in 1976 data were not collected for Year 7.

DISCUSSION

In comparison to 30 years ago, in England, formal algebra is taught to all students earlier. This is partly as a consequence of the introduction of a National Curriculum. The initial results of the study reported here suggest that, whilst this practice confers an initial advantage to students, this increased attainment may not be sustained. Our early analysis suggests that, by age 14, current performance in algebra is broadly similar to that of students in 1976. Moreover, it is worth noting again that the sample of students tested in 2008 is in general a relatively high attaining group. Hence, the data presented here suggest that increases in examination performance are not matched by increased conceptual understanding and, thus, add weight to the research reported earlier in this paper.

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PRESENTING EQUALITY STATEMENTS AS DIAGRAMS

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I describe a diagrammatic computer-based task designed to foster engagement with arithmetic equality statements of the forms a+b=c, a+b=b+a, and c=a+b. I report on six trials with pairs of 9 and 10 year old pupils, highlighting how they talked about distinctive statement forms and used these distinctions to discuss strategies when working towards the task goals. These findings stand in contrast to how pupils typically view and talk about equality statements as reported in the literature.

INTRODUCTION

The design of tasks that engage pupils with mathematical ideas in an open and exploratory manner presents a significant challenge. Constructionism offers a vision of mathematics learning in which learners explore, modify and create mathematical artefacts on a computer screen (Turkle, Papert and Harel 1991). The term “microworld” (Edwards, 1998) is often used to describe software that supports learners “discovering” mathematical rules through experimentation, mental reflection and discussion. The intention is to engage learners with mathematical ideas in a way that is meaningful to them. However, this can be difficult when the conventions of formal notation are the intended domain of learning because they are not so readily meaningful to learners. A way forward is offered by diagrammatic task designs in which learners explore, modify and create *notational* artefacts (Dörlfer 2006). This paper reports on trials with a diagrammatic computer-based task designed to engage primary children with arithmetic equality statements.

CHILDREN’S CONCEPTIONS OF EQUALITY STATEMENTS

In typical primary classrooms, arithmetical equality statements are presented and talked about as commands to work out a result. This leads most children to expect a term comprising numerals and operator signs on the left of the equals sign, and a single numerical result on the right (Behr, Erlwanger and Nichols 1976; Dickson 1989). This expectation can prove stubborn (McNeil and Alibali 2005), and lead to difficulties with equation solving (Knuth, Stephens, McNeil and Alibali 2006).

Presenting young children with a variety of statement forms leads to more flexible thinking about mathematical notation (Baroody and Ginsburg 1983; Li, Ding, Capraro and Capraro 2008). Interventionist studies have focussed on the careful selection of statements that appeal to structural readings, as in 50+50=99+1, 7+7+9=14+9, 246+14= __+246 and so on (Carpenter and Levi 2000; Molina, Castro and Mason 2008; Sáenz-Ludlow and Walgamuth 1998). The intention is that pupils can notice and exploit arithmetic principles in order to assess or establish numerical balance, without the need to generate results. Such interventions produce encouraging findings, but the long term impacts remain an open question (Dörfler 2008; Tall 2001).

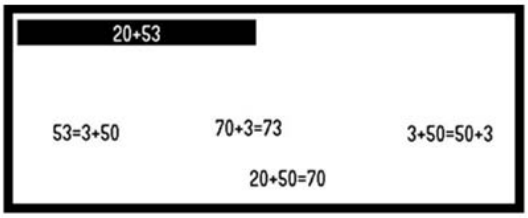

Figure 1: Screenshot from the computer-based task

A DIAGRAMMATIC APPROACH

An alternative to presenting statements as isolated questions of balance is offered by Dörfler's (2006) "diagrammatic" approach. The essence of diagrammatic notating tasks is learners manipulating conventional representations ("inscriptions") in an open, exploratory manner. This renders mathematical notating an empirical and creative activity, based in seeing potential actions (i.e. transformations). Generalisation can arise from noticing both visual patterns and patterns of repeated actions. As such, diagrammatic tasks offer learners an investigative, concrete notating activity that stimulates discussion, congruent with constructionist approaches. Note that "diagram" is being used here more loosely than everyday associations with "drawings" rather than "writings" would suggest. In another sense, however, it is more restrictive, referring only to those "inscriptions" that form precise mathematical structures with grounded rules for making transformations. From a diagrammatic perspective, arithmetic statements can be presented in parallel, forming relational systems akin to simultaneous equations (e.g. Figure 1). Numerals and their transformations, rather than numbers and arithmetic principles, are the intended "objects of the [learners'] activity" (p.100).

When pupils exploit shortcuts to establish the equivalence of presented statements they do engage in activities that are to some extent diagrammatic. Their attention is on the structural relationships of numerals, rather than computed results, and this can stimulate rich discussion (Carraher, Schliemann, Brizuela and Earnest 2006). However, such designs exclusively promote an "is the same as" meaning of the equals sign due to the task goal of establishing equivalence. There is no appeal to a "can be exchanged for" meaning, which is central to the nature of reversible equivalence relations (Collis 1975), and supports the transforming aspect of diagrammatic notating tasks.

The tasks used in the studies reported here presented pupils with sets of equality statements ("diagrams") on a computer screen. A screenshot from the task is shown in Figure 1 (an online example of the software is available at go.warwick.ac.uk/ep-edrfae/software). Each statement stands in isolation, but, as with an algebraic equation, can also combine with others in a collective, relational system. The task goal is to transform the term in the box at the top-left of the screen, 20+53, into a single numeral using the provided statements. For example, we might start by selecting 53=3+50 and using it to transform the boxed term into 20+3+50, then use 3 + 50 = 50 + 3 to transform it into 20+50+3, and so on until 73 appears in the box.

The tasks offer learners new ways to view and talk about statements. Working through notational diagrams (such as Figure 1) requires looking for matches of numerals across statements and the boxed term in order to determine where substitutions can be made, and this is quite distinct from viewing statements as isolated questions of numerical balance. Observing and predicting transformational effects (20+53 → 20+3+50 and so on), when a statement is selected and visually matched notation is clicked, promotes making distinctions of statements by form. Notably, $a+b=b+a$ can be seen as commuting the inscriptions a and b; and $c=a+b$ can be seen as partitioning the inscription c. If pupils articulate such distinctions when working towards the task goal this would stand in contrast to children's left-to-right computational readings of statements reported widely in the literature.

I report on six trials drawn from three studies. In each trial pupils were set a sequence of diagrams to solve, similar to that shown in Figure 1. These studies varied in the specific research questions addressed and the diagrams presented. The intention here is to present common and contrasting findings from across the trials (for a detailed discussion of the first two studies see Jones 2007, 2008).

METHOD

The method used was paired trialling and qualitative analysis for evidence of talking about mathematical ideas in novel ways (Noss and Hoyles 1996). Pairs of 9 and 10 year old pupils were presented with sequences of notational diagrams comprising statements of the forms $a+b=c$, $a+b=b+a$, and $c=a+b$. These began with simple diagrams comprising two or three statements of the forms $a+b=c$ and $a+b=b+a$, followed by more complicated diagrams comprising up to nine statements and including $c=a+b$ forms. Pupils were shown how to select statements and click on notation to see if a substitution occurs, and were given a few moments to get to grips with the software's functionality. I then set the task goal of transforming the boxed term into a numeral, and remained present to offer encouragement and ask for verbal elaborations ("what do you think?", "how did you know that would work?", and so on). Each trial lasted around 30 to 40 minutes.

Data were captured as audiovisual movies of the pupils' onscreen interactions and discussion. Data were transcribed and analysed using *Transana* (Woods and Fassnacht 2007). Occurrences of pupils computing results, looking for numeral matches and articulating the distinctive transformational effects of statement forms ("swap", "split" and so on) were coded. A trace of each trial was constructed to examine how such articulations arose, and how they were used by pupils in order to discuss strategies when working through the diagrams.

The six trials reported will be referred to as Trial A through to Trial F. The pupils in trials A to C were deemed mathematically able by their class teachers, and the pupils in trials D to F were deemed average. The trials can usefully be grouped as A, B, C and D, E, F in terms of the extent to which pupils (i) articulated distinct statement forms, and (ii) used these distinctions to work strategically with the diagrams.

FINDINGS

The data are presented here to illustrate the similarities across all trials, and the differences across trials A to C and D to F. I present a visual overview of the six trials, and offer illustrative transcript excerpts.

Visual overview

Figure 2 shows a time-sequenced map of codings across the six trials and was

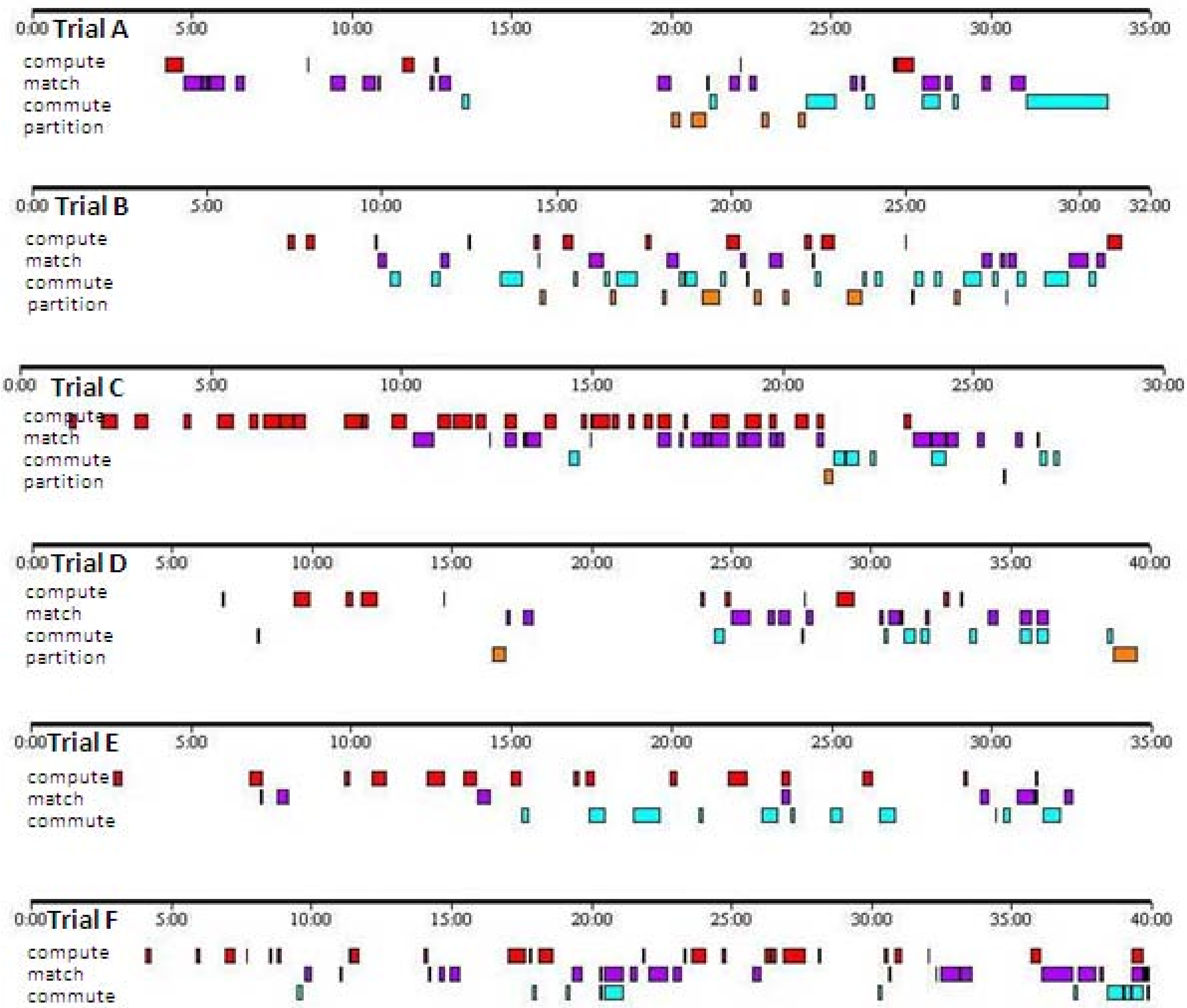

Figure 2: Time-sequenced coding of the data for computing results, looking for matches of numerals, and articulating commuting and partitioning transformations.

produced using *Transana*. Each block shows an occurrence of pupils computing results, looking for matches of numerals or terms across statements and the boxed term (Figure 1), or articulating the distinctive commuting ("swapping") or partitioning ("splitting") transformational effects of presented statements. The length of each block is somewhat arbitrary. For example, one block of (say) "commute" might reflect pupils working in a trial-and-error manner with one of them suggesting they "swap" numerals, but offering no reason. Another block of similar length might reflect pupils discussing which numerals to commute, and how and why, as part of a shared strategy. As such, Figure 2 provides a useful visual aid for summarising the trials, but does not convey the quality, or the precise quantity, of the pupils' articulations and strategising. Non-coded segments are those times when either I was speaking, or pupils' discussion was ambiguous ("Click that one", "Let's try this one, no, that one" and so on).

The first thing to note is how little the pupils computed results across the trials (with the exception of Trial C, in which the notably enthusiastic pupils appeared keen to impress me with their computational prowess). Conversely, the pupils did engage in looking for matching numerals, and articulating the commuting properties of $a+b=b+a$ statements. Figure 2 shows that "compute" was prominent in the first ten minutes of each trial (bar Trial A), but was less present than the other codes in the final ten minutes. This reflects how most pupils began by computing results, as would be expected, but changed, sooner or later, to more diagrammatic views.

"Partition" is less prominent across the trials, and does not appear at all in trials E and F. The pupils in trials A to C came, sooner or later, to articulate partitioning transformations as part of their shared strategy for achieving the task goal. After a little practice, they would generally begin a new diagram by identifying partitioning statements, then using commuting statements to shunt the numerals in order to compose them. However, the pupils in Trials D to F rarely articulated partition if at all, and did not use it strategically, instead relying on a less efficient approach characterised by trial-and-error statement selection. It seems, then, that articulating partition is key to strategic discussions when working collaboratively with the diagrams.

Illustrative transcript excerpts

Early on in the trials, after the pupils had been introduced to the software's functionalities, they articulated computational readings of statements. The following is from Trial E:

John: 9 add 12 add 1 equals 22.

Derek: 21.

John: No it's 22. 13 add 9.

Derek: Hm, no 9 add 12. 9, 13 add 12. No, 13 …

John: 12 add 1 is …

Derek: Yeah 22 because it's 9 add 12 add 1 is 22

Searching for matches of numerals arose across all the trials as the pupils discussed why the software sometimes allowed a selected statement to make a substitution and other times did not. Often they looked for matches of single numerals, rather than terms. The following is from Trial C:

Barbara: 31 plus 19.

Nadine: 19. What's that?

Barbara: 31 ... look for a 31 somewhere.

Nadine: Well I found a 19 and another 19.

Barbara: But we need something that will equal 19. Aha, I found a 31.

At other times pupils attempted near matches, such as trying to use 5+18=23 to transform 5+8+18 (Trial C). However, often these near matches were attempted doubtfully when pupils were momentarily stuck, and, overall, they showed greater confidence when attempting exact matches. With prompting, the pupils were often able to explain why a given substitution did not work. From Trial A:

Researcher: Why do you think that wasn't working?

Terry: Maybe because ... 1 and 9 is ...

Arthur: Oh, because it hasn't got that sum in it.

Researcher: What do you mean?

Arthur: Well, because that's got 1 add 9 but then the end of that's got 9 add 1.

Pupils across all the trials readily came to articulate the observed or predicted transformational effects of a+b=b+a statements as "swapping" or "switching" or "changing round". Some pupils did not initially see that this could be helpful for achieving the task goal. For example, when the pupils in Trial F used 31+35=35+31 to transform 31+35+8 → 35+31+8 they commented:

Colin: That just swapped it.

Imogen: Swapped it around.

However, most pupils came to see a use for commuting numerals sooner or later, as articulated by John (Trial E) when prompted to explain why 16+32=48 would not transform 13+32+16:

Researcher: It's not working. Why not?

John: Because we haven't got a 13 yet.

Derek: Yeah we have look.

John: No, in these.

Derek: No.

John: It equals 48. But there is 48 in some things. Yeah, there is in this one.

Researcher: That's not actually the reason. It's not because of that 13.

John: Hm. [Doubtfully] Is it because we went wrong on one of these?

Researcher: No, no.

John: Is it because it's the wrong way round? The 16 and the 32?

Researcher: Is there anything you could do about that?

John: Oh yes, yeah, yeah, yeah, yeah. I thought this was useless but now it's useful. These bits. Okay. Right, now we've just changed it round. Now try. There we go. Now, 13 add 48. Now that one.

All pupils, to a greater or lesser degree, came to articulate potential commutations one or more steps ahead in order to use further statements to make transformations. From trial B:

Yuri: If we can swap them two around.

Linda: Yeah.

Yuri: And swap them with the 33 so we can get the 50 and 11. Go on, that one.

Linda: Huh?

Yuri: That one. Now swap them two around. Now you can get 50 add 11.

At times, some pupils commented on the physical appearance of the boxed term when transformed by $c=a+b$ forms. From Trial C:

Barbara: Now change the 53 into 41 plus 12.

Nadine: Okay now it's a big sum.

However, partition was explicitly articulated only in trials A to D. For example, in Trial A, when the pupils first encountered a diagram containing the form $c=a+b$, Terry inferred its transformational effect, and its use for achieving the task goal:

Terry: Oh! That's the one that you do first! It has to be.

Researcher: Why?

Terry: Because it's splitting up the 40 and the 1.

In trials A to C, the pupils adopted a strategy of starting with $c=a+b$ forms to partition the numerals in the box, then using $a+b=b+a$ and $a+b=c$ forms to commute and compose the term into a single numeral. From trial B:

Yuri: Try splitting the 37 first. Um, you have to click on that. No, hit [i.e. click] all the numbers ...

Linda: 29 add 8.

Yuri: So, 73. 29 add 73 that said so, split, no wait. How do you get that for... Unless you got to switch them two around. So it's...

Linda: Which two around?

Yuri: Them two. Go, go on that. Now switch them two. Now you got that one.

Linda: 29.

However, in trials D to F, this start-with-partitioning strategy was not discovered or adopted by the pupils. They relied on trial-and-error when selecting statements to a greater extent than the pupils in trials A to C. The following example is from Trial D:

Zoë: Try that on the other one.

Kitty: No, it's just swapped them.

Zoë: Shall we try swapping and then we can try ...

Kitty: What shall we try?

Zoë: That one.

Researcher: Why that one Zoë?

Zoë: I don't know.

The contrast across trials was most marked in the later stages when the diagrams are more complicated and so strategic approaches are significantly more efficient.

DISCUSSION AND FURTHER WORK

The data show that the presentation of equality statements as transformational rules enables pupils to explore and talk about arithmetic notation in non-computational ways. Left-to-right readings of individual statements, as widely reported in the literature, are replaced by looking for matches of numerals across statements and terms. The task offered pupils a *utility* (Ainley, Pratt and Hansen 2006) for equality statements, namely making substitutions of notation towards a specified task goal. This utility arose because statements were presented as reusable rules for diagrammatic activity rather than isolated questions of numerical balance.

All the pupils distinguished the commuting transformational effects of $a+b=b+a$ forms, and used this distinction to discuss possible transformations one or two steps ahead. Only half the pupils distinguished the partitioning transformational effects of $c=a+b$ forms, and these pupils were able to use this distinction as part of a strategy that proved advantageous for later, more complicated diagrams.

When the pupils articulated commuting and partitioning effects this does not mean they had a conception of the underlying arithmetic principles. Baroody and Gannon (1984) found that young children can appear to exploit commutation to reduce computational burden, but are often merely indifferent to consistency of outcome. Trial B came from a study in which the last few diagrams contained some false statements, such as 77=11+33, and the value of the boxed term was not conserved across transformations. Interestingly, the pupils did not comment on this, and when asked afterwards if diagrams had contained false statements were unable to say (Jones, 2008).

This suggests pupils do not coordinate ‘sameness’ and ‘exchanging’ meanings for the equals sign when working with the task.

Current work is exploring how these two meanings for the equals sign might be coordinated using a constructionist approach to task design. Trials C, E and F are from a study in which the pupils subsequently went on to make their own diagrams using provided keypad tools. This requires ensuring numerical balance when inputting statements, and testing that these statements can be used to make substitutions when placing them into a diagram. A second aim of this current work is to find out whether pupils can translate verbalised calculations into notational diagrams. These calculations usually contain implicit partitioning and commuting (as in “34+23. 3 plus 4 is 7, and 30 plus 20 is 50, and 50 add 7 is 57”), which learners must identify and make explicit as statements on the screen in order to achieve the task goals. Early analysis suggests that again articulating partition is key to success.

A future aim, then, is to explore how the selection and sequencing of arithmetic diagrams can help all pupils to notice and articulate partitioning effects.

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APPROACHING FUNCTIONS VIA MULTIPLE REPRESENTATIONS: A TEACHING EXPERIMENT WITH CASYOPEE

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Abstract: *Casyopée is an evolving project focusing on the development of both software and classroom situations to teach algebra and analysis at upper secondary level. This paper draws on our current research in the ReMath European project focusing on the approach to functions via multiple representations. In this paper, we present the design of an experimental teaching unit for the 11th grade and some preliminary results.*

INTRODUCTION

The notion of function plays a central role in mathematics and for many authors technology can help students to learn about this notion especially because of the representational capabilities of digital environments. Recently, authors extended the range of representations by considering functional dependencies in a non symbolic domain. Falcade and al. (2007) proposed for instance to use Dynamic Geometry as an environment providing a qualitative experience of covariation and of functional dependency in geometry.

An aim of our team in the ReMath project is to develop a teaching unit taking advantage of a wealth of representations of functions offered by technology. In this aim, our software environment - Casyopée - has been extended, adding to the existing symbolic window a geometrical window with strong connections between them. Casyopée's symbolic window is a computer environment for upper secondary students. The fundamental objects in this window are functions, defined by their expressions and domain of definition. Other objects are parameters and values of the variable. Casyopée allows students to work with the usual operations on functions like: algebraic manipulations (factoring and developing expressions, solving equations ...); analytic calculations (differentiating and integrating functions); graphical representations; supports for proof …. The new window offers the usual dynamic geometry capabilities, like defining fixed and free geometrical objects (points, lines, circles, curves) and constructing others. It also offers distinctive features: geometrical objects can depend on algebraic objects and it is possible to export geometrical dependencies into the symbolic window, in order to build algebraic models of geometrical situations (Lagrange & Chiappini, 2007).

SOLVING A PROBLEM OF FUNCTIONAL DEPENDENCY WITH

CASYOPEE

In order to explain this extension, we expose now the type of problem whose resolution can take advantage of Casyopée, and how**.** This is an example:

Consider a triangle ABC. Find a rectangle MNPQ with M on [oA], N on [AB], P on [BC], Q on [oC] and with the maximum area

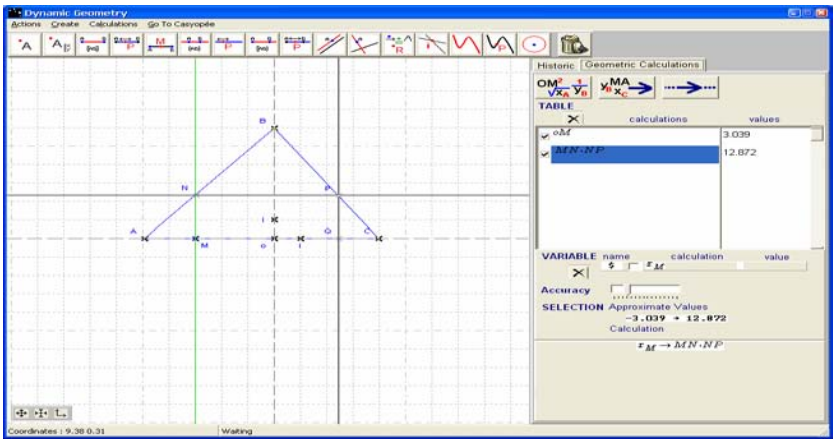

Fig. 1: The geometrical window of Casyopée

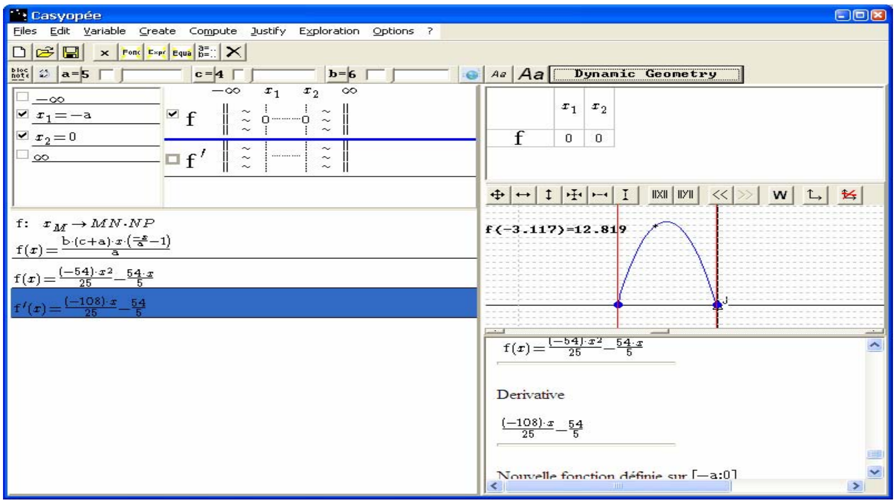

Fig. 2: The symbolic window of Casyopée

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Constructing a generic triangle ABC in the geometrical window can be done after creating parameters in the symbolic window. For instance, the points can be *A(-a;0), B(0;b)* and *C(c;0)*, a, b and c being three parameters. Then one can create a free point M on the segment [oA] (o being the origin) and the rectangle can be constructed using dynamic geometry capabilities.

In the Geometric Calculation tab (Fig.1) one can create a calculation for the area of the rectangle MNPQ and then define an independent variable. Numerical values of calculations and of the variable are displayed dynamically when the user moves free points. The user can then explore the co-dependency between these values. If this co-dependency is functional (i.e., the calculation depends properly on the variable) it can be exported into the symbolic window and Casyopée automatically computes the domain and the algebraic expression of the resulting function. Otherwise, Casyopée gives adequate feedback.

After exporting into the symbolic window, one can work on various algebraic expressions of the function and on graphs. For instance, one can use properties of parabolas, or algebraic transformations or Casyopée's functionality of derivate to find the answer to the question. One can also use the graph of the function to conjecture about the area maximum.

QUESTIONS AND THEORETICAL FRAMEWORKS

As the above example shows, Casyopée offers very varied functionalities and representation of functions:

- means for creating generic dynamic figures,
- geometrical calculations to express a range of quantities that can be considered as dependant variables,
- possibilities of choosing an independent variable like a distance or an abscissa involving free points,… feedbacks about this choice of a variable,
- means to observe numerical covariation between points and calculations, or between an independent variable and a calculation,
- means to export a functional dependency between the chosen variable and a calculation to the symbolic window, resulting in an algebraic form of the function,
- means for treating this algebraic form in various registers.

The overarching question addressed by the Casyopée team is: how to exploit these varied functionalities of representation in order to develop students' understanding of a functional dependency, particularly by articulating a geometrical situation with its algebraic model?

To investigate this question, we built an experimental teaching unit at 11th grade. In this paper, we present first the frameworks that helped us to build this experiment and to interpret our observations. Then we present the experiment and we report on the observation of the last session where students used the wider range of representations.

The first framework is based upon the notion of "setting" introduced by Douady (1986). According to Douady, a *setting* is constituted of objects from a branch of mathematics, of relationship between these objects, their various expressions and the mental images associated with. When students solve a problem, they can consider this problem in different settings. Switching from a *setting to another* is important in order that students progress and that their conceptions evolve. Students can operate these *changes of setting* spontaneously or they can be helped by the teacher. The setting distinguished here are geometry and algebra,

We also rely upon the notion of *registers of representations* from Duval (1993). Duval stresses that a mathematical object is generally perceived and treated in several registers of representation. He distinguishes two types of transformations of semiotic representations: *treatments* and *conversions*. A treatment is an internal transformation inside a register. A conversion is a transformation of representation that consists of changing of a register of representation, without changing the objects being denoted. It is important that students recognize the same mathematical objects in different registers and they get able to perform both treatments and conversions.

Here we distinguish the geometric and the algebraic settings corresponding to Casyopée's two main windows. In these two settings, the functions modeling a dependency are different objects: a relationship between geometric objects or measures in the geometric setting, and an algebraic form involving a domain and an expression in the algebraic window. In the above problem, students have to switch from the geometric to the algebraic settings and back, to be able to use symbolic means for solving questions that were formulated in the geometric setting. As explained by Lagrange & Chiappini (2007), we expect that, working in the geometric setting, students would understand the problem and the objects involved, and that after switching to algebra, this understanding would help them to make sense of the objects and treatments in the algebraic setting.

Inside each of these two settings the functions can be expressed in several registers. In geometry, especially with dynamic geometry, functions can be represented and explored in different registers: covariations between points and measures, or between measures, or functional dependency between measures. In algebra, functions can be expressed and treated symbolically, by their expressions, by way of graphs and of numerical tables. Mastering these expressions and treatments, and flexibly changing of register are important for students' ability to handle functions and acquire knowledge about this notion.

A third framework is the *instrumental approach*, based on the distinction between artefact and instrument. An actefact is a product of human activity, designed for specific activities. For a given individual, the artefact doest not have an instrumental value in itself. It becomes an instrument through a process, called *instrumental genesis*, involving the construction of personal schemes or the appropriation of social preexisting schemes. Thus, an instrument consists of a part of an artefact and of some psychological components. The instrumental genesis is a complex process; it requires time and depends on characteristics of artefacts (potentialities and constraints) and on the activities of the subject (Vérillon & Rabardel, 1995).

In the case of an instrument to do or learn mathematics like Casyopée, the instrumental genesis involves interwoven knowledge in mathematics and about the artefact's functionalities. Artigue (2002) showed how this genesis can be complex, even in the case of simple task like framing a function in the graph window. More generally, the many powerful functionalities of CAS tools have a counterpart in the complexity of the associated instrumental genesis (Guin & Trouche, 1999). We are then aware that we must take care of students' genesis when bringing Casyopée into a classroom. Moreover, Casyopée offers a multiplicity of representations in two settings and in several registers. Understanding and handling these representations involves varied mathematical knowledge. Students have then to be progressively introduced to these representations, taking into account the development of their mathematical knowledge.

Constructing the sessions of the experiment, we also used the Theory of Didactical Situations as basis for designing tasks. According to this theory, learning happens by means of a continuous interaction between a subject and a milieu in an *a-didactical situation*. Each action of the subject in milieu is followed by a retro-action (feedback) of the milieu itself, and learning happens through an adaptation of the subject to the milieu. Thus, with regard to Casyopée use, learning does not depend only on the representational capabilities of this software, but also on tasks and on the way they are framed by the teacher. Within this perspective, we looked for situations in which students interact with Casyopée and receive relevant feedbacks. For example, to solve the above problem, students have to choose between different independent variables to explore functional dependencies in the geometrical window and to export a dependency into the algebraic window. In case the variable is inadequate, the feedback they receive is a message from Casyopée. In other cases, the algebraic expression automatically produced by Casyopée can be more or less complex, which is another feedback: too complex expressions have to be avoided in order to ease the subsequent algebraic work.

Concerning the methodology, we use *didactical engineering* (Artigue, 1989), a method in didactic of mathematics, to organize and evaluate the experimental teaching unit, and to answer the research questions. The treatments and interpretations of collected data based on an internal validation which consists in confronting *a priori* analysis of the situation with *a posteriori* analysis. This method produces an ensemble of structured teaching situations in which conditions for provoking students' learning have been planned.

THE EXPERIMENT

Our experimental teaching unit consisted of six sessions. It was experimented in two French 11th grade classes. It was organized in three parts. Consistent with our sensitivity to students' instrumental genesis, each part was designed in order that students learn about mathematical notions while getting acquainted with Casyopée's associated capabilities:

- The first part (3 sessions) focused on capabilities of Casyopée's symbolic window and on quadratic functions. The aim was that students became familiar with parameter manipulation to investigate algebraic representations of family of functions, while understanding that a quadratic function can have several expressions and the meaning of coefficients in these expressions. The central task was a "target function game": finding the expression of a given form for an unknown function by animating parameters.

- The second part (two sessions) aimed first to consolidate students' knowledge on geometrical situations and to introduce them to the geometrical window's capabilities. The central task was to build geometric calculations to express areas and to choose relevant independent variables to express dependencies between a free point and the areas. It aimed also to introduce student to coordinating representations in both algebraic and geometrical settings, by way of problems involving areas that could be solved by exporting a function and solving an equation in the symbolic window.

- Finally, in the third part (one session) of the experimental unit, students had to take advantage of all features of Casyopée and to activate all their algebraic knowledge for solving the optimization problem presented above.

Below, we give some insight on how we are currently exploiting this experiment with regard to our question about Casyopée' potential for multi-representation. We limit ourselves to the final session for which the problem and the students' instrumental genesis should allow to take full advantage of this potential. We draw some elements of a priori analysis of this session and we compare with the a posteriori analysis of the functioning of a two student team.

THE SITUATION IN THE FINAL SESSION: ELEMENTS OF *A PRIORI* ANALYSIS

Tasks

The problem is presented by the teacher by animating a figure in Casyopée's geometrical window:

> *Let a, b and c be three positive parameters. We consider the points A(-a;0), B(0;b) and C(c;0). We construct the rectangle MNPQ with M on [oA], N on [AB], P on [BC] and Q on [oC]. Can we build a rectangle MNPQ with the maximum area?*

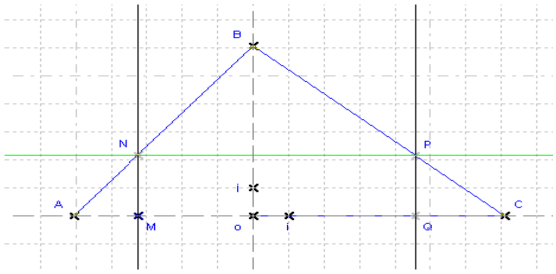

Fig. 3: The figure built in Casyopée

The tasks proposed to students are then:

- The construction of the rectangle MNPQ: students are required to load a Casyopée file with the parameters' definition and the triangle, then to complete the figure by building the segments [oA], [AB], [BC] and [oC] and to create the free point M and the rectangle's vertexes.
- To create a geometrical calculation for the area of the rectangle MNPQ: this can be obtained by the product of the lengths of two adjacent sides, e.g. MNxMQ
- To explore the situation by moving the point M on the segment [oA].
- To prove the conjecture by algebraic means.

The teacher also asks students to write the proof, indicating their choice of variable and using results displayed by Casyopée. Finally, students are expected to visualize the answer in the geometrical window.

Covariations and representation of functional dependencies

This situation involves two settings and different registers. Students can conjecture the answer to the question by exploring numerical values of the area in the geometrical setting. They can explore the variation of the area in different ways corresponding to different registers of representation. First, they can observe co variation between the point M and the area, looking at the values of the calculation they created for the area of the rectangle, noting that when M moves from A to B the value grows then decreases, with a maximum value when M is the middle of [oA]. They can also observe co variation between a measure involving the free point M and the area. For instance, they can observe together the values of the distance oM and of the area. Finally, they can choose an independent variable involving M and observe the functional dependency between this variable and the area.

In the algebraic setting students can apply different algebraic techniques to the algebraic form of the function in order to find a proof. Exporting a function with Casyopée, one obtains a more or les complex algebraic expression reflecting the calculation's structure. Students then need to expand this expression to recognize a quadratic function. They can then apply their knowledge about these functions to prove the maximum. It is possibly not easy for them, because of the three parameter involved.

They can also use the graphical representation in this algebraic setting to explore the curve, complementing the exploration they did in the geometrical setting: the parabola is familiar to the students and they can easily recognize a maximum.

The situation is partly a-didactical. In each setting, students interact freely with Casyopée and use the feedbacks to understand the situation. Nevertheless, some key points like passing from a co variation to a functional dependency are expected to be difficult for students, although the corresponding action (choosing an independent variable) has been presented in the preceding sessions. Passing from one setting to the other is expected to be far from obvious for students. The corresponding actions in Casyopée (exporting a function in the symbolic window, interpreting a symbolic value in terms of position of a point) have also been presented before, but it is the first time that students have to do it by themselves.

Students can choose their own independent variable between possible choices (oM, x_M, MN, MQ…) with consequences upon the algebraic expression of function. They can do it alone but it is expected that the teacher mediation will be necessary. It is also possible that they will want to change their choice of a variable in order to obtain a simpler algebraic expression of the function.

We expect a great variety of uses of representations reflecting students' free interactions with the situation. Some students can stay a long time exploring co variations and need teacher mediation to go to functional dependency while others pass more or less quickly to the algebraic setting to consider the function. In this setting, some can prefer to explore graphs, while others prefer working on algebraic expressions. It is possible that some students find too difficult to apply algebraic techniques to the general expression (i.e. with parameters) and prefer to work by replacing these parameters by numbers. In any case, we expect that students will consider several representations, make sense of them and make links between them.

ELEMENTS OF *A POSTERIORI* ANALYSIS: THE CASE OF A TEAM

During the experiment, we observed selected teams of students. In this paper, we focus on a team of two students, which according to the observation in the first five sessions had a favorable instrumental genesis. According to their teacher they were good students.

The explorations in different settings and registers

Creating a geometrical calculation for the area of the rectangle, they typed MNxMP instead of MNxMQ by mistake. They moved M and observed growing numerical values of this calculation, while, for some positions of M the area was visibly decreasing. This first feedback allowed them to correct the geometrical calculation.

Like most students they had difficulties in choosing an appropriate independent variable, confusing the independent variable and the calculation. They needed help from the teacher to activate the correct button. They chose at first NP. They moved for a long time the point M and observed how numerical values of this variable and of the area MN$\times$MQ changed. They found an optimal value and interpreted it: "(the optimum) is when N is the midpoint of [AB] I believe, and P is the midpoint of [BC]". The teacher asked them for a proof. A student suggested an equation in an interrogative tone. Actually, the problems solved in sessions 4 and 5 were about equalities of areas and have been solved by way of an equation.

The teacher guided them to export the function, but they found the resulting expression too complicated. Then they choose another independent variable MQ, and got the same expression after exporting again the function. Finally, they chose x_M as an independent variable, obtained the algebraic expression b(x-1/ax)(a+c-a(x-1/ax)-c(x-1/ax)) and expanded it into a quadratic polynomial.

Proving the maximum

The team graphed the function, recognized a parabola, and said that they do not know how to determine the maximum's x-coordinate. Then they wanted to apply an algebraic formula to get this x-coordinate and used Casyopée to expand the expression. For some reason they got a non parametric expanded expression, the parameters being instantiated. Then it was easy for them to obtain by paper/pencil a numerical value of the maximum's x-coordinate. Then they returned to the geometrical window, checked this result and generalized, saying that the maximum is for x_M=a/2.

They did not attempt to prove this generalized property by working on the parametric expression and then they only partially solved the problem. Other teams did, but had much difficulty to apply the formula to the parametric quadratic expression.

SYNTHESIS

The observation reported above is globally consistent with the a priori analysis. The students used more or less all registers of representation. The independent variable was recognized as the central feature of the solution, allowing connections between registers. Casyopée offered means for exploration and various feedbacks that helped this recognition. The students' instrumental genesis helped them globally to interact with Casyopée, but important actions like choosing a variable and exporting a function were still unfamiliar. They were influenced by the problems they solved before and it was difficult for them to have a clear approach of an optimization problem. Although they used parameters before and they understood the generalized problem, using parametric expressions was still difficult.

With regard to our question on how to exploit Casyopée's varied functionalities of representation, we can say that, in spite of remaining difficulties, the teaching experiment helped this team to develop an understanding of a functional dependency. We have of course not now a more definite conclusion and we are currently analysing the other teams' observation as well as productions after the experiment. We are especially sensible to the teacher's help to students. In the above observation, we saw this help in crucial episodes, like changing settings and we want to know whether this help was efficient for students' learning, beyond the solution of the problem.

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EQUALITY RELATION AND STRUCTURAL PROPERTIES [1]

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We present the results of a questionnaire on equality we administrated to a large and vertical sample of Italian students. Some of the questions were devised to investigate the presence of relational thinking.

INTRODUCTION – THE SCENARIO OF THE RESEARCH

This paper emanated from an international study of arithmetical misconceptions in primary schools (Cockburn & Littler, 2008) part of which considered equality (Parslow-Williams & Cockburn, 2008). One way to detect whether a wrong answer can be attributed to a misconception or a slip (Schlöglmann, 2007), is to analyse the persistence of the same wrong answer through a range of school grades. Here we focus on a questionnaire on equality administered to 1,147 Italian seven to sixteen and a group of university students in their first year. (cf. table 1 below).

THEORETICAL FRAMEWORK AND THE AIM OF RESEARCH

It has been well documented that an understanding of equality is crucial to the development of algebraic thinking (Alexandrou-Leonidou & Philippou, 2007; Attorps & Tossavainen, 2007; Puig, Ainley, Arcavi & Bagni, 2007).). Here we focus on formal number sentences, building on the work of Molina, Castro & Mason (2007) and, in particular, *relational thinking* – a term that Molina *et al*. (2007) borrow from Carpenter, Franke & Levi (2003). The student employs *relational thinking* if s/he

> "makes use of relations between the elements in the sentence and relations which constitute the structure of arithmetic. Students who solved number sentences by using relational thinking (RT) employ their number sense and what Slavit (1999) called "operation sense" to consider arithmetic expressions from a structural perspective rather than simply a procedural one. When using relational thinking, sentences are considered as wholes instead of as processes to carry out step by step." (Molina *et al*., 2007, p. 925)

The term *relational thinking* here is the opposite of procedural thinking. Although it sounds similar to Skemp's (1976) *relational understanding*, i.e. "knowing what to do and why" (Skemp, 1976, p. 21), in this context it focuses on different aspects of learning. In our opinion *relational thinking* is very similar to *relational interpretation* of equality detected by Alexandrou-Leonidou & Philippou (2007) and very closely related to *conceptual knowledge*, as proposed by Attorps & Tossavainen (2007) as opposed to *procedural knowledge*. The latter adopted the framework of Sfard (1991) and focused on the mathematical properties of the equality relation, i.e. reflexivity, symmetry and transitivity and, using a sample of 10 qualified and 75 pre-service secondary mathematics teachers, concluded that a lack of understanding of these properties impairs the development of the concept of equation. In Italy the structural approaches to arithmetic and algebra, together with equations, are usually introduced in grade 9. Early structural approaches and equations are, however, in the curricula for grades 6, 7 and 8. In the light of the above, this study investigated

- whether there was evidence of relational thinking in grades 2 - 5;
- how the structural notions taught of pupils in grades 6 - 11 influenced the responses;
- misconceptions about aspects surrounding equality amongst the students.

METHODOLOGY

The questionnaire

All pupils were given a written questionnaire containing a series of equality problems. Our questionnaire comprised simple number sentences using similar questions and symbols to those found in the literature (cf. Radford (2000), Hejný & Slezáková (2007), and Behr, Erlwanger & Nichols (1980)).

Zan (2000) suggested that misconceptions may exist in a sort of 'grey' zone beneath the complete consciousness of the person. Our questions were intended therefore to be sensitive enough to reveal misconceptions and relational thinking without being too direct, since this can make the subjects aware of their errors, resulting in an immediate correction before they commit themselves to writing an answer.

We decided to avoid the issue of having both signs '+' and '-', in the same calculation, as an awareness of both algorithms was required to find the solution. The questionnaire was four pages in length [2]: 2a and 2s presented addition and subtraction problems respectively, using mainly single digit numbers; in 3a and 3s numbers were between 20 and 100. The first six questions on each page were designed to build confidence and involve two given numbers, one operational sign, '+' or '-'. On all pages a firm knowledge of symmetry of equal relation can help solve the first six questions; in 2a form, two of them focus explicitly on the symmetry of the equality relation. The next four questions have three given numbers, two operational signs (cf. Behr *et al.* (1980), Sáenz-Ludlow & Walgamuth (1998) and Alexandrou-Leonidou & Philippou (2007)). These were followed by 'open' questions [3] with two operational sign, two boxes and two given numbers, as $a+\square = \square+b$; $\square+a = \square+b$; $a-\square = b-\square$ (we have yet to come across such examples in the literature). These were intended to reveal the possible use of reflexivity of equality, the commutative property of addition and awareness of 0 and its formal properties. We also tested the presence of the 'commutative property' of subtraction. Other less common open and closed questions were devised to detect the possible awareness of the transitive property of equality, with two-equality schema such as $a\pm b = c\pm\square = \square$ or with three-equality schemas such as $a\pm b = c\pm\square = d\pm\square = \square$ and $a\pm\square= b\pm\square =c\pm\square = \square$. These can be solved correctly by direct calculation showing the *non-RT behaviour* or by the use of structural properties, applying the different *RT behaviours* of Molina *et al*. (2007). For the reflexivity of equality in forms 2a and 3a we included a question of the schema $a = \square$.

The questionnaire instructions were intentionally open-ended: we asked "Can you complete these number sentences?", without specifying which type of number could be used (naturals, relative integers, rationals or reals), thus leaving the possibility that older students could apply their knowledge about the various numbers systems.

The sample

As we had to rely on volunteers teachers, our sample was determined by their response. The number of returned questionnaires was as shown in Table 1.

Grade	2	3	4	5	6	7	8	9	10	11	Univ.
No.	76	131	58	228	282	172	161	62	22	47	112

Table 1: The sample structure

The size of the sample (1,147 respondents giving 62,898 answers) and its breadth (11 different grades) allowed us to compare our data with the research literature; observe whether such findings might be extended to older students and detect any new phenomena. Due to the scope of the study, the conditions of the test administration were largely un-specified (time, day, duration of the test, surveillance during the proof, and so on) except in case of university students who were given 15 minutes to complete the questionnaire.

THE RESULTS

Interestingly, regardless of age, the majority of solvers only used natural numbers. Due to the lack of space we focus on sample questions (while retaining the original questionnaire 'numbering').

1. The first six questions on each page and symmetry of equal relation.

A-priori analysis. In the questions 2a. (b) 5+□=8 and 2a. (f) 8=5+□ the role of symmetry is evident, since the numbers involved are the same ('strict'). In other examples we can speak of a symmetry 'at large' for the structure of the number statements, but not for the numbers involved. This gave us the opportunity to examine whether some pupils were 'blind to the symmetric property of the equality' (Attorps & Tossavainen, 2007), in the 'strict' sense and/or the 'at large' meaning. For each pair the correct answers to both questions can be obtained by computation; in case of 2a. (b) and 2a. (f), the result is 3, for both. For this pair, a difference in the result or the lack of one answer can be attributed to an incomplete mastery of the formal property of equality. For the remaining pairs we presume that a right answer to one question of the pair and the firm awareness of equality relation symmetry may suggest a good strategy for solving the other question of the couple, even if the numbers are different: a solver of 79-□=25 who has trouble with 53=78-□, can think of this second task in the form

78-□=53 to find the right answer. A right answer of only one question of these couples can suggest an ‘at large’ non-application of the symmetry in the pair.

A-posteriori analysis. The case simplicity of 2a. (b) and 2a.(f) resulted in high success rates: 98,14% and 95,40% respectively. People responding differently to the two tasks, certainly gave an incorrect answer. Individuals who responded incorrectly, are highly likely to a lack of their understanding of symmetry. However, in the case of the other pairs, the situation is more complex since we cannot exclude wrong computations even if symmetry was being used. In table 2 we distinguish between the ‘strict’ symmetry non-application and the ‘at large’ non-application. Data in the latter case are obtained cumulatively for the other eleven pairs (sample no. 12,993).

	number of at least one wrong or missing answer		rate of symmetry non-application		rate of contemporary success	
	strict	large	*strict*	large	*strict*	large
Grades 2-5 [4]	*46*	891	*93.48%*	79,91%	*90.67%*	78.33%
Grades 6-8	*18*	1090	*83.33%*	74,86%	*97.07%*	83.90%
Grades 9-11	*4*	231	*75.00%*	62,77%	*96.95%*	83.94%
University		47		89,36%		93.01%
χ -test	*8.68E-6*	3.38E-94	*0.29*	1.27E-7	*3.29E-6*	3.04E-24
Global sample	*68*	2259	*89.71%*	75,92%	*94.51%*	82.61%

Table 2: The non-application of symmetry of equality.

Values of the χ-test less than 0.05 (0.01) show that difference among grade classes are statistically significant; the result 0.29 is consequence of small numbers.

Reference the sum of the numbers of all the wrong and missing answers to at least one of two tasks suggests that a lack of awareness of the formal property is the greater source of error.

2. The task 2a. (k) 5 + □ = □ + 7

A-priori analysis. The task is open with the choice of one of two missing numbers determining the other. The location of the boxes invites, possibly, the reflexive property of equality without the need for any sort of calculation e.g. 5+$\boxed{\mathbf{7}}$ = $\boxed{\mathbf{5}}$+7. The neutral role of 0 with addition could inspire the answer 5+$\boxed{\mathbf{2}}$ = $\boxed{\mathbf{0}}$+7. Other structural answers using the formal property of negative numbers (and 0) are 5+$\boxed{\mathbf{0}}$ = $\boxed{\mathbf{(-2)}}$+7 and 5+$\boxed{\mathbf{(-5)}}$ = $\boxed{\mathbf{(-7)}}$+7. Relational thinking offers a criterion for revealing a wrong answer: the given numbers are odd, therefore the two inserted numbers must have the same *even parity*. The repetition of a box could prompt (wrongly) younger pupils, in particular, into thinking that the numbers they are required to insert must be the same.

A-posteriori analysis. Of 1,143 students that were given this question, 1,057 responded, of which 842 gave the right answer (73.76%) suggesting that the task was relatively easy. Each answer given (right or wrong) used natural numbers. It is interesting to note the distribution of the structural answers by age of pupils. We suspect that the infrequent use of zero to solve the problems e.g. 5+$\boxed{2}$ = $\boxed{0}$+7 could be due to a 'fear' of 0 - i.e. the complex acknowledgment of 0 as a number - or, simply reflect that individuals were unacquainted with this mathematical character.

2a. (k) 5 + □ = □ + 7	correct response	presence of the answer 5+$\boxed{7}$ = $\boxed{5}$+7	presence of the answer 5+$\boxed{2}$ = $\boxed{0}$+7	commonest correct response (with frequency)
Grades 2-5	62.47%	7.26%	7.66%	5+$\boxed{4}$=$\boxed{2}$+7 (26.21%)
Grades 6-8	78.21%	6.86%	6.24%	5+$\boxed{3}$=$\boxed{1}$+7 (34.93%)
Grades 9-11	86.26%	2.65%	0.88%	5+$\boxed{3}$=$\boxed{1}$+7 (46.02%)
χ -test	4.79E-10	0.21	0.04	
Global sample	73.67%	6.41%	5.94%	5+$\boxed{3}$=$\boxed{1}$+7 (32.30%)

Table 3: The relational thinking presence and the commonest right answers to 2a.(k).

The commonest incorrect response was 5+$\boxed{2}$ = $\boxed{7}$+7 (with 18.14% of the 215 wrong answers). To interpret this we can consider the application of "Three First Numbers – TFN" and then "Answer After Equal Sign–AAES" modalities of Alexandrou-Leonidou & Philippou, (2007). The presence of two equal boxes, did not appear to be highly relevant as only the 8.37% of incorrect responses used the same number twice: 5+$\boxed{a}$ = $\boxed{a}$+7 (***a***=1 or ***a***=2 having the greatest frequency). The *even parity* criterion was found in all of the 842 exact answerers and in 26.98% of the wrong answers, giving a total rate of 85.15% of the answers. We have also an *echo effect*: when the given numbers are odd, the percentage of correct answers using a pair of odd numbers is 62.59%.

3. The task 2s. (k) 6 - □ = 8 - □

A-priori analysis. This task is also open with the first number determining the second. Moreover, if restricted to natural numbers, the subtrahend must be less than minuend. The location of boxes may invite the following answer 6-$\boxed{6}$ = 8-$\boxed{8}$, a solution using 0 as result of both members of equality. Alternatively the neutral role of 0 when subtracting could be employed e.g. 6-$\boxed{0}$ = 8-$\boxed{2}$. For other aspects the a-priori analysis of this task is similar to the previous one. We expected a wrong relational thinking answer in the 'commutativity' of subtraction, i.e. the answer 6-$\boxed{8}$ = 8-$\boxed{6}$.

A-posteriori analysis. 1,056 students were given the question; 953 responded, 762

2s.k) 6 - □ = 8 - □	rate of success	rate of response 6-$\boxed{6}$ = 8-$\boxed{8}$	rate of response 6-$\boxed{0}$ = 8-$\boxed{2}$	rates of commonest right answer 6-$\boxed{2}$=8-$\boxed{4}$
Grades 2-5	61.09%	3.16%	2.76%	31.58%
Grades 6-8	76.06%	2.78%	1.05%	38.12%
Grades 9-11	80.15%	1.90%	3.43%	28.57%
χ-test	9.33E-7	0.24	0.24	0.09
Global sample	72.16%	2.76%	2.86%	35.17%

Table 4: The relational thinking presence and the commonest right answers to 2s. (k).

did so correctly (success rate 72.16%), suggesting that this task was relatively easy even if slightly harder than 2a. (k). Table 4 summarises the use of relational thinking. The *echo effect* appeared to be present as 56.17% of the right answers used pairs of even numbers. The *even parity* criterion is present in 87.20% cases. In this case the commonest correct answer is similar for all grades. Again we could argue that the commonest right answers were influenced by the fear of using 0 combined with the *echo effect*. The commonest wrong answer was 6-$\boxed{\mathbf{2}}$=8-$\boxed{\mathbf{2}}$ (12.04% of the 191 wrong answers) and we could consider this kind of response motivated by application of TFN twice assuming that the second box is filled in first. Of the wrong answers, the structural, but incorrect, response 6-$\boxed{\mathbf{8}}$=8-$\boxed{\mathbf{6}}$ was given in 4.19% cases. The value 0.09 of the χ-test show that the differences among grades classes are not statistically significant.

4. The task 3a. (k) □ + 21 = □ + 11

A-priori analysis. As above the task is open and has ‘freedom grade one’. The location of boxes may invite the use of commutative property of addition, i.e. $\boxed{\mathbf{11}}$+21 = $\boxed{\mathbf{21}}$+11. Moreover the neutral element of addition could reduce computation e.g. $\boxed{\mathbf{0}}$+21 = $\boxed{\mathbf{10}}$+11. Questions 2a.(k) and 3a.(k) have the same quantity of given numbers and addition symbols, but the boxes are differently placed: in 2a.(k) reflexivity of equality is at stake while in 3a.(k) the commutativity of addition is involved.

3a.k) □ + 21 = □ + 11	rate of success	rate of response $\boxed{\mathbf{11}}$+21 = $\boxed{\mathbf{21}}$+11	rate of response $\boxed{\mathbf{0}}$+21 = $\boxed{\mathbf{10}}$+11	rate of commonest right answer $\boxed{\mathbf{10}}$+21 = $\boxed{\mathbf{20}}$+11
Grades 3-5	60.52%	9.22%	2.84%	26.24%
Grades 6-8	72.17%	10.76%	4.48%	28.92%
Grades 9-11	80.92%	11.32%	4.72%	31.13%
University	94.64%	11.32%	6.60%	42.45%
χ-test	1.04E-10	0.94	0.57	0.03
Global sample	73.03%	9.14%	4.51%	30.54%

Table 5: The relational thinking presence and the commonest right answers to 3a. (k).

A-posteriori analysis. This task was administered to 1,094 students from grade 3 to first year of University: 979 responded with 799 of them giving the right answer (success rate 73.03%), comparable with the success rate for 2a. (k). Here *RT* appears to become more evident with increasing age. The use of 0 as the neutral element in addition is similar to that in task 2a. (k) but the commutativity of addition is more prevalent. Multiples of ten - excluding 0 - were found in 53.82% of the correct answers. The *even parity* criterion occurred in 83.86% responses. In 3a. (k) question the *echo effect* was not evident as 60.33% of the right answers had a pair of even numbers. The commonest wrong answer is $\boxed{\mathbf{32}}$+21 = $\boxed{\mathbf{53}}$+11 (6.11%). We hypothesize that the first box is filled in when the task is interpreted as □ = 21+11, in a sort of “Left Side Sum-LSS” modality. The completion of the second box is suggested by AAES modality (Alexandrou-Leonidou & Philippou, 2007).

In our opinion, the presence of two digit numbers had a double effect: the attempts decrease from 92.48% of 2a. (k) to 89.49% of 3a. (k) and this may be significant as the latter sample excluded 2nd graders but included first year university students. Secondly it may be that the presence of two digit number in this task activates a more attentive approach to the computation (the answers to other questions support this) and we could attribute to this attitude the greater presence of *RT*.

5. The tasks of type a = □.

A-priori analysis. Behr *et al.* (1980) include examples of the type $a = a$, with given numbers and so we incorporated 2a. (l), 9 = □, and 3a. (n), 42 = □. To solve them one needs only apply the reflexive property of equality. These are closed tasks and do not require computation.

We anticipated that the absence of operational symbols would be destabilizing and

	9=□ success rate	9=□ with operational signs	42=□ success rate	42=□ with operational signs
Grades 3-5	76.67%	35.00%	70.82%	43.48%
Grades 6-8	73.17%	59.68%	72.98%	39.60%
Grades 9-11	67.94%	86.67%	67.94%	57.89%
University			92.86%	100%
χ-test	0.10	2.11E-6	1.53E-5	0.02
Global sample	74.01%	54.70%	73.70%	44.77%

Table 6: Comparison of results of the tasks 2a. (l) and 3a. (n).

The result in no answer or the use of operational symbols (cf. Behr *et al*., 1980). location of the two tasks in their form allowed us to explore if there was a *tiredness effect*, influencing the rates of answer and success.

A-posteriori analysis. Task 2a. (l) was given to grades 2 - 11 (1,239) with 1,151 responding with 917 of correct (74.01%). The majority of incorrect answers (54.70%) express the result with operational symbols and the computation on the proposed numbers gives 9, showing a procedural interpretation of the sign =. The commonest answer of this kind is 9 = $\boxed{\mathbf{3^2}}$, in 44.53% of all ‘operational’ answers and was given by the majority of 6th graders and above.

Task 3a. (n), 42 = □, was given 1,190 grade 3-11 and 1st year university students, 1,049 responded with 877 of them giving the right answer (73.70%). The ‘operational’ answer rate is 44.77% and the commonest ‘operational’ responses were, globally, 40+2 (19.48%) and 21+21 (18.18%).

6. The task 2a. (m)   5 + 4 = □ + 6 = □

A-priori analysis. This task is the first which presents more than one equality sign. It is a closed task. The ‘chain’ of equality asks for the transitive property of equality.

Wrong answers suggest a lack of awareness of it. The most probable incorrect response is 5+4 = $\boxed{\mathbf{9}}$+6 = $\boxed{\mathbf{15}}$ (cf. Alexandrou-Leonidou & Philippou, 2007).

A-posteriori analysis. 1,104 students responded with 718 giving the right answer (62.82%). As was expected the commonest wrong answer (70,47%) was 5+4 = $\boxed{\mathbf{9}}$+6 = $\boxed{\mathbf{15}}$. This suggests either the pupils filled in the second box before completing the first or that they worked step by step from left to right. In either cases such results bring into question their intuition as Semadeni (2008) states:

> "The transitivity of equality: "if A = B and B = C then A = C" was regarded by Fischbein (1987, pp. 24, 44, 59) as intuitively true. Piaget et al. (1987b, p.4) regards transitivity as an example of a systematic type of necessity…Transitivity is part of the deep intuition of equality (for numbers, for geometric points, for sets), involved in a multitude of deductive inferences." (p.10)

7. The task 3s. (m) 48 - □ = 47 - □ = 46 - □ = □

A-priori analysis. This task is complex: it is open-ended, involves two-digit numbers, three subtraction signs and three equalities. Despite having four boxes to fill, it has 'freedom grade one'.

3s.m) 48-□=47-□=46-□=□	rate of success	48-$\boxed{\mathbf{48}}$=47-$\boxed{\mathbf{47}}$=46-$\boxed{\mathbf{46}}$=$\boxed{\mathbf{0}}$	48-$\boxed{\mathbf{2}}$=47-$\boxed{\mathbf{1}}$=46-$\boxed{\mathbf{0}}$=$\boxed{\mathbf{46}}$	commonest right answer rate 48-$\boxed{\mathbf{3}}$=47-$\boxed{\mathbf{2}}$=46-$\boxed{\mathbf{1}}$=$\boxed{\mathbf{45}}$
Grades 3-5	56.99%	2.73%	9.09%	20.00%
Grades 6-8	56.91%	4.29%	12.29%	27.14%
Grades 9-11	60.77%	0%	7.59%	48.10%
University	83.04%	2.15%	6.45%	61.29%
χ-test	3.64E-6	0.22	0.29	6.5E-12
Global sample	60.19%	3.16%	10.28%	33.54%

Table 7: The presence of relational thinking regarding 0 and the commonest right answers to 3s.m).

To solve these questions correctly an explicit awareness of transitive property seems to be required. The task allows simple solutions involving *RT* and formal properties of 0 in many ways: 48–$\boxed{\mathbf{48}}$ = 47 – $\boxed{\mathbf{47}}$ = 46 – $\boxed{\mathbf{46}}$ = $\boxed{\mathbf{0}}$, or 48 – $\boxed{\mathbf{2}}$ = 47 – $\boxed{\mathbf{1}}$ = 46 – $\boxed{\mathbf{0}}$ = 46. It is also possible to apply negative numbers, or fraction and so on, but no one did.

A-posteriori analysis. 896 – out of a possible 1,050 - responded with 632 giving the right answer (60.19%). The commonest correct answer reveals that the learners are at different levels of understanding, growing with age, taking care of the additive decomposition of numbers by fives: 48 = 45+3, 47 = 45+2 and so on. The structural properties of zero were most common in first eight grades of schooling. 41 pupils gave incorrect answers (22.65%) applying the transitive property of equality only once. 47.51% of those who were incorrect responded 48-$\boxed{\mathbf{1}}$ = 47-$\boxed{\mathbf{1}}$ = 46-$\boxed{\mathbf{1}}$ = $\boxed{\mathbf{45}}$.

CONCLUSIONS

The questionnaire enabled us to explore a phenomenology linked to relational thinking expressed by the reflexive, symmetric and transitive property of equality, the roles of zero respect to addition and subtraction and the commutativity of addition.

Our study is peculiar in the variety of schools and age range sampled. In this sense other similar experience known in literature took place in smaller school, segments. Another feature of our paper is that we are interested here in the right answers, even if sometimes we quote, also, wrong answers. Our research would have been more rigorous had we selected the sample statistically. Therefore our paper cannot be used for drawing general conclusions, statistically sound, about relational thinking, nevertheless in our feeling it might open a new trend of study about the equality, pointing out that this subject needs an attentive reflection regarding the way and the time in which the concept of equality is presented (in itself), let it grant that is introduced somewhere and somehow.

Overall primary school pupils were slightly better (even if in many cases differences are not statistically significant) than the older respondents in their application of relational thinking in specific tasks, but the presence of two-digit numbers appeared to hindered them. Nevertheless, a small but significant group demonstrated structural thinking provoking the question of how to extend such thinking to others. The transmissive teaching methods in Italy may explain why relational thinking does not appear to improve between grades 6 and 11 even if the structural properties of operations are taught explicitly, suggesting a parallel presence of relational and procedural thinking, independent from teaching. For symmetry our pupils confirmed the Attorps & Tossavainen (2007) results with prospective teachers.

There was a global score progression with increasing age. Addition questions were easier than subtraction; generally, pupils responded more appropriately to one digit answers than to two digit problems. Answering more complex questions under conditions of stress (e.g. tiredness) suggests that the students possessing a 'reified understanding' (described by Sfard (1991) as 'being able to see something familiar in a different light') of formal properties have an important tool which saves time and mental energies. Students who were aware of formal properties tended to cope better than others under conditions of complexity and stress. The prevalence of such knowledge was low however and in some cases appeared to decrease with age despite such topics being introduced in Italian Secondary School. Few participants (even from University) reificated the reflexive property of equality, and the function of zero in addition and subtraction. The commutative property of addition was more apparent. The more complex nature of the statements of symmetry and transitivity of equality do not necessarily indicate their presence, but only their absence. The sub-sample of university students appeared to have the awareness of these arithmetic tasks, but, surprisingly, more than 1/5 of the sub-sample responded to 3s. (m) incorrectly with more than 1/3 of them revealing a lack of a global view, answering 48-$\boxed{\mathbf{1}}$ = 47-$\boxed{\mathbf{1}}$ = 46-$\boxed{\mathbf{1}}$ = $\boxed{\mathbf{45}}$, and of the transitive property of equality!

NOTES

[1] The authors gratefully acknowledge the support of the British Academy (Grant no. LRG-42447) which provided a platform for this study.

[2] The questionnaire presents 54 questions divided in four forms: 2a, 2s, 3a, 3s (the digit refers the grade of primary school and the letter ‘a’ is for addition and ‘s’ is for subtraction). The integral version of questionnaire and the report of results are available at the web-site http://www.unipr.it/arpa/urdidmat/M2ip.

[3] When a solution is uniquely determined, e.g. 32 + 25 = □ + 16 = □ we use the adjective ‘close’; whenever the solver is free to choose the suitable numbers, e.g. 48 - □ = 47 - □ = 46 - □ = □ we use ‘open’.

[4] Italian children start school 6-years-old. Primary school comprises five grades; stage one of secondary school, grade 6, 7 and 8, and the final stage of secondary school 9 to 13.

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STRUCTURE OF ALGEBRAIC COMPETENCIES

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This paper reports a research study that aims at understanding interrelationships between algebraic abilities. Theoretical considerations drawn from the literature suggest various interconnections. To gain empirical evidence a test was developed and the findings analyzed by fitting different statistical models.

INTRODUCTION

Ideally, algebra lessons lead students to develop a profound understanding of algebraic concepts and the ability to see algebra as a central and connected branch of mathematics and the ability to apply algebra to a wide range of topics. If this happens, then students can be said to have a high algebraic competency. Even with this aim in mind, it is not clear how to design algebra courses. There are many approaches to the teaching of algebra (see e.g. Bednarz et al. 1996) and they obviously differ in the algebraic concepts that are given priority. The field of algebraic concepts is very broad, e.g. mastering the concept of an equation is a long process in which various aspects of the equation concepts are learned and they all interact with other algebraic concepts. To help in planning the algebraic learning process, it would thus be useful to gain more insight into the inner structure and dependencies of these algebraic concepts.

Such insight can be expected from empirical studies of various designs. Interpretative studies are valuable and some have been performed, especially as they allow to link theory and observations. However, they usually focus on a small number of students and it often remains unclear, how representative they are. Quantitative studies, on the other hand, often lack a deeper connection to theories.

The quantitative study reported here tries to apply advanced statistical models on a test that was developed to reflect certain theoretical assertions about the learning of algebra. In this paper, only results from a single use of the test are reported but this study is part of a larger research project that will collect longitudinal data as well.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

Algebra deals with a lot of objects, including numbers, variables, expressions, functions and relations, and each of these can play many different roles. School algebra thus is composed of many ingredients. Several theories have been developed that give some structure to this large field and we will mention some of them that were used implicitly in our study.

Variables play the central role in our investigation because they are a link between most of the other objects mentioned. Variables are used in many different ways in algebra. Küchemann (1979) gave six ways of using variables. From the perspective of integrating these modes of variable usage into a scheme we found that these modes, although useful in explaining students results, are bit unhandy.,When looking at algebra problems from textbooks we found that his “Letter ignored” is not of great importance and test items regarding it seem always a bit artificial. Moreover, it may be subsumed to the aspect of a variable as generalized number. The use of a variable as a reference to a non-arithmetical object “Letter as object” is (which restricts itself to standard school algebra) an important misconception that is viable only in a very limited subset of algebra. As a misconception it should not be included into the structure of abilities that are to be mastered by the students. Malle (1993) gave a short list of three aspects which proved a bit coarse when classifying textbook problems and test items. A synthesis of these approaches that works well for the classification of the role of variables in different problems turned out to be very similar to the one found by Drijvers (2003) in his empirical study, see below. It is worth to make explicit the operations that are linked with the different roles of variables. This shall emphasize the fact that the role of a variable is not only determined by the algebraic context but also by the subject working with it, e.g. the x in $2x+1=4$ may be viewed as an unknown which is to be determined or as a placeholder were one can insert numbers or expressions.

- Placeholder P (operation: substitute (not only numbers but general expressions))
- Unknown U (operation: determine)
- General number G (undetermined; operation: expressing relations)
    - Ga: General number used in analyzing expressions
    - Gm: General number used for modeling (describing)
- Variable as changing quantity V (operation: change the value)
    - Vi: independent variable (operation: change at will)
    - Vd: dependent variable (operation: observe change)
    - Vr: variable in a relation without predetermination what variables is changed independently as in Ohm’s law $U=RI$.
- Variable as a symbolic element of the symbolic algebraic calculus: C (i.e. operation: use as structure-less object in symbolic manipulations)

Different researchers have advocated the point of view that mathematical objects are constructed from operations (Sfard 1991, Dubinsky 1991, Gray & Tall 1994). While the theories of these authors differ in detail, the broad picture seems similar and naturally explains e.g. the creation of symbolic expressions as encapsulated calculation sequences. It is not as clear to which processes the concept of a variable is linked. Therefore, we associated the above mentioned processes to each aspect of variables. Obviously, different operations lead to different objects, but nevertheless, mathematicians look at variable as a single concept which can be used under different aspects. It is therefore interesting to note that the operation of substitution has tight relations to all the other operations except those operations associated with the last aspect of the above list. We therefore formulate the hollowing hypothesis:

Substitution is a central operation in algebra and the competence to use it properly is at the heart of algebra in the sense that it makes other operations easy as well, with the exception of the symbolic calculus aspect. Put in more technical language, this states that the ability to use substitutions should be a good indicator variable for performance in other algebraic tasks.

Checking the validity of this hypothesis is one of our research questions. The next question is much more open: To what extent do these aspects of ‚variable' depend on each other?

METHODOLOGY

There exist many tests for algebraic achievement but most items test syntactic term rewriting or formal equation solving capabilities. Far fewer test items exist that assess algebraic understanding and algebraic concepts developed by the students. A notable exception is Küchemann's work in the late 70s and early 80s. For this study we developed a new test that is somewhat in the spirit of Küchemann and uses many of his items, but most items were developed to reflect the various aspects of variables described above. In addition, there were test items on the relation between equations and functions.

The study was conducted at the beginning of grade 11 (age approximately 16 years) of a German high school (Gymnasium).There were 141 students from six classrooms in the study. Unlike most other German schools this particular high school starts at grade 11 and thus collects students who were recently at a large number of different schools. Although this sample is not representative of German students, it can be expected to span the breadth of the population better than samples from classes that had the homogenizing effect of a common school culture. However, the mean achievement level is supposed to be below that of an average grade 11 high school.

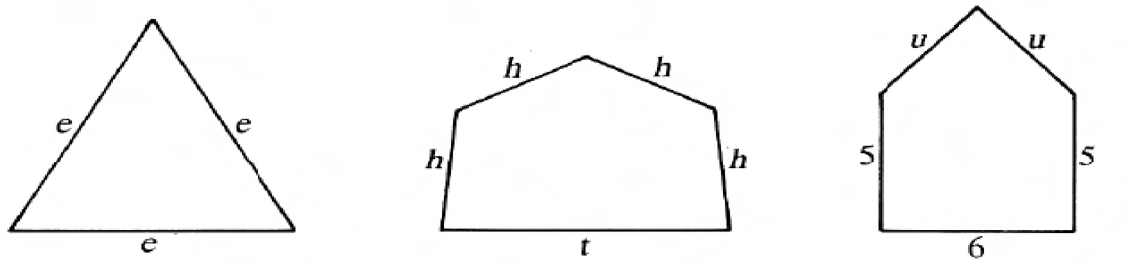
The test was compiled for this study but most of the items had been used in our research group before. The test consists of 43 items, two of which are multiple choice items, while the others ask for a free form response. The answers were rated on a point scale as the following example of a rating rule indicated:

> Item 2a (from Küchemann 1979): Give a short answer and explanation: What is greater? n+2 or 2n?
> 0 Points= no response; false response without argumentation
> 1 Points= example; some explanation; wrong answer with detailed explanation
> 2 Points= example with explanation; detailed explanation without case distinction
> 3 Points= almost correct with case distinction
> 4 Points= completely correct

Some examples of the test items are shown below; their association to aspects of variable' are shown in square brackets:

Item 4: (based on Küchemann 1979) Let r be the number of rolls and c the number of croissants bought at a bakery. A roll costs 30ct, a croissant is 70ct.
a) What is the meaning of $30r+70c$? [G]
b) How many parts have been bought all together? [G]

Item 6a,b,c (from Küchemann 1979): Work out the circumference of the following figures:



[G]

Item 9: a) Assume that the equation $a=b+3$ always holds. What happens to a if b is increased by 2? [V] (from Küchemann 1979)
c) Assume that the equation $a=2b+3$ always holds. What happens to b if a is increased by 2? [V]

Item 13: It is known that $x=6$ is a solution of $(x+1)^3+x=349$. How then can one get a solution of $(5x+1)^3+5x=349$? [G] (from Küchemann 1979)

Item 14: Simplify the following expressions a) $(a-3)^2-a^2$ b) $(x-x^3)\cdot(x+x^3)$ c) $\sqrt{36+4a^2}$ d) $\frac{1}{n}-\frac{1}{n+1}$ [C]

Item 16: Given the examples $7\cdot 9=8^2-1$ and $11\cdot 13=12^2-1$, formulate a general rule and justify it. [G]

Item 17: A function is defined by: $f(x)=x^3-2$. Determine
a) $f(2)=$ b) $f(y)=$ c) $f(x+1)=$ d) $x\cdot f(x)=$
[P]

Item 19: What must be substitute for x in the expression $2(x^2-1)$ to obtain the desired result? [P]

Desired Result	Substitute $x=$….
6	
-2	
$2((a+1)^2-1)$	
$2(b^2+2b)$	

The test items were classified by the aspects of variables they involve and by the relevance of the abilities to handle functions (Fun), relations (Rel), syntactical expression manipulation (Syn), working with unknowns (Unk), handling substitutions (Sub) and translating between algebra and geometry (Geo). Of course, this classification is build upon assumptions about typical solution strategies.

Besides more traditional statistical methods, this study uses structural equational modeling as a tool to model dependencies. While this technique is frequently used in many empirical sciences, it seems that its use in the mathematics education community not as widespread and I know of no application of this technique to gain insight into concepts of algebra. However, I believe that this statistical tool is appropriate here, because it allows us to work with hidden variables that cannot be observed directly (e.g. the person's understanding of a variable as a general number) and to model relations among latent and observed variables.

RESULTS AND INTERPRETATION

The test contained several items developed and used by Küchemann 30 years ago. Despite the passage of time, our results were very similar, thereby underpinning the validity of his study. The order of empirical difficulty of the items turned out to be precisely the same as that found by Küchemann. Also the percentage of students that solved the items were remarkable close (despite the fact that we tested 16 year old students while Küchemann tested 14 year olds), with one interesting difference regarding the 'letter as object' aspect. We found Item 6a was solved by 74% while Küchemann found 94% (for 6b and 6c we found 74%, 58%, Küchemann found 68%, 64%). These numbers become interesting when combining with the result that item 4a was solved only by 14% and 4b only by 7%. Most students that failed on 4a showed a clear object interpretation reading $30r+70c$ to mean 30 rolls and 70 croissants. However, many more students were able to solve 6a and 6b, which are described by Küchemann as items that can be solved successfully using 'letter as object'. Using a variable as reference to an object should be differentiated into two aspects: The misconception that a variable can stand as shorthand for any object, and the conception that a variable stands for some measureable quantity, such as the length of a segment. This latter interpretation is at the heart of an approach to algebra by Davydov, Dougherty and others (see Gerhard 2008) that is suitable also for younger children. Interestingly, the sum of points of 6a and 6b show a correlation with the total test score of r=0.62 indicating that the ability to solve these items show much more than a misconception.

Next we gather some results from analyzing cumulative variables as described above. Together, these variables accounted for approximately 70% of all test items. According to the Kolmogorov-Smirnov-Test they can be considered to be normally distributed. Then a multivariate regression of the total score to these Variables was performed. The standardized beta-weights (with standard errors) were:

Variable	**Standard. Beta(SE)**
Syn (syntactic manipulation)	0.15(0.03)
Geo (geometry)	0.28(0.03)
Sub (substitution)	0.26(0.04)
Gen (working with general numbers)	0.22(0.04)
Fun (functions)	0.07(0.04)
Rel (relations)	0.38 (0.04)

The interpretation of these numbers must of course take into account that they reflect to some extend the composition of the test. There were eight items that were taken together to form the Rel variable, but only four that formed the Fun variable. Yet this can't explain the dramatic difference in beta weights. We conclude that understanding of algebraic relations is an important component of algebraic competency. It is also interesting that the Geo variable that consists of only five items is that important. One may draw the conclusion that expressing relations among quantities is at the heart of algebra. It is therefore justified to exercise this extensively in introductory algebra lessons.

Then an analysis of covariance gave first insight into interdependences. The interesting findings were: There is almost no correlation between the syntactic manipulation (Syn) and Geo (r=0.09), Sub (r=0.09), Gen (r=0.10), Fun (r=0.13), Rel (r=0.02). The scale Syn consists of item 14 (which has two more sub-items than shown) on the simplification of expressions and of two items on solving linear equations. The result means that syntactic manipulation and conceptual understanding are two different dimensions. The assumption implicit in some teachers position on teaching algebra that learning the symbolic algorithms will lead to insight seems thus to be false. To further support this point we give the following two-way table:

Number of students		Score on syntactical items	
		Above average	Below average
Score on other items	Above average	37	27
	Below average	36	41

The χ^2-test gives p=0.19 on that, compatible with the assumption of independence (which is certainly not correct, but there is only a very weak relationship.)

This almost-independence result was stronger than expected and future studies should investigate this again. An interesting observation is that the connection is somewhat stronger for higher achieving students.

On the other hand the highest correlation (r=0.63) is between Rel and Subs. Subs also correlates with Geo (r=0.44), Gen (r=0.54) and Fun (r=0.54). All of these correlations are highly significant (p<0.01). This supports the hypothesis about the fundamental role of substitution given above.

Next, we report some results from the path model study. Although this interpretation was not intended by Drijvers (2003) we made up a structural equational model (more specific, a path diagram) from his diagram given below (Fig. 1). The model fit was acceptable according to Hair's (Hair et al. 1998) recommendations with CMIN/df=1.96<2.0 and Parsimony-Adjusted Measure PCFI=0.56 . We found that the concept of placeholder loads most on the changing quantity (our role V of a variable; path weight and standard error: 1.14(0.48)), then on Unkown (U, weight 0.37(0.12)) and negligible on the generalizing aspect (G, weight 0.08(0.04)). The other arrows carry small weights as well. While the first two results are plausible, the question arises what influences the important aspect of a variable as a generalized number if not the placeholder aspect.

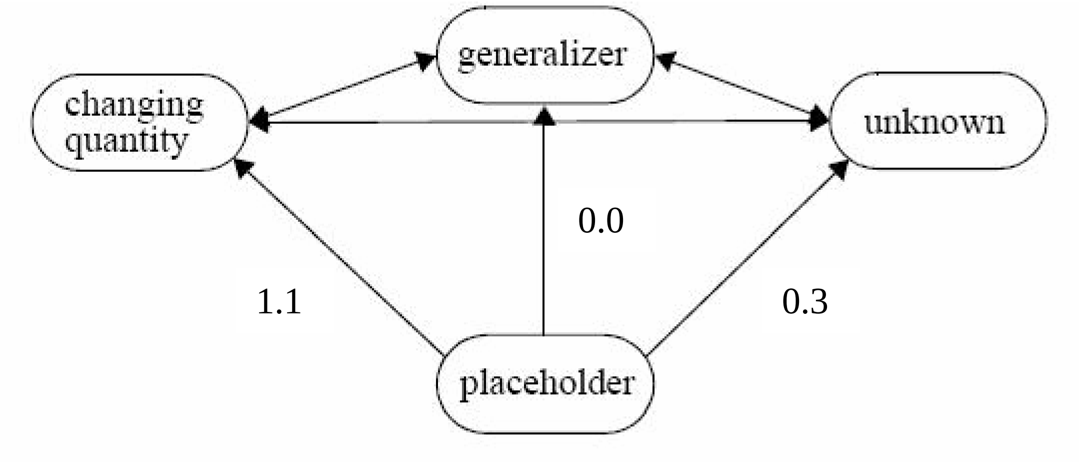


Fig. 1

The following model (Fig. 2) includes all of our five variable aspects. The latent variables are named by the short cuts of the variable aspects defined in the theory section. This model provides almost good model fit CMIN/df=1.53, PCFI=0.67. Nevertheless many of the estimates for regression weights are rather small and we will refine and modify the model shortly to get better results. Nevertheless this model shows some interesting results. First the arrows that relate the calculus aspect C with other aspects carry small weights. This feature is common to all models we tried and reflects the fact mentioned above, that syntactic manipulation is almost independent from the rest of the test. Another interesting fact is that there is a substantial (and significant) weight for the arrow from G to V. This is naturally interpreted as the implication that a general number can be viewed as standing for changing numbers. On the other hand, students learning algebra may first master the aspect of changing quantities and only later develop the general concept of a variable that stands for a general number without reference to a particular number. Therefore we omit this arrow in later models.

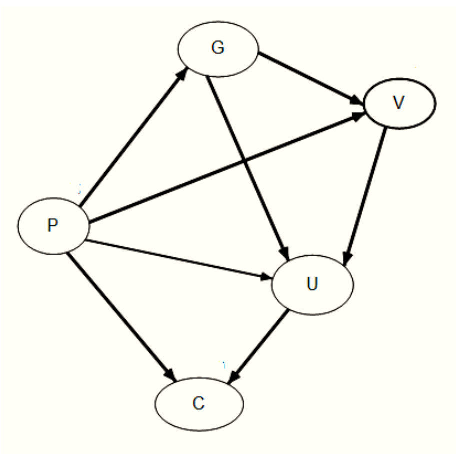


Fig. 2

The above path-model can be refined by splitting the aspect of general number as indicated in the theory section into the aspect of using the general number for analyzing or for modeling. Furthermore, we will omit the syntactic aspect of a variable as an element of algebraic calculus, because it is essentially independent from the rest. With these decisions made we tried out many linear structural equational models but concluded that the following one is the best choice. Some other models provide a slightly better model fit, but this model (Fig. 3) has two important properties: It is plausible from the theoretical point of view and can therefore be easily interpreted. Its advantage from the statistical point of view is that most of its path coefficients are either significant or close to significant. The model fit is adequate with CMIN/df=1.92 and PCFI=0.55. The estimates for regression weights (with standard errors in parentheses) are:

Place holder P	→	Unknown UK	0.15 (0.07)
Place holder P	→	General Number Ga	0.031 (0.024)
Place holder P	→	General Number Gm	-0.003 (0.341) ≈0
Place holder P	→	Variable V	-0.76(0.42)
Unknown UK	→	General Number Ga	0.30 (0.16)
Unknown UK	→	General Number Gm	-0.64(2.8) ≈0
Unknown UK	→	Variable V	6.6 (2.0)
General Number Ga	→	General Number Gm	2.5 (1.9)
Variable V	→	General Number Gm	0.077 (0.44) ≈0

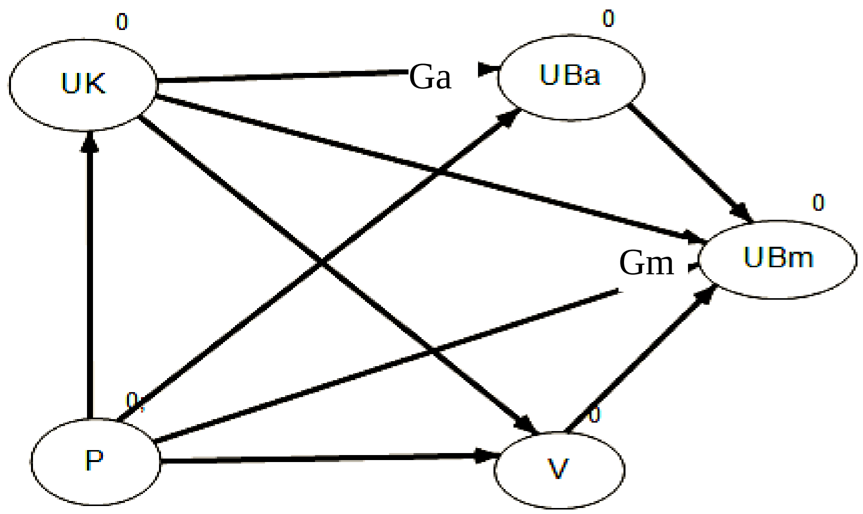


Fig. 3

Compared to the above model based on Drijvers diagram it may seem strange that the arrow P→V has a negative weight. This result does not claim that there is a negative correlation between these abilities but only that the direct influence is negative taking into account the large influence from the arrows P→UK and UK→V which both have positive weights. In fact, when omitting the UK→V arrow from the model, the arrow P→V gets massively positive (1.6). The negative weight in our model is therefore plausible: Learning to handle variables as placeholders may pave the way to seeing

variable as unknowns and this in turn helps develop the full concept; however students who can only deal with placeholders are unlikely to see variables as quantities that can change because a placeholder once filled with a number is constant.
The path weight for Ga→Gm was 2.5(1.9). When reversing the arrow it became negligible. This can be interpreted to mean that learning to analyze situations with variables is a prerequisite to modeling situations that are initially free of algebraic symbolization. On the other hand the aspect V is not helpful for algebraic modeling. This may give a hint that at the level of modeling situations by algebraic equations one is working at a rather high level where individual values of variables and their change is not considered. We hypothesize that the aspect of change is not important in forming the model but in its validation. But this conclusion can't be drawn from the data of this study.

Is it possible to assign students a single latent variable "algebraic competence"? To test this we fitted two simple models to the data. One model with only one latent variable "algebraic competence" and one model with latent variables "Univariate" and "Multivariate". The model with two latent variables has a model fit of CMIN/df=1.78, while the model with a single latent variable has a model fit of CMIN/df=2.99. This substantial difference may be seen as support for the hypothesis that algebraic competency is a higher dimensional construct, because here we have a higher dimensional modeling that fits the data better. Nevertheless, the test as a whole fits the assumptions of the one-dimensional Rasch model. Hence we conclude, that structural equational models can reveal detailed results.

CONCLUSION AND OUTLOOK

The findings of this study lead to two different kinds of conclusions. The first kind concerns the results from analysis of covariance and fitting the structural models. They indicate that the activities of describing general geometric situations algebraically are good indicators for overall performance. Similarly, substitution is a fundamental operation in algebra that shapes the meaning of algebraic constructs.

The second kind of conclusion concerns the level of algebraic competency reached in grade 11 and this is more specific to the situation in Germany (although the study does not claim to be representative for all German schools). While some areas (in particular, solving linear equations and using binomial formulas) show acceptable results, other parts of algebraic thinking, especially those that serve as a backbone in introductory calculus courses, reveal a serious lack of competence. Either a solution has to be found to cure the algebra decease or one should consider curricular changes in grades 11 and later that eliminate the need for those kinds of algebraic thinking; however, this would mean dropping calculus from the curriculum.

The future work of this research project is aimed at improving the situation. In collaboration with schools we aim to use this test as diagnostic instrument to help us assign tasks that will improve the construction of algebraic meaning. This includes the use of new algebraic technology (Oldenburg 2007) and the use of experiments (Ludwig & Oldenburg 2006).

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GENERALIZATION AND CONTROL IN ALGEBRA

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This study addresses the importance of a pedagogical approach that contemplates generalizations students make spontaneously, due to the high value generalizations have in the learning of algebra and the construction of mathematical rationality. I consider the problem of the control of spontaneous generalizations, from the perspective of both didactic interventions and student's learning. I analyze the problem of the internal validation in the case of algebraic writings. I show various examples of pre-university students' (17-18 years) spontaneous generalizations and handling of control. The study suggests the necessity to face this problem from the beginning of the secondary school.

INTRODUCTION

Algebra constitutes a domain which favours the progress of mathematical rationality from the beginning of secondary school, through reasoning involving generalization. Moreover, generalization processes are of a great value in the production of knowledge (personal and scientific) (Garnham & Oakhill, 1993).

The ability to generalize is a common faculty of human reasoning, not specific of any content, which raises (not content-specific) learning questions. However, the ability to generalize in a particular domain involves specific learning problems within this domain. Various authors have considered the question of generalization in algebra, and favouring generalization activities is now seen as being an approach to algebra (see Bednarz, Kieran, Lee, 1996). Specially, justification related to generalization processes has been considered by Radford (1996) and, from a different perspective, by Balacheff (1987, 1991), amongst others.

However, students do not generalize only when faced to **generalization activities** (so as to find numerical or geometrical patterns, laws governing numbers, or the construction of formulas, etc). They also make generalizations in the context of tasks which do not require finding any regularity. This is what we call **spontaneous generalizations** (Panizza 2005a, 2005b).

From the point of view of the teacher's interventions, this sets the problem of anticipation. How can the teacher be attentive to the emergence of such spontaneous processes? Moreover, the student perceives differently the necessity to justify generalization, according to the more or less spontaneous character of the generalization, inasmuch as mathematical rationality is under construction.

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On the other hand, algebraic environment differs clearly from numerical and geometrical environments from the point of view of the *feedback* given to the student's activities.

It is important to consider this question in a systematic way through the various approaches to algebra (described in Bernardz, Kieran, Lee, 1996), which provide very different contexts for the emergence of such processes; in particular, from the point of view of the possibilities of control within algebraic environment or by means of conversion to other semiotic 'registers' (Duval, 1995, 2006).

I claim that such a pedagogical approach in the domain of algebra may favour the construction of mathematical rationality in secondary school.

RESEARCH METHODS

The data presented in this paper were obtained trough qualitative methods: observation of regular classrooms and case studies, focusing on student’s reasoning when analysing statements written in symbolic language. The research was conducted within four different pre-university (17-18 years) algebra courses.

The observations were conducted in a systematic way. A set of tasks was selected to be administrated in class by the teacher, in order to observe the procedures of students when analysing statements written in symbolic language, especially when trying to determine conditions under which algebraic statements are true. Special attention was directed to: the verbal and symbolic descriptions students produced, based on their observations and descriptions of objects of reference of statements (instantiations); its influence on the processes of statements (re)formulation; the *treatments* (in the sense of Duval) they do within the algebraic writings register and the capacity for going over from the formulation of statements in symbolic language to a representation of the statement in other register (*conversions*, in the sense of Duval), very especially the use of this capacity for control. The data consisted of notes from classroom observations and the student’s written works.

The study allowed identifying some phenomena among which the different kinds (according to its origin) of *spontaneous generalizations* presented in this paper.

For the case studies, four students that were considered representatives of the studied phenomenon were chosen from the algebra courses (their real names have been changed in this paper). The intention was to find specific features related to spontaneous generalizations, through mini-clinical interviews, all of them audio recorded. The reactions of students facing counterexamples provided by the interviewer in the context of their spontaneous generalizations, together with their perception (or lack of it) of the necessity of control and their processes of control inside or outside the register of algebraic writings, were observed.

The study showed that students often do (new) spontaneous generalizations based on the counterexamples provided by the interviewer and that their spontaneous generalizations are based on *local associations* of *few* examples which are not representatives of the objects of reference of the statements.

SPONTANEOUS GENERALIZATIONS: WHICH? WHY? WHAT? WHERE? HOW?

What are the spontaneous generalizations? Why it is important to take them into account in the class of mathematics? In what contexts do they emerge? How?

Spontaneous generalizations: which?

Let us see some examples, taken from the observations in the algebra courses:

Faced to the problem "Find the real values of x such that $x^2 \geq x$", Belén and María answered that "*$x^2 \geq x$ is true for every real number*" without solving the equation, but they arrived there by different ways. Inquired by the teacher, Belén argued "*it is evident, the square of any number is always greater than the number itself!*". María, instead, argued "*I have tried with several examples*, 1, 2, 3, -1, -2, -3, *and so…*"

Belén seems to have generalized to real numbers the property valid for natural and integer numbers (*extension of schemes of knowledge*, see Vergnaud, 1996). María seems to have done an *induction process*.

I wish to point out that both have done a generalization **even if the activity was not a generalization one**. It is also important to notice that both arrived to the **same conclusion** by **different ways** of reasoning. I will come back to this point. Nevertheless, both examples are very familiar. But let us turn to another one.

The problem:

"Decide if the following implication is true or false:

$\forall x \in \mathbf{R}$: $(2x^2 > x\,(x+1) \Rightarrow x > 1)$"

was given in class in order to analyze the algebraic competence of students to decide the relation between the solution sets of two inequalities - in an implication context -. Brenda's production is especially illustrative of the "problem" of **spontaneous generalizations** arising within the frame of a task.

When solving it, Brenda considers diverse examples, $x = 0$, $x = 1$, $x = 2$, $x = 3$, $x = -1$, $x = -2$, $x = -3$, $x = -4$ analyzing the value of truth of the antecedent and the consequent in each case. She concludes, correctly, that the statement is false, because "*it is possible to find values of x smaller than 1 that fulfil* $2x^2 > x(x+1)$"

The professor asks her to explain how she arrived at the answer.

Brenda says that "–2, –3, -4 *are counterexamples, because for them the antecedent is true and the consequent is false*".

According to the task, Brenda could have finished there, but she adds, immediately:

"*Ah, it was* $|x|$ *what we should have put!, what is true is*:

$\forall\ x \in$ R: ($2x^2 > x(x+1) \Rightarrow |x| > 1$)”.

According to my interpretation, Brenda makes a spontaneous generalization of the set of counterexamples used by her to argue (x = -4, x = -3, x = -2), and proposes a statement that she considers true. It is to note that the task did not require to find any regularity. Brenda does it spontaneously, perhaps with the intention of finding a true statement (Balacheff, 1987).

I want to draw attention to the fact that from the point of view of the logical complexity, Brenda could have analyzed the value of truth of her statement, since the original task was correctly solved and both statements required the same logical competences. Even though we can think about a greater difficulty to find the counterexamples - in as much these are in the interval [-1,0)-, I want to point out that Brenda does not consider it **necessary** to analyze her statement, she does not even consider it at all. She displays her affirmation beyond. So?

So, spontaneous generalizations: why?

Because a large part of the learning achievements resides in the capacity to generalize. By generalizing students construct knowledge. The emergence of these processes in the class is most important, as much for the learning of algebra as for the development of the mathematical rationality.

But conclusions require validation. This necessity –as it is well known -, is acquired, if it ever is, in the very long term.

On the other hand, when the generalization is a spontaneous one and therefore it is not directly related to the task to be solved- as in the cases of Brenda, María and Belén- it is difficult for the professor to anticipate it. In addition, a same result can come from different processes of generalization, as in the case of María and Belén. This is about something that usually occurs in the class of mathematics, and it is difficult for the teachers to have appropriate resources of intervention. So?

So, spontaneous generalizations: what?

This problem has led me to consider the generalization trying to deal with this phenomenon in its diverse manifestations. To do so, I tried to find the student’s processes of generalization in there amplest sense, such as those of transference of a domain to another one (see Sierpinska, 1995). I also consider extension of knowledge schemes as generalization, as it has been studied by Vergnaud (1996) in the domain of mathematics, by Leonard and Sackur (1990) through the notion of local bits of knowledge; and by Harel and Tall, -quoted by Mason (1996)- through expansive, reconstructive, and disjunctive generalization. So?

So, spontaneous generalizations: where?

I consider that the different contexts of use, the nature of the task, the forms that are used for representation, the meaning granted to the letters, can originate different types of spontaneous generalizations.

The contexts provided by different approaches to algebra must be studied from this point of view: these contexts, give rise to specific spontaneous generalizations? Are there particularities of these contexts in relation to the control possibilities? (Balacheff, 2001). So?

So, spontaneous generalizations: how?

Up to now, I have found a lot of spontaneous generalizations, and I find it fruitful to consider them as of different kinds. According to its origin (for a particular student in a particular moment), a spontaneous generalization may be of nature:

2. conceptual (based on the content to which the statement refers to), as Belén did in extending the range of an existing scheme (“*it is evident, the square of any number is always greater than such a number!*”);
3. logic (based on an inadequate understanding of logical connectors or rules of reasoning), as María did when considering that with several examples she had arrived at a true conjecture (“*I have proved it with several examples,* 1, 2, 3, -1, -2, -3, *and so…*”)
4. semiotic (based on an analysis of the content of the semiotic representation (Duval, 1995, 2006).

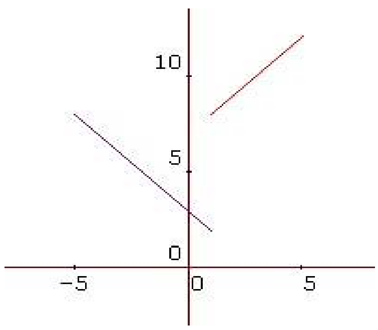
I think that this typology is interesting because it helps the teacher in the identification of leading elements of spontaneous generalizations on the part of the students, in the possibility of interpreting them and making them evolve.

Let us see an example of the later (semiotic) kind

Problem: Study the properties of the function

$$f(x)=\begin{cases}-x+3 & \text{if } x<1\\ x+7 & \text{if } x\geq 1\end{cases}$$

Taking into account the habitual scales that students use to plot functions I posed the hypothesis that -looking at those graphs- :



students would decide the *injective character* of the function. And it is what 40% of the group of students actually did. They generalized the *content* of the graphic semiotic representation and decided that it was representative of the function in its complete domain.

As in the case of Brenda, the students who responded to the problem in agreement with our anticipation did not consider it even necessary to make a control.

In order to advance in this point, clinical interviews were made. Let us see the processing of control that Ana Paula makes, faced to a counterexample provided by the interviewer. Ana Paula had stated that the function is injective, having done an incomplete analytical study (she analyzed each branch ($x < 1$ and $x \geq 1$) of the function in isolation) and looking at the plot.

Let us see (minor episodes have been skipped):

The researcher suggests her to analyze the pair of values $x_1 = -6$, $x_2 = 2$

Ana Paula does some calculations

Ana Paula:	Oh, yes, it's true...it is not injective... (*she thinks*)…What should I have put to see it was not injective? A negative number and a positive one?
Researcher:	I don't know, you find out.
Ana Paula:	I am searching so that they are the same... (*she thinks*)
Ana Paula:	Of course, as $-x$ changes the sign it is as if I had two positives, one adds up 3 and the other 7, I must get the same result... (*she equals to* 10, *she thinks and finds* –7 and 3)
Ana Paula:	-7 and 3...-(-7) +3 = 3 + 7, and thus I prove it is not injective
Researcher:	Wasn't it proved with –6 and 2?...
Ana Paula:	Yes, of course I had already verified it (*she still searches for counterexamples*)
Researcher:	Why are you searching other counterexamples?
Ana Paula:	Because if I had to do it again I would do it wrongly once again, because before I did it analytically, I verified it in the plot and I got the same result in both of them. Even more, I did a value table and I didn't put –6 and 2. I don't understand where was my mistake (reviewing her previous works).
Researcher:	aha...
Ana Paula:	Has the difference between x_1 and x_2 to be constant? Let's see, $x_1 - x_2$ equals to image
Researcher:	Which image?
Ana Paula:	Of both!… (*she gets at a loss in the calculations*).
Ana Paula:	Oh no! There are going to be infinite providing the image is greater or equal to 8. What can I do to find them?

Researcher: The image of x_1 has to be the same as that of x_2.

Ana Paula: I've already said it, it is the definition.

Researcher: You've said it but you didn't use it...

Ana Paula: Aha! (*she finally does some calculations and arrives to the equation*).

$-x_1 + 3 = x_2 + 7$

$x_2 + x_1 = -4, x_1 < 1, x_2 \geq 1$

To make control, Ana Paula analyzes the problem in various representations (graphical, algebraic, by tables) without integrating them. This example is representative of what happens with many students. Next I set out to analyze this problem, specially the problem of control related to the **algebraic writings**.

PROBLEMS OF CONTROL

Two aspects seem essential; on the one hand, the problem of the recognition of the **necessity** of control of the conclusions; on the other hand, supposing that the student has this ability, the problem of the **possibility** of making this control is posed (Panizza 2005b).

The problem of the necessity of control

In relation to the first point, perceiving the **necessity** of control is different according to whether generalization is a spontaneous one or it is obtained as asked for by the task. In the latter case, necessity of control is intrinsic to the task. Indeed, when someone **must** make a generalization, a suitable representation of the task should include the control necessity, that is to say the need to adjust the conjecture to the data. In addition, as Radford (1996) indicates "representations (in generalization) as mathematical symbols are not independent of the goal. They require a certain anticipation of the goal". That means, according to my interpretation, that in the **generalization activities** the control occurs like a process, during the resolution itself, through the re-representations that are made on the data, based on the analysis of the goal. On the contrary, for **spontaneous generalizations** the necessity of control is not intrinsic to the task, since generalization is not directly related to the goal. The examples of María, Belén, Brenda and Ana Paula are representative of this claim. However, many students may perceive this necessity. Ana Paula, faced to a counterexample provided by the interviewer, tries to control by shifting to other representations (graphical, algebraic, by tables). Anyway she does not succeed. This leads us to the problem of the **possibility** of control.

The possibility of control within the algebraic writings register

I claim that the **possibility** of control within the algebraic writings register is difficult as the retroaction does not work in the same way that in the arithmetical writings register or the material geometrical figures domain (Panizza & Drouhard, 2002).

In fact, in the arithmetical writings register, when students arrive by reasoning at an equality of the type 2 = 3, this writing in itself gives them information that plays the role of an element of control.

In the same way, in the material geometrical figures domain, when, faced to the famous problem of extension of a puzzle of Nadine and Guy Brousseau (1987), the pupils make inadequate extensions, the fact that the resulting pieces do not fit, constitutes an element of control.

Algebra is quite different. As Drouhard (1995) shows, when students arrive at $(a + b)^2 = a^2 + b^2$ they believe that the teacher just "prefers another rule", for instance $(a + b)^2 = a^2 + 2ab + b^2$ ("*You made a transformation and I made another one*...").

This example illustrates a general problem: that the register of the algebraic writings does not offer the students good elements of feedback and control.

Rojano (1994) establishes a similar conclusion (quoting Freudenthal), when analyzing the differences of feedback of the errors in arithmetic and natural language - provided by numerical contexts and daily communication -, unlike the feedback in the register of algebra. However, these characteristics of algebra are not sufficient to determine the conduct of control of a particular student in a particular context. The possibility that certain information can act as a feedback also depends on:

3. the student's abilities to "see" such information;

4. his possibilities to enter in contradiction (see Balacheff, 1987);

5. his capacity to deal with different types of statements (of existence, individuals, generals);

6. his linguistic skills on letters (syntax and semantics) (see Kirshner, 1989, Duval, 1995, Durand Guerrier, 1996, Panizza, Sadovsky & Sessa, 1998, Drouhard, Panizza, Puig & Radford, 2006);

7. his conceptual and operating skills on numbers, variables, unknowns and parameters (see Janvier, 1996).

I consider that an education that contemplates the fact that these skills are developed in parallel and in an interrelated way, must find didactic strategies for helping students to develop control means *inside* and *outside* the register of the algebraic writings. I adhere to the didactic frame of reference provided by Duval (ibidem) with the notion of *conversion* between different semiotic representation registers, especially for what control possibilities concerns.

CONCLUSIONS AND PERSPECTIVES

This study shows that pre-university students make different types of *spontaneous generalizations* in contexts of explanation, proof or discovery, without neither having acquired conscience of the necessity of justification of the conclusions, nor abilities for making control. From my point of view, this suggests the need of a pedagogical approach at secondary school that considers educational interventions in front of the students' spontaneous generalizations, in order to help them to improve mathematical reasoning.

I think that much more research is still needed for that. Specially, concerning the spontaneous transferences -such as analogies and metaphors- of algebra domain to another one, and the different approaches to algebra as contexts of emergence of spontaneous generalizations, their particularities and problems of control.

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FROM AREA TO NUMBER THEORY: A CASE STUDY

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In this paper we examine the way two 10^th^ graders cope with a tiling problem that involves elementary concepts of number theory (more specifically linear Diophantine equations) in the geometrical context of a rectangle's area. The students' problem solving process is considered from two perspectives: the interplay between different approaches relevant to the conceptual backdrop of the task and the range of executive control skills showed by the students. Finally the issue of the setting of modeling problem solving situations into number theory tasks is also commented.

INTRODUCTION

Modeling problem solving situations into generalization tasks related to number theory is useful for learning mathematics and includes two stages: modeling and solving the number theory tasks that emerge. On the one hand, solving generalization tasks dealing with number theory serves as a tool for developing patterns, as a vehicle towards appreciation of structure, as a gateway to algebra, as a rich domain for investigating and conjecturing at any level of experience (Zazkis, 2007). However despite of their significance number theory related concepts are not sufficiently featured in mathematics education. Consequently many issues related to the structure of natural numbers and the relationships among numbers are not well grasped by learners (Sinclair, Zazkis & Liljedahl, 2004). On the other hand according to Mamona-Downs and Papadopoulos (2006) when students have an accumulated experience on problem solving they can affect changes in approach and are able to take advantage of overt structural features appearing within the task environment. Moreover they can show a deeper understanding of the nature of mathematical generalizations. In their work which lasted 3 years they followed some students from the 5th grade up to their 7th with emphasis on problem solving techniques relevant to area. Three years later we follow two of these students who currently attend the 10th grade (15 years old) during their effort to cope with a non-standard task concerning problem solving activity relevant to elementary number theory concepts. The case is interesting since it displays executive control skills related to the way the students proceed when they have to work on a new domain and to the handling and establishment of a 'model' that could lead to the generalization. This is why we try to explore in this paper the interplay of the students among different approaches during their problem solving path towards generalization and at the same time to refer to the actions of the students concerning decision making and executive control. In the next section we present the task and describe the students' background. After that in the next two sections we present the problem solving approaches followed by our students (Katerina for the first, Nikos for the second). These are followed by a discussion section trying to shed light on these two axes (i.e., the interplay and the control issues) and finally the conclusions.

DESCRIPTION OF THE STUDY AND STUDENTS' BACKGROUND

Katerina and Nikos were 10^{th} graders and they had participated in an earlier study conducted by Mamona-Downs and Papadopoulos (2006) aiming to explore and enhance the students' comprehension of the concept of area with an emphasis on problem solving techniques for the estimation of the area of irregular shapes. Their participation in this resulted in the creation of a "tool-bag" of available techniques as well as in an accumulated experience on the usage of these techniques. The conceptual framework now mainly lies in number theory. However in the official curricula (for 10^{th} graders in Greece) the only reference to number theory concepts is a tiny one commenting the divisibility rules for the numbers 2, 3, 5, 9, 10.

This is the problem we posed to the students:

> Which of the rectangles below could be covered completely using an integral number of tiles each of dimensions 5cm by 7cm but without breaking any tile?
>
> Rectangle A: dimensions 30cm by 42cm
>
> Rectangle B: dimensions 30cm by 40cm
>
> Rectangle C: dimensions 23cm by 35cm
>
> Rectangle D: dimensions 26cm by 35cm.
>
> For each rectangle that could be covered according to the above condition show how the tiles would be placed inside the rectangle.
>
> Now, we want to cover a rectangle with an integer number of (rectangular) tiles. Each tile is of dimensions 5cm by 7cm. What could be the possible dimensions of the rectangle?

The mathematical problem is: define a set of necessary and sufficient conditions on a, b so that there exists a rectangle of dimensions a by b, that can be covered completely with tiles of dimensions 5 by 7. Look at the side of length a: if there are s tiles that touch it with the side of length 5 and k tiles that touch it with the side of length 7, then $a= 5s+7k$. The same reasoning applied to b gives $b=5s'+7k'$, where s, k, s', k', are non negative integers. Now if c denotes the total number of tiles used then the area ab of the rectangle should be 35c. Therefore 35 divides ab. Thus, there are three cases: i) 35 divides a, ii) 35 divides b, or iii) non of the previous, but since 35 divides ab, 7 must divides a and 5 divides b (or vice versa). Consequently, a and b should satisfy one of the following necessary conditions: i) $a = 35 m$, $b=5s'+7k'$, ii) $b=35n$, $a= 5s+7k$ ii) $a=7q$, $b=5t$ (or vice versa). It easy then to be shown, that these conditions are also sufficient. Thus, even though the context of the task seems to be geometrical with its relevance to area, however a crucial aspect in solving the task is the usage of a Diophantine linear equation $ax+by=c$ where the unknowns x and y are allowed to take only natural numbers as solutions. The task consists of two parts. In the first part four rectangles have been carefully selected to help the solver when finishing the first part to be able to reach the generalization asked in the second part.

The problem solving session lasted one hour, without any intervention from the researchers, and the students were asked to vocalize their thoughts while performing the task (for thinking aloud protocol and protocol analysis, see Schoenfeld, 1985). Protocol analysis gathered in non-intervention problem-solving session is considered especially appropriate for documenting the presence or absence of executive control decisions in problem solving and demonstrating the consequences of those executive decisions (Schoenfeld, 1985). The students' effort was tape-recorded, transcribed, and translated from Greek into English for the purpose of the paper.

THE FIRST PROBLEM SOLVING APPROACH - KATERINA

Katerina's first criterion for deciding whether the four rectangles can be covered completely by the tile was based on whether the dimensions of the four rectangles were multiples of the dimensions of the tile. This is why her answer was positive only for the rectangle A (since 30=5*6 and 42=7*6) and negative for the remaining three ones. She used the quotient of their areas (E1/E2, E1 the area of rectangle A and E2 the area of the tile) as a way to determine the number of the tiles required for the covering and not as a criterion to decide whether the tiling is possible). She tried then (according to the task) to show how the tiles will be placed inside the rectangle. The visual aspect of this action made the student to realize her mistake and to re-examine the four rectangles:

K.1.23. The tiles could be placed in any orientation in the interior of the big rectangle.

K.1.24. It is not necessary to be placed all of them in a similar orientation.

After that she verified that the rectangle A could be covered according to the task's statement. For the rectangle B she worked with an interplay between an arithmetical and geometrical-visual approach and she realized that the case of tiles with different orientation could mean that she could work with an 'equation' since she was not able to proceed geometrically. Now, it is the first time a linear combination is involved:

K.1.37. It could be ….. 5x+7y=30

K.1.38. It must be a rectangle with length of 30cm and this has to be expressed with tiles of length 5cm and 7cm.

She was not able to express her thought using proper mathematical terms. Her intention was to say that this equation did not have integer solutions (the case for an unknown to be equal with zero is excluded). So she decided to use terms such as 'round numbers' to show that it is needed for x and y to be integer numbers:

K.1.42. However this case is not possible… (*the above mentioned equation)*

K.1.43. We could not expect to have 'round' numbers for x and y.

For the rectangle C she decided to rely on the question whether the length of the side of the rectangle could be written as a linear combination of the dimensions of the tile. The lack of relevant knowledge on this domain provoked a certain technique for overcoming this difficulty. She worked with successive multiples of 7 plus the remainder (expressed in multiples of 5). She followed the same line of thought for the rectangle D. The criterion of the linear combination was already established and by the technique of the successive multiples she founded that:

K.1.67. For the side of 26cm it is necessary to have 3 tiles of length 7cm and 1 tile of 5cm.

Immediately she turned to the visualization in order to verify that indeed this can be done, working independently on each dimension of the rectangle D (Fig. 1, left).

For the second part of the task she started with two steps that according to her opinion could help her:

K.1.74. I will use drawings because it seems to me easier in that way

K.1.76. How could I use the findings of the first part of the task?

She rejects the condition of E1 being an integer multiple of E2 as the unique criterion since:

K.1.87. …it might be necessary for a tile (or some tiles) to be split.

Her model for finding the possible dimensions of any rectangle that could be covered by tiling using an area unit (tile) with dimensions 5 by 7 includes two cases exploiting her previous findings of the first part of the task.

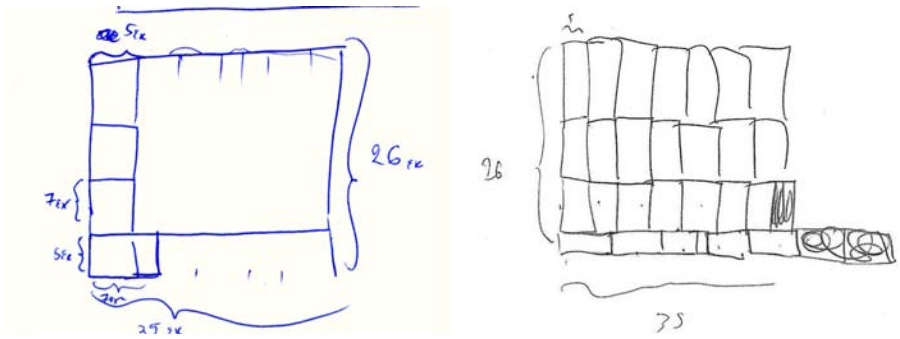

Fig.1 Katerina's (left) and Nikos's (right) visual approach on rectangle D

So, in the first case:

K.1.92. If all the tiles are oriented uniformly then the asked dimensions of the rectangle could be multiples of 5 or 7.

K.1.93. I will make a draw

K.1.94. It is a shape whose length is multiple of 7 and its width multiple of 5.

The second case resulted mainly as a consequence of the rectangle D and two conditions must be satisfied: one side must be multiple of the Least Common Multiple of the dimensions of the tile and the second dimension linear combination of them.

K.1.101. Length must be common multiple of 7 and 5 whereas width must be sum of tiles that are oriented some of them horizontally and some vertically.

She tried then to refine her model asking for a rule that governs the common multiples of 5 and 7 (i.e., of 35). For the number 5 she knew the divisibility rule (the last digit must be 0 or 5). However she could not give any rule for the 7 or the 35. Finally she concluded with a recapitulation of her model trying to describe in a more formal way the second case of the model:

K.1.110. The rectangle in the second case should have one of its dimensions common multiple of both 5 and 7 and the other one sum of multiples of 5 and 7 at the same time.

THE SECOND PROBLEM SOLVING APPROACH - NIKOS

Nikos's first step was to interpret the statement of the problem in terms of conditions for the correct tiling: a) there is a rectangular region that has to be covered and b) the tile is a structural element of the task:

N.1.5. It means that each rectangle must be covered and for the measurement I must use an integer number of tiles

N.1.6. So we could consider this rectangle of 5 by 7 as a measurement unit

In his work and for each one of the four rectangles we can distinguish a concrete line of thought. For the rectangle A, his criterion was (as in Katerina's case) the proportionality of the sides, i.e. whether the dimensions of the rectangle were multiples of the dimensions of the tile. We have to mention here that his way of reading the task was non-linear in the sense that he did not follow the instructions of the task in the given order. Thus, he did not initially give answers for all the rectangles but after deciding for each rectangle, he proceeded to the specification of the way the tiles could be placed in the rectangle. In case there was not proportionality among the lengths of the sides of the rectangle and the tile -as it happened in the rectangle B- he used the criterion of E1/E2 as a way to ensure a negative answer. This quotient was not an integer number and this meant that there could not be coverage according to the task's statement. As he explained:

N.1.20. Because the ratio of their areas is not an integer

Now, in the rectangle C, the E1/E2 was an integer but the dimensions were not proportional. It is interesting the fact that his decision about E1/E2 is justified by the fact that E2(=35cm^2) is a factor of E1(=23*35), a relationship often overlooked even by pre-service elementary school teachers (Zazkis & Campbell, 1996). In their study and in an analogous quotient, teachers first calculated the product and then divided. At that point, Nikos asked for the linear combination that satisfies one of the dimensions since the second is multiple of 5:

N.1.24. When the area is 23 by 35, then obviously this product is divided by 35 which is the area of the unit (*tile*)

N.1.27. The point is **the way** the tiles must be placed

N.1.29. We could have 3*7+2, 2*7+9

N.1.34. 5+5+5+8, 4*5+3,….

N.1.35. For the 23 cm I can't make any **combination** of 5s and 7s.

In the rectangle D, he applied directly the rule of the linear combination that could satisfy the side of 26cm since the other one (35cm) was multiple of 5 (Fig.1, right). Trying to describe how the tiling will take place he worked initially independently on each side. However the way the tiles will be placed in one dimension affects the way the tiles will be placed in the second. This made him to turn towards a consideration of both dimensions at the same time. Despite this method could be considered adequate for him to give an answer for each rectangle, he preferred to re-check all the given rectangles, to verify his answers before making his final decision.

For the second part of the task he started with an impressive conjecture:

N.1.83. Obviously, if we want to cover a rectangle with this specific unit of dimensions 5 by 7, then the rectangle's sides must be the sum of multiples of 5 and 7 at the same time.

N.1.84. The case of 0*5 and 0*7 must be included in this.

However he still considers the two dimensions separately. Trying to figure out what would be the general case for the asked dimensions of the rectangle he created some arithmetical examples, fulfilling the need for linear combination for each dimension, without considering the fact that there is an interrelationship among the two dimensions since the area of the rectangle must be a multiple of 35:

N.1.102. We could say that a=5x+7y (where 'a' is one of the rectangle's dimensions)

N.1.103. and similarly b=5z+7w

N.1.104. The product of these dimensions a and b will be the area

N.1.105. I can choose for a and b any sum of multiples. For example, a=5+14=19, b=15+28=43. So, the area is 19*43

N.1.106. However in that case I have for the area a number that is not divided by 35.

N.1.107. So, 35 must divide the product a*b which is the area of the rectangle.

N.1.112. Thus, a=5x+7y, b=5z+7w and the quotient ab/35 must be an integer.

Trying to establish a model that would describe all the possible cases he was also influenced by the four rectangles of the first part of the task. He decided that his model would include two types of rectangles:

N.1.141. The first type concerns rectangles with one side multiple of 5 and the other multiple of 7. So, a=5x and b=7y, which is a=5x+0*7 and similarly b=0*5+7y.

N.1.142. Consequently the area of such a rectangle divided by 35 gives an integer number as quotient.

N.1.154. And it is in accordance with the general form I conjectured earlier

For the second type he decided that:

N.1.159. One of the rectangle's side will be a sum of multiples of 5 and 7 at the same time

N.1.160. whereas the second side will be a multiple of 35

N.1.171. that is a=5x+7y and b=35z

N.1.172. I think that these latter conditions form the most general form for the dimensions of any rectangle able to be covered with rectangular tiles 5 by 7.

After that, Nikos applied this most general form for each of the four rectangles examined in the first part to check the validity of this form. Furthermore he made clear that the first type of rectangles could be incorporated in the second:

N.1.188. …to incorporate the first type which essentially is a special case in the second type which is more general..

Finally Nikos proceeded to a refinement of his model determining the circumstances that do not allow a rectangle to be covered according to the task giving a certain counterexample:

N.1.213. The second side must be always multiple of 35 and it can be constructed using either 5s or 7s.

N.1.218. This is the only solution because 35 is the Least Common Multiple of 5 and 7

N.1.219. This means that it is not possible to have a rectangle for which both its dimensions are linear combinations of 5s and 7s.

N.1.220. When I say that a is a linear combination of 5s and 7s, I mean that a=5x+7y but not a multiple of 5 or 7.

DISCUSSION

In relevance to our research questions we could make some comments on our fieldwork.

1. Interplay among differing modes of thinking

During their attempts to solve the problem the students worked in tandem with two pairs of modes. The first pair included the arithmetical mode and visualization. Both students started arithmetically even though the context of the task was relevant to area that is geometrical. Katerina from the very beginning used the visual aspect as a tool. She started arithmetically but when she was unable to proceed with numbers she preferred to make drawings that would help her (K.1.74). In the same spirit some times she moved from the visual context to algebra. At some point she clarified that the tiles could be posed not necessarily with the same orientation. However she was not able to proceed geometrically and she preferred to turn to algebra asking for an equation (K.1.37). Nikos did not choose to work with this pair of modes. He mainly worked arithmetically and he turned to the visual aspect only to show the way the tiles could be placed in the interior of the four rectangles in the first part of the task. The second pair of modes has to do with the way students dealt with the dimensions of each rectangle. Working with the first mode dimensions were considered by the students separately as two unconnected objects (arithmetical mode). Thus, they made calculations (they summed, multiplied, divided) to determine the way the tiles should be placed in one dimension. In the second mode the dimensions were interrelated (geometrical mode, relevant to area). The fact is that the way the tiles will be placed in the first dimension influences the way the tiles will be placed in the second dimension. Working independently in two dimensions does not guarantee that the total area of the rectangle will be integer multiple of 35 which is the tile's area. Both students made successive movements between these two modes. Their initial approach was to work separately for each dimension and only then they made the connection about the interrelation of the two dimensions. For example in Nikos's work (N.1.102-N.1.112) it is clear that his working on the two dimensions separately resulted in a rectangle that could not be covered with integer number of tiles since its area was not multiple of 35.

As a result of this interplay emerges -for Nikos in particular- the issue of putting forward a set of conditions (N.1.112) that are evidently realized as being *necessary* and later an equivalent set of conditions (N.1.172) that are seen as *sufficient* (because the covering of the relevant rectangles can be explicitly constructed).

2. Executive control and decision making issues

The students realized many actions that indicate interesting executive control and decision making skills. Katerina rejected her initial approach which was based only on the criterion of proportionality among the rectangle's and the tile's dimensions. This was because her turn to visualization made her to realize that it was not necessary for the tiles to be placed in a uniform orientation. This turn seemed to be in practice an important act of control. The task's statement did not give any direction concerning the way the tiles could be placed inside the rectangle. It was up to her to interpret correctly the statement. Later when she tried to solve the Diophantine equation she applied the technique of the successive multiples. According to this technique if one has to solve the equation ax+by=c starts with positive multiples of a and then examines whether c minus ax is multiple of b or vice versa (i.e., one starts with multiples of b). This is an act of control since the solving of the equation was dealing with the task's limitation to use an integer number of tiles without breaking any of them. When she decided to deal with the second part of the task her first thought was to use her previous results (K.1.76). Moreover, an important act of control was the 'model' she proposed for estimating the possible dimensions of any rectangle that could be covered with an integer number of tiles according to the statement of the task (K.1.92, K.1.110). She exploited her previous findings (the four rectangles of the first part), and progressively she established this 'model' checking step by step its accordance with these rectangles as also with examples generated by herself. The choice of examples is especially important since not every example facilitates a successful generalization. Nikos also made an analogous proposition of a 'model'. He was also based on the four rectangles of the first part of the task. The steps followed by his line of thought reveal presence of control: First look if there is proportionality among the dimensions. See also whether E1/E2 is not an integer. This means that your answer has to be negative. It is not necessary always to make the long division E1/E2. Instead, see whether E2 is factor of the E1(N.1.24). Now if sides are not proportional and E1/E2 is an integer, then construct the Diophantine equation and apply a strategy to find integer solutions. He also used to check always the consistency of his generalization model against particular examples and this is important. The continuous checking of their steps that both students showed is especially significant as an act of control since students checking is not usually part of the algebraic thinking of the students when they make generalizations (Lee and Wheeler, 1987). A capable problem solver recognizes a correct approach and insists on it. This evaluation of a specific approach could also be considered as an act of control. Nikos recognized the applicability of the linear combination and he used it to check the plausibility of his answers always according to task conditions (N.1.154). This often turn to the tasks' statement was a common pattern for both students. However, perhaps the most important act of control of both students was their effort to refine their model regardless of whether they succeeded. Katerina tried without success to achieve a condition for the second side to be common multiple of 5 and 7. Nikos however did manage to refine his 'model' determining whether it is impossible for a rectangle to be covered according to the task's requirements (N.1. 219). Such an asking for a counterexample actually is an important act of control.

CONCLUSIONS

According to Douady and Parzysz (1998) when a problem allows the solver to move between different modes during the problem solving process then an interplay between these different modes is caused. They claim that the effort of the solver to reach the solution results to the relations of these modes as well as to the usage of some tools that belong to each of them. Additionally "...this interaction provides new questions, conjectures, solving strategies, by appealing to tools or techniques whose relevance was not predictable under the initial formulation..." (p. 176). Both of our students were able to apply this interplay among two pairs of modes. In the first pair (arithmetical-visual) this interplay was used as a way that allowed overcoming difficulties about how to proceed or for verifying or checking the validity of an argument. In the second pair of modes the one mode (arithmetical, working on one dimension) was indicative of a surface understanding of the structural elements of the task. However it seemed that finally the students did show a deeper understanding of these elements through the other mode considering both dimensions at the same time (geometric, interrelated dimensions).

'Executive control' and 'decision making' constitute in general the issue of control in problem solving. Executive control is concerned with the solver's evaluation of the status of his/her current working vis-à-vis the solver's aims (Schoenfeld, 1985). In general, this requires mature deliberation in projecting the potential of the present line of thought, married with an anticipation how this might fit in with the system suggested from the task. In our study and despite their age, these 15-years old students showed considerable control skills in relation to the task's requirements on the one hand and the specification of the 'model' they proposed for solving the task on the other. The existed experience enabled students becoming capable to make generalizations.

Concluding we could refer to some final remarks that emphasize the significance of our results. It is common thesis that the task design is a crucial parameter for teaching and learning algebra at every level. So, in reference to our work, we could claim that the setting of modelling problem solving situations into number theory tasks allows students to:

5. transfer knowledge from one domain to another during their successful interplay among different modes of thinking (algebraic thinking and geometrical one).

6. construct and propose a 'model' that possibly describes the situation and facilitates the generalization

7. generate examples that check the consistency of their model, and

8. generate counterexamples that result to the refinement of the proposed 'model'.

Obviously it would be an exaggeration for these conclusions to be generalized since we dealt with two students and this study could be better considered as a case study. However these finding were encouraging enough to call for a design of a future research on these aspects of problem solving.

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ALLEGORIES IN THE TEACHING AND LEARNING OF MATHEMATICS

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This paper explores how the concept allegory from literature theory can be used in the teaching and learning of mathematics. A cognitive allegory theory is developed in analogy with the metaphor theory of Lakoff and Johnson (1980). The theory differs from the traditional view. For instance an allegory is also a cognitive mapping and not only a narrative. The paper draws upon data from a study of how teacher training students learn the concept of linear congruence equations. The students are given word problems which were translated to congruence equations and later used to solve other word problems.

INTRODUCTION

Researchers like Lakoff, Núñez, Sfard and Presmeg have elaborated the role of metaphors in mathematics and mathematics education, see for instance Lakoff and Núñez (2000), Sfard (1994) and Presmeg (1997). Allegory is another concept from literature theory which so far has been sparsely used in mathematics education. In this paper we suggest that the concept of allegory can be applied to this field. Our contribution is to develop allegory as a part of mathematics education theory, in a way similar to how metaphors have been used in the tradition initiated by Lakoff and Johnson (1980).

METAPHORS AND ALLEGORIES

Traditionally a metaphor is a figure of speech in which a phrase denoting one kind of object or idea is used in place of another. An example is "You are straight on target with your reply." In this view of metaphors "straight on target" is a figure which means something else. The phrase can be translated to literal speech, for instance "precise and relevant". In cognitive metaphor theory metaphors are not as in the old traditional view, seen as isolated phrases, but as systems structuring concepts and thought. Such systems map one domain into another such that the target domain inherits structure from the source domain. An example is "argument is war", Lakoff and Johnson (1980, p. 4).

$$\boxed{\text{Warfare}} \longrightarrow \boxed{\text{Argument or discussion}}$$

Target is a concept from warfare. If an argument is compared to an arrow or a bullet, we can characterize the argument by describing how the arrow aims at the target. But, the metaphorical mapping can also express lots of other things. An example is “The teacher went into a defensive position when faced with critique”. ‘Defensive’ is also part of military jargon, just like ‘targets’ is. In the tradition initiated by Lakoff and Johnson it is stressed that metaphors create or modify abstract concepts. The metaphor “argument is war”, is modifying or giving a special interpretation of what argument is. In other cases metaphors create a complete new concept.

Allegories are similar to metaphors, but have the structure of narratives and are usually more extensive. The New Encyclopædia Britannica has this definition:

> ...allegories are forms of imaginative literature or spoken utterance constructed in such a way that their readers or listeners are encouraged to look for meanings hidden beneath the literal surface of the fiction. A story is told or perhaps enacted whose details when interpreted – are found to correspond to the details of some other system of relation (its hidden, allegorical sense) (Fadiman, 1986, p.110)

Like metaphors, an allegory maps one domain onto another one, but the source domain is a narrative. Different parts of the source narrative are mapped into different parts of the target domain. An example from the Bible is Galatians 4:24, in which the word ‘allegory’ appears in the King James Version of the Bible. Two covenants are compared to the first two sons of Abraham by a freewoman and a bondwoman. An allegory maps objects and persons of a narrative to a more abstract domain. Each woman is mapped to a covenant, and the story told by the Apostle Paul gives flesh and meaning to the rather abstract concepts of a new and an old covenant. Both this allegory of Paul and the parables of Jesus have clear didactical purposes. They are designed by a teacher. These kinds of allegories are the focus of this paper, but of course mathematical ideas and conceptions are the goal, not spiritual ones. The word ‘conception’ is used to avoid non-cognitive interpretations of the alternative word ‘concept’, see Sfard (1991, p. 3) and Rinvold (2007, p. 4). A conception is a cognitive network in which several allegories and metaphors can be nodes. It isn’t uncommon to think that concepts are primarily given by formal definitions. Such definitions are just an aspect of conceptions and not at all a complete description.

We restrict our attention to allegories which include a timeline. This means that the narratives move in time. All the parables of Jesus are like that and so are most narratives. In mathematics education many text problems have the form of narratives. Such problems will be called narrative text problems. In this paper ‘text problem’ will always mean ‘narrative text problem’. On the other hand, narrative is a wider concept than text problem. A narrative is neither necessarily a problem nor given by a text.

Not all narrative text problems are allegories. This is only the case if a narrative text problem is going to represent or create something else, which usually is more abstract. Consider the following text problem: “John was hiking in the mountains. The first day he walked 20 km and the next day 25 km. What is the total distance he walked these two days?” This problem isn’t likely to represent something outside itself. Most students will solve the problem, forget it, and go on to the next one. The following narrative is different:

> Peter has an urn containing balls. On each ball it's written a prime number. The urn may contain more than one ball with the same number. Peter asks Andrew to draw as many balls as he wants. Then Andrew is asked to find the product of the numbers on the drawn balls. When Andrew has told what the product is, Peter starts calculating. After a while he says: "I know which balls you have drawn". How is this possible? What would happen if composite numbers had been written on the balls?

With possible guidance from a teacher, this story can help the students to understand unique factorization in prime numbers and the role of such numbers. Drawing of a ball represents a factor. The information that Peter is able to tell which balls Andrew has drawn, corresponds to uniqueness of prime factorization. The fact that he isn't able to tell the order the balls were drawn, represents the commutative law. The narrative is used to create understanding of an abstract property of numbers. The problem isn't just a problem among many, but may have a lasting effect.

Even if the teacher tells a story intended to be an allegory, the learners don't always understand it in this way. Relating to a constructivist epistemology, it is the learners themselves who develop allegories. An allegory may be idiosyncratic and has elements of individual variation.

METHOD

This paper uses data from a study of how teacher training students learn the mathematical concept of linear congruence equations. The study was conducted in March 2008 by the authors on our own students. The data consist of participant observation of a teaching and learning session and three videotaped and partially transcribed interviews. One of the researchers interviews the teacher of the lesson, who at the same time is the other researcher. Both researchers together interview groups of either two or three students. As part of the sessions, students work together with a text problem. The researchers then ask questions helping the students to describe their reasoning process. The student interviews were conducted two days after the lesson, and the teacher interview the day after that. The transcriptions, descriptions and interpretations of the teacher interview were read by the interviewee, discussed with the researcher and then adjusted. Later, in the process of writing the paper, the teacher sometimes remembered thoughts and events from the lesson which can't directly be read from the data. Such thoughts aren't presented as data, but have without doubt influenced interpretations and directions of the paper.

CONGRUENCE CALCULUS

The lesson is based on several text problems given to the students. The first problems concern week days. Exercise 1 asked them to calculate the weekday of 31st March, given that 1st January was a Tuesday. The teacher gave comments and discussed solutions in between. The students themselves made tables resembling calendars.

0	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30				

After their work the teacher showed them the table above on a blackboard. Then he introduced the signs '≡' and 'mod' for congruent numbers. From 1st January to 31st January is 30 days. He pointed to the numerals 2 and 30 in the table and connected them with a red line. Then the teacher said that 2 and 30 are in the same column and wrote $30 \equiv 2 \pmod 7$. Mathematically this means that 30 and 2 have the same remainder upon division by seven. In other words, the difference between 30 and 2 is an integer multiple of 7. Practically, the meaning is that 30 days from now and 2 days from now differs with a number of integer weeks. The identity was followed by $29 \equiv 1 \pmod 7$ and $31 \equiv 3 \pmod 7$ since 2008 is a leap year and March has 31 days. Finally the teacher wrote

$$30 + 29 + 31 \equiv 2 + 1 + 3 \equiv 6 \pmod 7.$$

The move of six days forwards from a Tuesday gives a Monday, so that is the weekday of 31st March.

NARRATIVE TEXT PROBLEMS

After three other text problems having to do with calculation of week days, the students were given what we call the Duckburg problem:

> A ship arrives at the harbour of Duckburg today, which is a Monday. Then the ship arrives at the harbour every third day. Some days later the ship arrives at Duckburg harbour on a Wednesday (two weekdays later). How many arrivals later can this be?

According to our observations, all students made a table with the weekdays from Monday to Sunday in the first row. There was some variation in the content of the tables, but in some way all students marked the days when the ship arrived. They all discovered the first solution of the problem, and some even found a formula for the number of arrivals when the ship arrives on a Wednesday. The student work was followed up by the teacher in a plenary session. As support for the introduction of congruence equations, he made a protocol for the arrivals of the ship by writing the identities

$$3 \cdot 1 \equiv 3 \pmod 7,\ 3 \cdot 2 \equiv 6 \pmod 7,\ 3 \cdot 3 \equiv 2 \pmod 7,\ 3 \cdot 4 \equiv 5 \pmod 7,\ \ldots$$

He simultaneously said things like "three times four is in the same column as five", referring to the table. Then the teacher related the text problem to the mathematical formulation "which multiplies of three are in the same column as two when divided by seven". Finally the congruence equation $3x \equiv 2 \pmod 7$ was presented as a translation of the Duckburg text problem. The lesson continued with the demonstration of algebraic techniques for solving the equation. These techniques are part of the motivation for the translation, but we don't discuss the solving process in the paper.

The Duckburg narrative is built upon the culturally shared concepts of days, weeks and calendars and the well-known phenomenon of ships regularly arriving at harbour cities. The name Duckburg, which is the domicile of the Disney figure Donald Duck, is used to make it clear that we are talking about a fantasy world in which details can be changed. Duckburg is a name which is easy to remember and with positive associations for most students. Also, this cartoon city is placed close to the sea, Grøsfjeld (2007), so arrivals of ships are relevant.

Later in the lesson the students were given the running track problem:

> An athlete runs intervals of 300 m on a 400 meters running track. She starts at the starting line, runs 300 meters and stops. She continues this way. After a while she stops 100 meters after the starting line. How many 300 meter intervals has she run?

The students at first worked with the task themselves using a table. A drawing of the track was introduced afterwards by the teacher in the plenary. He used the drawing to simulate the intervals of the runner. This problem also corresponds to a linear congruence equation, but the situation is sufficiently different from the Duckburg problem to supplement it.

FROM NARRATIVE TO PROTOTYPE

Lakoff and Johnson (1980) claim that usually we place things and phenomena in categories by comparing with a typical or prototypical member. A prototypical bird has wings, is able to fly, lay eggs and has a beak. A picture of a blue jay is used by some dictionaries when defining birds. The blue jay is a candidate for a prototypical bird in countries where this bird is well known. We will use interview data to argue that the Duckburg problem has the potential to be a prototypical text problem for linear congruence equations. The argument is based on the way the Duckburg problem is used by the group of three students to solve the following text problem which also corresponds to a linear congruence equation:

> Oda is sick and has to take a tablet every fifth hour, both day and night, in order to get well. She takes the first tablet at five in the morning. A friend calls her when her watch has just passed one o'clock. Her watch is analogue, that is, has rotating hands. How many tablets has Oda taken? There are several correct answers.

The students work for about twenty minutes with the problem and are then interviewed. In the interview one of the students was passive and seemed to participate only to a restricted degree. The active ones were Kari and Lise. A reason why they used so much time is that the problem is structurally more different from the Duckburg problem than we intended. In particular, Lise mentioned several times in the interview that she was confused because the problem was unclear. In fact, one of the researchers had forgotten to specify that Oda had just taken a tablet when the friend called. However, the students demonstrated understanding of the problem and were able to solve it with the extra constraint when asked to. Some statements by Kari support the claim that the Duckburg problem and some of its structure were used in the solution process. One example appears when Kari and Lise had written the congruence equation $5x \equiv 8 \pmod{12}$ on their sheets. When asked why the right hand side is 8, Kari said:

> Kari: I remember when we worked on the problem with the ships which arrived at the harbour, we started with a Monday. Then we were going to find Wednesday, which was two days later, so we would have two there.

The student refers to the Duckburg problem which corresponds to the equation $3x \equiv 2 \pmod{7}$. The numeral 2 on the right hand side corresponds to 2 days later. In analogy, one o'clock is 8 hours later than five. We think this is the reason why the students wrote 8 on the right hand side of the equation $5x \equiv 8 \pmod{12}$. This is supported by another statement from the interview:

> Kari: Then we draw a table with 12 columns. We started with the hour she took the first tablet, which was at five o'clock. (...) Then we counted every fifth hour...

The students made the same type of table as in the Duckburg problem. Five was the first column in the tablet case, as Monday was the first in the Duckburg problem. They counted how many hours after the first tablet she takes the next and would have got the same result if the first one was taken at for instance two o'clock. Another argument is this mentioning of the running track case:

> Kari: Recognizing the running track task. Then 0 and 12 were the same. It was the starting line. Do we have to start with 0 then? But, now 0 is at 5 o'clock. If she starts at 5 there, then…

Without doubt, Kari now uses five o'clock as the zero point. The students however, didn't notice a minor difference between the questions in the problems. In the Duckburg problem the question is how many days *after the first* arrival the ship arrives at Wednesday. In the tablet case we asked how many tablets she has taken, *including the first* one. If Oda had taken the first tablet at hospital, and we had asked how many tablets Oda has taken at home, their equation had been correct. Then $5 \cdot 1 \equiv 5$ would have meant that she took the first tablet at home 5 hours after the one at hospital. The identity $5 \cdot 4 \equiv 20 \equiv 8$ would have meant that she took the fourth tablet at home 20 hours after the one at hospital. In some sense the wrong equation is more convincing than $5x \equiv 1 \pmod{12}$, which has $x \equiv 5$ as solution. In the latter case the students could just have put in the numbers 5, 1 and 12 given in the problem, without any understanding.

The students' use of the Duckburg problem and its structure is an argument that the Duckburg problem is on its way to becoming a prototype for a category of narratives. A more thorough study would have been necessary in order to claim with strength that some text problem has been established as a prototype. A possible weakness in our study is that only one student orally indicates this kind of reasoning. However, the students wrote the equation $5x \equiv 8 \pmod{12}$ collaboratively and Kari said that "we worked with the problem". This may indicate that at least Lise also shared her ideas.

ALLEGORIES AND GENERALIZATION

The transformation of a narrative text problem to a prototype for a class of such problems is an important step in giving a problem lasting value in mathematical thinking. This is one aspect of making the special case represent something general. In the Duckburg problem we can change the involved numbers without changing the structure of the narrative. Clearly, there is nothing special in "every third day" or "two weekdays later". The general is represented by the special case. The related "principle of generalization" is investigated in Rinvold (2007). To change the numbers of weekdays from seven to something else is also possible, but needs more imagination because weeks with seven days are so deeply established in our culture.

We think that the narrative of Duckburg has the potential of becoming an allegory for linear congruence equations with one unknown. When the narrative is turned into a prototype, each part of the narrative represents a part of a generalized narrative. For instance "arrives every third day" represents "a tablet every fifth hour" and "runs an interval of 300 meter" in the two other example problems. But, the parts of the Duckburg narrative also represent parts of a formal linear congruence equation. These representations can be made clearer with the help of mappings. In the latter case the source domain is the Duckburg problem and the target domain is the class of congruence equations.

$$\boxed{\text{Duckburg narrative}} \longrightarrow \boxed{\text{Congruence equations}}$$

"Every third day" is mapped onto $3x$, "two days later" onto 2 and the number of weekdays is mapped onto 7 in the equation $3x \equiv 2 \pmod 7$.

In the lesson the students were given some context free congruence equations and told how to translate these into Duckburg problems. When given the equation $2x \equiv 3 \pmod 8$, one of the groups introduced a new weekday and drew a table. They quickly realized that the problem had no solution. With eight columns, steps of two weekdays can't lead to the same place as a change of three weekdays. The students said that it was a cheating exercise since there was no solution. In the beginning of the interview the students were asked about their experience of the lesson.

Kari: When we used the practical situations as starting points, we could in the end see a congruence equation, and then the numbers gave meaning. We could know what 4x really represents. When I recalled the ships, it gave meaning.

This could refer to the equation $4x \equiv 1 \pmod 7$ which was one of the translation problems from the lesson. The formal congruence equation in the beginning seems to give little meaning to the students. The ships were part of the Duckburg problem and are used by the student to refer to that problem. We infer that translation of context free congruence equations into variants of the Duckburg problem was a main source of the meaning which emerged.

ALLEGORIES AND THE SOLVING OF TEXT PROBLEMS

When solving text problems allegories can be intermediate stages between the given narrative and a mathematical model.

Text problem → Allegory → Mathematical model

The idea of prototypes means that new text problems given to the students won't be directly mapped to a mathematical model, but first to a prototype like the Duckburg problem.

Narrative → Prototypical narrative → Mathematical model

A prototypical narrative in the learning of mathematics is a mathematized narrative. The given text problem or narrative also has to be mathematized to some degree in order to be mapped onto the prototype.

A crucial question is which qualities these mappings have for the students. Certainly, their versions of the mappings can differ from the intentions of the teacher. At best, the mappings reflect the mathematical structures effectively, but the mappings may also be based on superficial aspects of the text problems. Clement (1982) identified a syntactic and a semantic way of thinking when students tried to solve word problems for equations. The syntactic variant consists of a word by word translation of the text problem to algebraic language. Another kind of syntactic translation is based on possibly superficial similarities between the text problem and other text problems known to give a specific mathematical model. When working with the tablet problem, the student Kari made the following utterance:

Kari: We thought that it was 5x because it was every fifth hour she had to take the tablet and that was because the ship arrived every fifth day.

We see that the phrase "every fifth" appears in both problems. This may be interpreted as a sign that the student compared the appearance of words in the two problems. However, the ship in the Duckburg problem arrives every third day, not every fifth. If fact, some of the students, certainly including Kari, during the lesson also solved a variation of the Duckburg problem corresponding to the congruence equation $5x \equiv 3 \pmod{7}$. This at least indicates that she compared with the appropriate version. Another argument that the translation has a semantic flavour is that in the lesson Kari, together with a group of students, generalized the Duckburg problem. They investigated what happens when the interval between arrivals or the number of weekdays ahead were changed.

Part of our theoretical thinking is that allegories are one of the sources for semantic meaning. When one text problem has been transformed to an allegory, the comparison with other text problems will no longer be just syntactical. Clement's semantic way of thinking means a mapping from a narrative or text problem to a mathematized version of the problem, and then a mapping to the congruence equation.

Narrative → Mathematized narrative → Mathematical model

This is similar to the mappings of Parzysz (1999):

Real situation → Pseudo-contrete model → Mathematical model

In the case of the Duckburg problem, the emphasis on the table, the columns and the introduction of mathematical signs means that the teacher intended to support the development of a mathematical structuring of the narrative. In the lesson the teacher explicitly sets up a mapping from the pseudo-concrete model to the congruence equation. For instance "the numbers which are multiples of three" was translated to '3*x*'. The text problem is still a real situation for the student, but mathematical language has been introduced in order to change the students' interpretation of the situation, making the translation to formal mathematics precise and smooth.

The term "real situation" is not as clear as commonsense language may suggest. We interpret 'real' as "real for the student", as in RME, the Dutch approach to mathematics education (see van den Heuvel-Panhuizen, 2003). One point is that 'real' doesn't have to mean practical or related to everyday life. The student Kari used the phrase 'practical situation' referring to the Duckburg problem, but imagined situations such as weeks with eight days can also be 'real'. A situation isn't something objective, but an experienced or imagined phenomenon. A narrative may create a situation in the mind of the student, but the process of mathematization also has a role in shaping the situation for the student. The degree of mathematization and semantic interpretation decides the quality of the mappings.

QUESTIONS FOR RESEARCH

This paper introduces the idea of cognitive allegories in mathematics education and supports this by discussions based on one limited empirical study. Obviously there is a need for more studies to establish that the concept of allegories is a fruitful one for the use of narratives and text problems for conceptual learning in mathematics. It is necessary to have more thorough studies to establish that students transform introduced narratives into prototypes and allegories and how they do this. Other mathematical concepts and other potential allegories have to be studied. We also need to develop criteria for the design of such narratives. Another interesting task is to study how several allegories can be used for the same concept. We think that a single allegory usually isn't enough to develop a rich intuition. In our study the running track problem is a candidate for a complementing allegory, but only very limited evidence for this can be inferred from the data.

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ROLE OF AN ARTEFACT OF DYNAMIC ALGEBRA IN THE CONCEPTUALISATION OF THE ALGEBRAIC EQUALITY

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In this contribution, we explore the impact of Alnuset, an artefact of dynamic algebra, on the conceptualisation of algebraic equality. Many research works report about obstacles to conceptualise this notion due to interference of the previous arithmetic knowledge. New meanings need to be assigned to the equal sign and to letters used in algebraic expressions. Based on the hypothesis that Alnuset can be effectively used to mediate the conceptual development necessary to master the algebraic equality notion, two experiments have been designed and implemented in Italy and in France. They are reported in the second part of this paper.

***Keywords**: Alnuset, semiotic mediation, conceptualisation of algebraic equality*

INTRODUCTION

The research reported in this paper is carried out in the framework of the ReMath project (http://remath.cti.gr) addressing the issue of using technologies in mathematics classes "*taking a 'learning through representing' approach and focusing on the didactical functionality of digital media*". The work is "*based on evidence from experience involving a cyclical process of a) developing six state-of-the-art dynamic digital artefacts* [DDA] *for representing mathematics […], b) developing scenarios for the use of these artefacts for educational added value, and c) carrying out empirical research involving cross-experimentation in realistic educational contexts*". This paper presents the research concerning Alnuset, one of the 6 DDA developed within the project. First, some theoretical considerations related to the notion of algebraic equality, at stake in this paper, are presented. Next, our research hypotheses are discussed and Alnuset is briefly presented. Finally, two experiments involving this artefact are described and the main results are discussed.

THE NOTION OF ALGEBRAIC EQUALITY

Important conceptual developments are needed to pass from numerical expressions and arithmetic propositions to literal expressions and elementary algebra propositions. As a matter of fact, in arithmetic only numbers and symbols of operations are used and the control of what expressions and propositions denote can be realized through some simple computations. In elementary algebra, instead, letters are used to denote numbers in indeterminate way and new conceptualisations are necessary to maintain an operative, semantic and structural control on what expressions and propositions denote (Drouhard 1995; Arzarello et al. 2002). The necessity of this conceptual development emerges clearly with the construction of the notion of algebraic equality. On the morphological plan, equality is a writing composed by two expressions or by an expression and a number connected by the "=" sign. On the semantic plan, equality denotes a truth value (true/false) related to the statement of a comparison. When the expression(s) composing the equality is (are) strictly numerical, it is easy verifying its truth value through some simple calculations (e.g., 2*3+2=8 is true while 2*3+2=9 is false). Experiences with numerical equality contribute to structure a sense of computational result for the "=" sign. This sense can be an obstacle in the conceptualisation of algebraic equality as relation between two terms, as highlighted by several researches (Kieran 1989, Filloy et al. 2000). When the expression(s) composing the equality is (are) literal the equality can present different senses because the value assumed by the letter can condition differently its truth value. In these cases the "=" sign should suggest to verify numerical conditions of the variable for which its two terms are equal. There are cases where the two terms could never be equal whatever the value of the letter is, as in 2(x+3)=4x-2(x-1). In other cases to interpret equality on the semantic plane, it is necessary to distinguish if it has to be considered as equation or as identity. The "=" sign assigns to the equality the sense of equation when its two members are equal only for specific values of the letter. For example, the equality 2x-5=x-1 is true only for x=4 and it is false for all other values. Instead, the "=" sign gives to the equality the sense of identity when its two members are equal whatever the numerical value of the letter is, as in 2x+1=x+(x+1). In order to master algebraic equality, a conceptual development of notions of equation, identity, truth value, truth set and equivalent equation is necessary. Moreover, to express the way in which a letter can condition the truth value of an equality, it is necessary to develop a capability to use universal and existential quantifiers, even though in implicit way.

RESEARCH HYPOTHESIS

Traditionally, conceptual construction of algebraic equality is pursued through solving equations using techniques of symbolic manipulation. Empirical evidence and results of research have highlighted that in many cases this approach does not favour a construction of an appropriate sense either for the notion of algebraic equality or for that of solution of equation. In more recent years, a functional approach to algebra has been introduced within the didactical practice allowing to articulate algebraic and graphical registers of representations (Duval 1993). Even in this approach difficulties emerge. These regard the interpretation of a graph. For example, for the solution of equations of the type ax+b=cx+d, the intersection of the two lines in the graph has to be interpreted as indicator of the fact that the equation has a solution. Moreover this solution has to be read on the x-axis in correspondence of the intersection point of the lines. As Yerushalmy and Chazan (2002) observed, this approach is not devoid of obstacles: students can interpret the graph as comparing two functions (y=ax+b and y=cx+d) or as a solution set of a system of two equations in two unknowns, instead of an equation in a single variable. Our research hypothesis is that Alnuset, an artefact of dynamic algebra recently developed, can be effectively used to mediate conceptual development necessary to master the notion of algebraic equality. Further in the paper we discuss this hypothesis referring to some results of two experimentations.

SHORT DESCRIPTION OF ALNUSET

Alnuset is constituted of three components, Algebraic Line, Symbolic Manipulator and Functions, strictly integrated with each other. They enable quantitative, symbolic and functional techniques to operate with algebraic expressions and propositions.

The main characteristic of Algebraic Line component is the representation of an algebraic variable as a mobile point on the numerical line, which can be dragged with the mouse along the line. This feature has transformed the number line into an algebraic line where it is possible to operate with algebraic expressions and propositions through techniques of quantitative and dynamic nature. These techniques focus on numerical quantities indicated by an expression when its variable is dragged along the line or on numerical quantities that make true a proposition. These techniques make a dynamic algebra possible. The main characteristic of Symbolic Manipulator component is the possibility to transform algebraic expressions and propositions through a set of particular commands. These commands correspond to basic properties of operations, properties of equality and inequality, logic operations among propositions, operations among sets. Another characteristic is the possibility to create a new transformation rule once it has been proved. These characteristics support the development of skills regarding the algebraic transformation and they contribute to assign a meaning of proof to it. The main characteristic of Functions component is the possibility to operatively integrate Algebraic Line with Cartesian Plane, where graphs of expressions can be represented automatically. Moreover, dragging the point corresponding to the variable on the algebraic line makes the expression containing the variable move accordingly on the line. On Cartesian Plane, the point defined by the couple of values of the variable and of the expression moves on the graph. These characteristics support two integrated conceptions about the notion of function: a dynamic conception developed on Algebraic Line and a static one associated to the graph on Cartesian Plane. For a more detailed description of Alnuset, we refer to the work of Chiappini and Pedemonte presented in this edition of CERME within the working group 7.

EXPERIMENTATIONS

As we mentioned above, the development of DDAs was followed by a design of learning scenarios involving these tools and the implementation of these scenarios "*in realistic contexts*". ReMath partners decided that each DDA would be experimented not only by the designer team, but also by an other team that did not participate to the DDA development. Such "cross-experimentation" of the DDA was intended to highlight the impact of theoretical frameworks and of contextual issues on the design of both DDA and learning scenarios. Indeed, each team was free to set up educational goals taking account of institutional constraints and to choose theoretical approaches to frame the scenario design process. Thus, the experiments involving a given DDA were not meant to be compared, but rather to validate design choices related both to the DDA and the learning scenarios.

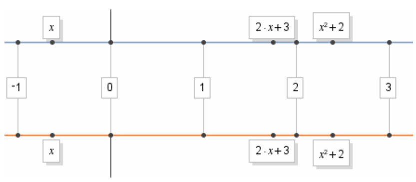
Italian experimentation

The experimentation activity reported below, lasting 1h40, has involved a class of 15-16 year-old students (Grade 10) attending a Classic Lyceum. The students worked in pairs using Alnuset. Previously, they had carried out 6 activities with Alnuset centred on notions concerning algebraic expressions. The whole teaching experiment lasted about 20 hours. The activity considered in this paper is centred on solving a 2^{nd} degree equation. In the previous school year, students had learnt to solve 1^{st} degree equations through symbolic manipulation. In this activity notions of conditioned equality, solution of an equation, equivalent equations, truth value of an equality and truth set of an equation are addressed. The didactical goal is the conceptual development of these notions while the research goal is the study of Alnuset mediation in this conceptual development. The activity comprises several tasks. The first task aims at allowing students to explicit their own conception of the algebraic equality notion.

Task: *Consider the following two polynomials: x^2+2; $2x+3$. Explain what it means putting the equal sign between them, or, in other words, how you interpret the following writing $x^2+2=2x+3$.*

Many students attribute to the "=" sign the meaning of computation result. Nevertheless they were already faced with 1^{st} degree equations. A typical students' answer is: "*To put the equal sign between two polynomial expressions means that these expressions have the same result*". For many students inserting the equal sign between two expressions suggests the idea that the computation result of the two terms has to be equal when a value is assigned to the letter.

In the following task students were asked to represent the two expressions on the algebraic line of Alnuset to verify their answers. Dragging the mobile point x along the line (and observing that the points corresponding to the two expressions move accordingly), all students noted that there are only two values of x for which the points of the two expressions are close to each other, almost coincident. Through this exploration students experienced that equality of two expressions is conditioned by numerical values of the variable, which is crucial to develop the conditioned equality notion. In previous activities with Alnuset, students experienced that every point of the algebraic line is associated to a post-it that contains all expressions constructed by the user denoting that point. In order to verify equality of two expressions, the students tried to find values of x for which the two expressions belong to the same post-it. Since these irrational values had to be constructed on the line, the students could not verify this directly: "*we don't*



understand what is the number…it will be 2 point something…even if we use zoom in we don't understand …". The technique mediated by Alnuset to find these irrational numbers requires transforming the equation into its canonical form (x^2-2x-1=0), representing its associated polynomial on the line and using a specific command to find roots of this polynomial. Our hypothesis was that this technique could favour a conceptual development of notions of equivalent equations and of truth value of an equation. The transformation was realized in the Symbolic Manipulator and was guided by the following task:

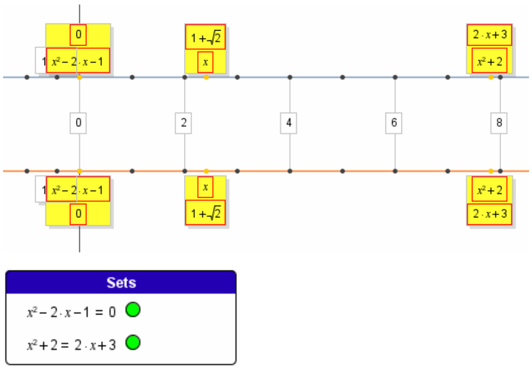
Task: *Select the equation and use the rule A=B ⇔ A-B=0 to transform it. Translate the result produced by this rule into natural language.*

This task focuses on the rule A=B ⇔ A-B=0 of the manipulator through which it is possible to transform the equality preserving the equivalence. We report two students' answers: "*If two terms are equal, then their difference is zero*"; "*it means that if two expressions are equal, subtracting them the result will be zero*". The conditional form of these sentences reflects a construction of an idea for the notion of conditioned equality used to justify the result produced by the rule. This does not mean that the students have understood the equivalence between the two equations in terms of preservation of the same truth set. Such understanding is the aim of the whole activity and its achievement requires several conceptual developments. First of all, students have to understand that the values of x for which x^2+2 is equal to 2x+3 are the same for which x^2-2x-1 is equal to 0.

The following task was assigned to favour exploring such quantitative relations:

Task: *Make a hypothesis about the relationship among the three polynomials x^2+2; 2x+3; x^2-2x-1 imagining what you could observe if you represented them on the algebraic line and if you dragged x. Use algebraic line to verify your hypothesis.*

A posteriori, we realized that the formulation of this task was misleading since it oriented the students to search for a relation among the three polynomials rather then between couples of terms of the two equations. Some students dragged the variable to explore if there were values of x for which the three polynomials could denote the same value on the line. They verified that such a value does not exist. Even if this exploration was not expected, it proved an important reference to overcome the following misconception, quite common in the students, concerning the equivalence of equations: two equations are equivalent if all their terms are equal for some values of the variable. A new formulation of the task by the experimenters allowed students to focus on couples of terms of the two equations. Exploiting the drag of the variable x they understood that, in order to find values of x for which x^2+2 is equal to 2x+3, it is sufficient to find values of x for which x^2-2x-1 is equal to 0. Subsequently they used the command E=0 to find the irrational roots of the polynomial x^2-2x-1 and to automatically represent them on the line (the student drags x to approximate the polynomial to 0 and the system automatically produces the exact value of the root). Through this experience an idea of equivalent equation begin to emerge. This idea will be consolidated through the exploitation of a new dynamic feedback offered by the system. We note that in the algebraic line environment expressions are represented on the line while equalities are represented in a specific window named "sets" and they are associated to a marker (a little dot) whose colour is managed automatically by the system. The marker is green if, for the current value of the variable on the line, the equality is true and, conversely, it is red if the equality is false. Dragging the variable allowed students to explore the truth of equalities and to construct a meaning for this notion, as shown in the following dialogue.



Student: If I drag x on $1+\sqrt{2}$ and on $1-\sqrt{2}$, the expressions of the first equation belong to the same post-it, namely x^2-2x-1 and 0 are coincident for these values of x.

For the same values of x even x^2+2 and 2x+3 belong to a same post-it.

Student 1: When x is $1-\sqrt{2}$ the two expressions are equal and these [dots] are green. So, since the solution of this equation is $1-\sqrt{2}$ then also for the other equation is the same.

Student 2: and for the other value [$1+\sqrt{2}$] it is true the same

Student 1: yes, for these values the two equations are true

To support the conceptual development necessary to master the notion of truth set of an equation, two other operative and representative possibilities of the algebraic line were exploited: a graphic editor to construct the truth set of an equality and a new feedback of the system to validate it. The graphic editor allows to operate on the line to define a numerical set that the system automatically translates into the formal set language associating it to a coloured marker. We note that the green/red colour of the marker means that the current variable value on the line is/is not an element of the set. As expected, students used this feedback to validate the defined numerical set as truth set of the equation, verifying the green colour accordance between equation marker and set marker during the drag of the variable on the line: "*for the values* $1+\sqrt{2}$ *and* $1-\sqrt{2}$ *the two equations* $x^2+2=2x+3$ *and* $x^2-2x-1=0$ *have the same truth set. In our opinion, the two expressions from one side and the other side of the = sign belong to the same post-it when x assumes the values of their solutions*".

French experimentation

Let us remind that the French team that experimented activities described in this section was not involved in the development of Alnuset. Therefore, a preliminary step before designing a learning scenario with Alnuset consisted in an analysis of the tool from the usability and acceptability point of view (Tricot et al. 2003). This analysis brought to light main functionalities supposed to enhance learning of functions and equations, notions at the core of the Grade 10 math curriculum: dynamic representation of the relationship between a variable and an expression involving this variable and possibility to articulate different registers of representation of algebraic expressions (Krotoff 2008). In addition, praxeological analysis (Chevallard 1992) of the above mentioned mathematical objects allowed identifying types of tasks and comparing techniques available in Alnuset with institutional techniques identified in the Grade 10 textbook. This analysis shows that while institutional techniques are based on algebraic transformations on algebraic expressions, Alnuset techniques rely on visual observations of expressions (their position on the algebraic line, colour feedback…), and (almost) no algebraic treatment is needed when applying these techniques (Krotoff 2008). Thus, Alnuset seemed to be an appropriate tool to help students develop conceptual understanding of notions of function and equation, without adding difficulties linked to algebraic treatment that many students do not master well enough.

Although the French experiment was designed independently from the Italian one presented above, both experiments shared some didactic goals, in particular conceptual understanding of notions related to the notion of equation: meaning of a letter as variable or as unknown and of the “=” sign, understanding of what a solution of an equation means. Therefore, below we present only activities and results related to these common concerns. Our research goal was both to investigate to what extent the new representation of algebraic expressions provided by Alnuset contributes to the conceptual understanding of the notions at stake, and to study instrumental geneses (Rabardel, 1995) in students when interacting with Alnuset.

The experiment took place in a Grade 10 class with 34 students (15-16 years old), during two sessions lasting 3 hours altogether, held in a computer lab where students worked in pairs on a computer. Their work was framed by worksheets describing tasks and asking questions the students had to answer. Written productions are one kind of gathered data. Moreover, a few student pairs’ verbal exchanges were audio recorded and this data provided us with the possibility to carry out case studies, namely as regards studying instrumental genesis in students. Results reported below draw mostly on these case studies.

The first task involving equations was finding solutions of $f(x)=4$, with $f(x)=x^2$, after having studied the function f with Alnuset. The task was intentionally quite simple: the students could either solve the equation algebraically and verify the result with Alnuset, or solve the equation with the tool by dragging x along the algebraic line and looking for values for which x^2 coincides with 4. Both strategies appeared to almost the same extent. However, students who used the exploration strategy to find solutions with Alnuset succeeded better than those who used the tool just to verify the results found by solving the equation algebraically, since these often provided only one, positive, solution. Alnuset turned out to be an efficient tool helping students to overcome their conception $x^2=k^2 \Leftrightarrow x=k$.

The next task, solving the equation $x^2=3x+4$, was proposed to prompt students to use Alnuset technique of dragging x on the line and searching for values for which the equality is true. Indeed, the students did not know yet algebraic techniques for solving such 2nd degree equation. Using the Alnuset technique requires to make sense of the "=" sign as meaning that the two expressions have the same value for some value of x, and thus also to distinguish between a letter standing for a variable and for an unknown. The students were first asked to determine whether 1, –1 and 2 are solutions of the equation. This question was intended to reveal students' conceptions of the notion of solution of an equation. Almost all students succeeded the activity. However, the following dialogue between two students reveals the student's S1 conception of a solution linked to the arithmetic sense of the "=" sign:

S1: You have to find 1. No, 3x+4 must be equal to 1, the solution.

S2: No, you have to put x on 1 and the… what do you call it [*pointing at 3x+4*]… Because x^2 should be equal to… the thing, equation and this isn't the case (Fig. 2a).

S1: But it's the result this [*pointing at 1*].

Indeed, it seems that S1 considers a solution of an equation to be the "result" or the value of the expressions: if 1 is a solution of $x^2=3x+4$, then ($x^2=$) $3x+4=1$. This conception emerged also when the students checked for -1. The student S2 grasped the targeted technique: *"On the other hand, -1 is the solution since f(-1) equals this equals this equals this*" (Fig. 2b), and explains it to S1: "*To find the solutions, you drag x until x^2 and 3x+4 overlap"*.

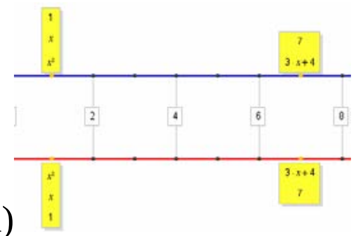
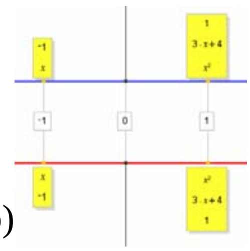
(a)  (b) 

Figure 2. (a) 1 is not a solution since x^2 and 3x+4 do not overlap when x is on 1; (b) –1 is the solution.

The students were then asked to find other solutions of the equation if there are any. This task was much more difficult for the students. Only half of the pairs succeeded it. The main obstacle was the fact that when x=4 (the other solution), the expressions x^2 and 3x+4 went out of the screen. The students did not spontaneously resort to using "tracking" functionality allowing to keep visualising the expressions taking bigger values, which the students had used previously. Teacher's intervention was necessary to remind the availability of this functionality, which helped the students to successfully finish the task. Such observations point to the issue of instrumental genesis in students, which can be a rather long-term process, especially in the case of innovative functionalities such as “tracking” or “E=0” command as we will see in the following example.

Next, the students were asked to find solutions of the equation $x^2=x+3$. This equation has irrational roots, therefore the technique based on dragging x and making the expressions overlap is not efficient anymore. The aim was to introduce the E=0 command allowing to find irrational roots of the expression x^2-x-3 and thus bring the idea of equivalent equations A=B and A-B=0. Most students used first the strategy relying on dragging x on the line and either provided approximate values of solutions (e.g., 2,3 and –1,3) or framed the solutions by integers (e.g., $-2<x<0$ and $2<x<4$). Teacher intervention was necessary to clarify that exact solutions were to be found and suggest using the E=0 command. Students encountered two main difficulties with using this command. The first difficulty was making a link between the expression E(x) they needed to find to be able to solve the given equation of the type A=B (the question intended to guide them was “What equation of the type E(x)=0 allows solving the given equation? Explain.”). The teacher had to state more precisely that Alnuset only provides a tool for solving equations with the right side equal to 0, and that it is then necessary to transform the given equation in a way to have 0 on the right side. Such intervention helped most students to find an adequate expression and use the E=0 command. The other difficulty was linked to the use of the E=0 command. In fact, to solve an equation with Alnuset, one has to use this command as many times as the equation has solutions. Although the students were aware that the equation has two solutions (most of them provided two approximate values at the beginning of the task), they did not think of using the command twice in order to find both solutions, and thus provided only a single solution. This difficulty is linked to the development of a scheme of using the E=0 command, which supposes to anticipate the number of solutions of a given equation and to be aware of the fact that applying the command gives a single solution at a time. This is quite unusual comparing to traditional algebraic techniques.

CONCLUSION

These two experimentations enable a first evaluation of the mediation offered by Alnuset. In both experiments Alnuset was exploited both as a tool to verify already developed conjectures and as a tool to explore algebraic phenomena in order to arise and validate new conjectures. It allows designing learning scenarios with characteristics that are deeply different, according to given contexts (institutional, cultural, social…) and educational goals to be pursued. The two experimentations lasted differently and this allowed to evidence that: (i) the instrumental genesis of the Alnuset instrumental techniques may be quite short for some of them (e.g., using drag mode for determining equivalence of two expressions) and longer for others (e.g., using E=0 command to solve polynomial equations and interpreting associated feedback); (ii) the instrumented techniques can be controlled by mathematical justifications and previous knowledge, correct or not. On the other hand, the French experiment showed that when the previous mathematical knowledge is rather fragile and the students are not very confident with it, resorting to the tool can help them carry out successfully the tasks they would not succeed without using the tool; (iii) the instrumented techniques produce representative dynamic events that can be easily related to algebraic notions and meaning involved in the activity.

Both experiments evidenced the importance of teacher’s role in supporting the development of students’ instrumental genesis at the beginning of the activity with Alnuset. Moreover, the role of the teacher remains very important during the whole activity to orient discussions and considerations about instrumental issues that have to be intertwined with algebraic knowledge involved in the activity.

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COMMUNICATING A SENSE OF ELEMENTARY ALGEBRA TO PRESERVICE PRIMARY TEACHERS

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This article reports on a university course for preservice primary teachers on 'patterns and structures in primary school to prepare algebraic thinking'. We believe, if arithmetic is taught with an algebraic awareness, e.g. looking for patterns within arithmetic problems, algebraic thinking could be enhanced in primary school and the 'cognitive gap' between arithmetic and algebra would be reduced. In order to teach with an algebraic awareness the teachers must have developed such awareness themselves. We present the design of a course with which we contributed to this. The course serves us as a pilot experience for gaining hypotheses on the needs of teacher students and on good teaching interventions. We conclude the article with research questions in this field of teacher education.

THEORETICAL FRAMEWORK AND FOCUS OF THE PAPER

It is well known that there are many-facetted difficulties in learning algebra (see for example the contributions in Bednarz et al., 1996). Also the working group on algebraic thinking of CERME 5 has considered many features constituting elementary algebra and problems of learners. Some of the contributions are concerned with problems of constructing new mathematical objects (as formal or as abstract, cognitive objects) when dealing with algebraic expressions (e.g. Dörfler, 2007; Fischer, 2007a; Lagrange, 2007). Others point to students' often limited or inappropriate ways of interpreting symbolic arithmetic or algebraic expressions (e.g. Alexandrou-Leonidou and Philippou, 2007; Molina et al., 2007; Papaieronymou, 2007). What do these many-faced difficulties have in common with the learning of algebra? The working group agreed on one central theme of algebra underlying all other aspects discussed: 'expressing generality' (Puig et al., 2007). However, students often do not experience this feature in their algebra classes.

One reason for these difficulties is the so-called 'cognitive gap' between arithmetic and algebra. Herscovics and Linchevski (1994) highlight some aspects of it. Features like the manipulation of variables occurring twice or more in a formal expression demand truly new cognitive abilities or constructions as compared to an arithmetic viewpoint. Similarly, they suggest a new viewpoint is required to comprehend formal arithmetic expressions as entities in their own right, or to look for patterns and structures in arithmetic problems. As a consequence of the observed gap, students have to cope with several changes to their habit of solving problems, their ways of interpreting signs, their ideas on what mathematics is about.

In this article we propose that some of the features of this gap between arithmetic and algebra are not so much due to the given characteristics of the two areas of mathematics, but to a tradition of teaching arithmetic common to many countries. This tradition focuses on ways of interpreting arithmetic expressions and treating them, which cannot be extended to the algebraic sign system. What is more, the tradition of teaching arithmetic narrows the focus of mathematics to calculations and results, giving little scope for the search for general patterns and the discussion of structures. Things can be done differently. The way formal expressions are interpreted in algebra can also be used for interpreting arithmetic expressions. For example the expression 3+4 need not only be understood as a description of an activity but also as a sign for a number. Many other characteristics of algebra could effectively first be established within arithmetic contexts. A lot of research exists on including algebraic activities in mathematical learning environments for primary school children. For example several studies (e.g. Carraher et al., 2008; Fischer, 2007b; Söbbeke, 2005) report on the understanding of arithmetic or geometric patterns by young children who are not yet familiar with the conventions of the formal algebraic sign system. When they become familiar with activities of this kind in primary school children might be better prepared for the step to algebra.

But how can primary school teachers be persuaded to teach these issues? For a pilot experience we designed a university course aimed at preparing (future) primary teachers for integrating algebraic aspects in the math classes. In this article we will explain our grounds for the design of the course and report on our experiences. At the end we suggest ideas for further research to help evaluate the course and develop it further.

A central issue for our course was how to persuade primary school teachers to engage in algebraic ideas. Understandably, primary school teachers tend to focus on the goals set by curricula for the first school years. Often they are not aware of the consequences of their attitudes for the children’s learning of further mathematical concepts. Moreover, many of them do not see a connection between learning mathematics in primary school and algebra in secondary schools. And those who do are not aware of different ways of dealing with arithmetic. Therefore, we consider it a necessary prerequisite to help (future) primary teachers look at the mathematics in primary school from an algebraic perspective and to show them how they can integrate pre-algebraic thinking without loosing track of their primary goals.

Mason (2007) gives some ideas on how teachers can learn to deal with the subject of expressing generality. One central point is the highlighting of typical mathematical processes involved in the search for general patterns and in their representation and use. This is one important connection between the general goals of mathematics and our specific interest in advancing algebraic thinking in primary school. We recognised different though interwoven aspects of ‘algebraic awareness’:

- *Experience with problem solving activities*, e.g. analysing and describing patterns and structures, continuing patterns, using structures for calculations and problem solving,

- *Knowledge of different mode of representations and structures* of problems, solution methods and solutions,
- The *disposition* to look for patterns and structures in arithmetic problems and to argue with them and to perceive arithmetic expressions as processes and as objects.

All of these aspects can be provoked within arithmetic and geometric contexts in primary school (grade 1 to 4).

CONCEPTUAL DESIGN OF THE COURSE

In the course we had four main goals:

- The students experience algebraic thinking within arithmetic and geometric contexts. They are encouraged by personal success and gain a broadened view on mathematical tasks.
- The students understand challenges of (pre)algebraic thinking as part of mathematics fitting in the goals of primary school.
- The students design and analyse mathematical problems concerning arithmetic or geometric patterns in a context of primary school either within a case study or while analysing schoolbooks.
- The students reflect upon learning mathematics themselves and by children.

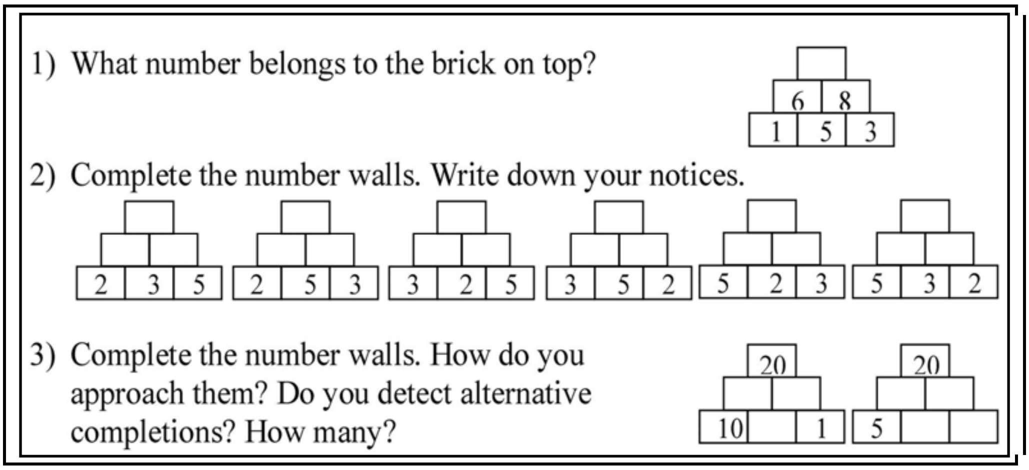
Organisational frame

The class met three hours each week for one semester (14 weeks) and was open for advanced students who had already taken some mathematics and mathematics education for primary school. Twenty three students attended the course. To obtain credits each student had either to undertake and write a report of a short empirical study with one or more children, or write a theoretical theses comparing two series of schoolbooks.

Progression

1. Introducing the course subject

During the first weeks of the course the students were presented with mathematical problems, which comprised different aspects of algebra and algebraic thinking. With this activate approach the students experienced algebraic thinking instead of dealing with a theoretical definition. We chose problems which highlighted characteristic aspects of algebraic thinking. Quite a number of these problems dealt with the discovery and expression of patterns. The students had to solve them with their preferred problem solving strategy and with at least one strategy that children in primary school might use. The class reflected upon the solutions, the solution methods and different ways of presenting both. Furthermore, problem solving strategies were elaborated and differences were highlighted between problems which appeared to be very similar at first sight but turned out to have very different algebraic potentials.



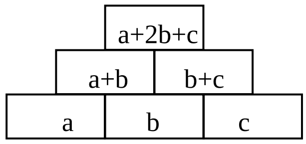


figure 1 **figure 2**

Figure 1 shows problems from a worksheet on “number walls”. Three-layer number walls involving additive structures within integers are an often used format in German school books. They are constructed as indicated in figure 2 (where a, b, and c are integers).

The first task on the worksheet presents a typical arithmetic task: the sum of integers has to be calculated. Note, however, that if used to *introduce* number walls, this already demands some degree of structural analysis. The second task also starts of with the calculation of sums. But the request to write down observations leads to a closer examination; the different walls have to be compared. Describing differences and commonalities of the six walls with the same integers in the bottom bricks demands a careful study of the walls. Verbalising the observation and explaining the findings helps the discovery of a mathematical pattern. Finally, the number walls of the third task cannot be worked out in the same straightforward way. They present disconnected problems (one of them is not solvable within integers) which can be tackled in different ways. Asking for the approach implies an explicit reflection on it; asking for other solutions and for the number of other solutions guides students towards a structural approach to the task.

Other problems given to the students offer different views of symbolical terms like the equal sign and expressions like the sums of two numbers. Given “3+4=”, say, whereas one view sees the equal sign as an instruction to calculate (3+4 adds up to 7), another promotes the view of the equal sign as a balance and of the sum as being a number (3+4 is the same number as 2+5). Cognitively the latter demands a view of an arithmetical expression as a number as well as a process (cf. Gray and Tall, 1994). Furthermore, the students were given problems on number sequences, geometric visualisations of such, arithmetic laws and (dis)connected arithmetic word problems.

Although the problems were basically taken from German schoolbooks for classes 1 to 4, the students had numerous difficulties solving them. Many of them made very formal use of variables, often with little or no understanding of the meaning. This caused mistakes on the one hand and impeded discussion of mathematical relations on the other hand. Moreover, the students frequently had difficulties to think of strategies without using variables. Often they thought of only one alternative strategy: systematic trial and improvement. Yet, they did not always acknowledge this as a valuable mathematical strategy.

Working on the given problems, the students were surprised by their experiences:

8. There are mathematical tasks with different ways of solving them, some problems can even have different solutions.

9. Strategies can be found which do not involve the formal algebraic sign system are possible. But to find such strategies requires insight into the structure.

10. The inherent structure of similar looking problems can be very different.

11. These problems offer challenges on different levels. Some of these challenges are revealed to the students only when working on them.

These experiences were facilitated by questions attached to the mathematical problems, which emphasised mathematical activities like visualising, comparing and arguing.

Besides solving the problems the students reflected upon the mathematical activities required by the children. Through this, we raised ideas of what algebraic thinking is about.

We concluded the introductory unit by taking a more theoretical standpoint. In class we discussed the paper of Lorenz (2006) on possibilities and challenges in using geometric representations of arithmetic patterns for illuminating the structure and solving problems about them. The claims of the text could well be investigated through some of the examples the class had worked on in the previous weeks.

The class then developed a notion of ‘good’ mathematical problems in general and in respect to algebraic thinking. The class agreed on the following features to constitute ‘good’ problems:

A ‘good’ mathematical problem must be

9. open to different approaches or different solutions,

10. given with a mathematical goal,

11. easy enough for every child in class to start solving the problem and to obtain a (partial) result, but also

12. challenging even for high achieving children.

The feature specifically relevant for the course is the encouragement of algebraic thinking. We listed the following characteristics of algebraic thought which can be found within arithmetic or geometric contexts:

- unknowns not only at the end of an expression,
- equal sign as balance sign,
- arithmetic expressions as representations of numbers,
- describing patterns,
- calculating big numbers effectively using structures instead of extensive calculations.

These criteria are neither original or exhaustive. But they reflect the views the students had developed at this point on the course and used as basis for their own work. Throughout the rest of the course these criteria served as an orientation for the students when developing and evaluating mathematical problems for primary school.

2. Preparing and realizing the individual projects

The students then started with their own projects. Seven carried out a case study with a child in primary school. Each of them prepared a short sequence of problems he or she was going to use in the interview. This sequence had to be analysed with respect to its algebraic potential. There was opportunity in class to have these sequences discussed in small groups and to work them through before they were used in the interviews.

After the interviews were accomplished the students had to transcribe interesting parts and analyse the children's performance. The students in Frankfurt have plenty of experience with carrying out interviews and analysing them with respect to interaction. Therefore we decided not to elaborate on these issues. Nevertheless we devoted one lesson to tools for analysing transcripts. We focused on gaining mathematical knowledge through working on representations. For this we read a paper on the epistemological triangle of Steinbring (2000). In this text two analyses are presented in which students explain and develop ideas on a mathematical problem. However this text turned out to be very difficult. It is too theory laden for our students to enable them to extract general principles and apply them for their own analyses.

Students who aimed for a theoretical thesis each had to analyse two series of schoolbooks for classes 1 to 4. Each student had to select two formats of problems like a sequence of problems with a common pattern or number walls recurring in his or her schoolbooks in different classes. He or she had to give an analysis of these formats pointing to their algebraic potential. On the ground of this analysis he or she had to evaluate the way the schoolbook makes use of these formats and compare the two series of schoolbooks. The students of this group, too, were given the opportunity to have some examples from their schoolbooks discussed in class. In addition, throughout the whole course such formats served as examples for different aspects.

The individual projects were mainly worked on at home. Meanwhile, we were able to introduce several theoretical articles on mathematics education which discuss issues related to our subject. Our main focus was to interrelate educational theories with the students' own mathematical activities as well as with their design and analysis of problems. Through this, we also deepened the students' algebraic understanding.

We covered topics like learning, practising and problem categories. In particular, we compared learning mathematics via instruction to learning via discovery (cp. Wittmann, 1994) and related the findings to previous class sessions. Practising – not only algorithms of calculation but also mathematical processes like problem solving, representing mathematical ideas, argumentation – was connected to the different learning theories (cp. Winter, 1984) and discussed for one specific problem. The task of determining whether problems are open (for different solutions and solution methods) informative (regarding the learner's thinking) and process-oriented (which means, if they support mathematical activities like discovering, arguing and further elaborations; Sundermann and Selter, 2006), leads to reflecting on problems, varying and exploring them.

These articles addressed general principles of teaching mathematics in primary school. We found plenty of opportunities to interpret and understand them in respect to our subject of inducing algebraic thinking. Thus this subject appeared in the general context of teaching mathematics in primary school not as an exotic theme but as one way of complying with these general goals that are commonly shared.

3. Presenting the students' projects

In the last unit of the course the students presented some of their results. Those writing a theoretical thesis chose examples of their analytical work and some theoretical aspects related to it. Those doing an empirical analyses presented crucial aspects of their interview analyses. All of them were asked to look for ways of presentation that would actively involve the class.

The students who analysed schoolbooks had to think of criteria for their analysis first. It turned out that they used the criteria listed in the introduction only as a starting point. In order to build their criteria most of them chose one or more topics on learning mathematics we discussed during the second part of the course. It is pleasant to see that they altogether made careful analyses covering important aspects of algebraic thinking which proved a good insight into the formats.

For example one student gave an overview on which pages the formats occur in the schoolbooks before she went into quantitative and qualitative analyses. She did not only list the pages but stated the type of task linked to it, like discussing calculation rules, completing the format and comparing numbers of neighboured formats. This affected her quantitative analysis: She put the frequency of a format into perspective with the aligned task. While she noted that in one book the format was used more often she also claimed that a lot of the tasks merely practise calculating.

At the beginning of the term another student commented on a schoolbook she had seen in use in primary school. She reported that the school children would love to work on the book and do their work autonomously. Her submitted analysis of this schoolbook shows that she gained a broadened view on mathematics teaching. She stated that this particular schoolbook is based on a theory of mathematics education of tiny steps but little structural understanding of mathematics problems.

CONCLUSION AND FUTURE PROSPECTIVES

Overall we are satisfied with this course since we met our goals for most part. The students gained (more) competencies solving mathematical problems with an algebraic notion. They intend to integrate (pre-)algebraic thinking in their mathematics classes through designing adequate mathematical tasks and an appropriate attitude. They gained competencies in judging maths problems in school books and their own, as well as reflecting on their interventions. Our evaluation corresponds well with the students' feedback.

It turned out that the aspects of algebraic thinking were best understood when they were directly linked to their own experiences – and more than once – and reflected upon afterwards. For example the students had to solve a variety of problems with patterns during the first sessions which were originally designed for primary school. We reflected upon them: The students had to present their results, find different solution methods, vary the tasks, compare it with other tasks, etc. The attitude to look for patterns became an important issue for the group and the focus on patterns can be traced to the students' projects. In contrast some algebraic characteristics were not understood quite as well, like the notion of the equal sign as a balance sign. This is perhaps because we did not mention those characteristics quite as often, or because we looked at them from a more theoretical perspective.

We believe that it was not only the students who learnt a lot about (pre-)algebraic thinking: we also benefited from this course. We learnt something about the thinking of university students, gained perspectives on teaching them and at the same time got deeper insight of the potential of mathematical tasks for teaching algebraic thinking.

This teaching experience serves as a pilot study for us. On the basis of this experience we see several research questions that would be worth following up.

- The course seems to indicate that student teachers do need help to get an algebraic awareness, even though they have used much algebra in their own time at school. A quantitative empirical study of teachers' performances in observing patterns and structures in geometric or arithmetic contexts should give hard evidence on this issue. One could also investigate how, during a course like ours, students' ideas about arithmetic lessons change.

- We do not know very much about the inner representations student teachers have of principles of algebraic notation and algebraic argumentation. A qualitative empirical investigation on this issue might help us to better understand some of the underlying difficulties. In connection with this, the effects of some of the principles we applied during the course should be evaluated by empirical studies. The results of these studies might inform the development of curricula for teacher education.
- An underlying assumption of our course is that children who work on describing and using patterns in the context of arithmetic problems will be better prepared for algebra than students who only do calculations in their arithmetic classes in primary school. This conforms with theoretical positions on the nature of algebraic thinking in scientific literature. However, more empirical evidence is needed to investigate this claim.

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CONCEPTION OF VARIANCE AND INVARIANCE AS A POSSIBLE PASSAGE FROM EARLY SCHOOL MATHEMATICS TO ALGEBRA

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Change and invariance appear at the very early stages of learning mathematics. In this theoretical paper, examples of topics and tasks from primary school mathematics with various kinds of interplay between variation and invariants are presented. Application of this approach might be a tool that helps to improve non-formal algebraic thinking of students. We present some examples of pre-service teachers' reasoning in terms of variances and invariance.

INTRODUCTION

For over fifty years, mathematics educators have studied ways of teaching algebra. Beyond viewing algebra as generalized arithmetic, various classifications for meaning of algebra, algebraic symbolism, procedures and skills have been proposed (Usiskin, 1988). In algebra, students have to manipulate letters of different natures such as unknown numbers (Tahta, 1972), parameters, and variables. Required skills include specific rules for manipulating expressions and an ability to construct and analyze patterns. These components form the basis for the structure of school algebra, which appears to students to be abstract and rather artificial. Through dealing with transformation of algebraic expressions, students can hardly recognize the core ideas of algebra, such as application of standard arithmetic procedures to unknown or unspecified numbers.

From the point of view of primary school teachers, algebra is comprised of letters, rules of operations with expressions, and formulas to solve equations. Moreover, the term *pre-algebra* in the school math curricula stands for some "advanced arithmetic" topics that are linked with future algebra, mostly chronologically but not conceptually.

Since 2005, the awareness of pre- and in-service teachers about algebra has been one of the "hot" issues of annual conferences on training primary school math teachers in Israel. In order to match the course *Algebraic Thinking* to the needs of pre-service primary school mathematics teachers, a systematic study on their vision of algebra has been initiated. Preliminary results of this research show that only a few of these students are aware of non-formal components of algebra (Sinitsky, Ilany, & Guberman, 2009).

What mathematical concept could help pre- and in-service teachers to construct relevant algebraic comprehension? School algebra is a combination of generalized arithmetic, calculations with letters, and properties of operations (Merzlyakov &

Shirshov, 1977). In general, it requires reasoning on connections and relations between objects, for example, finding similarities and dissimilarities between objects. The question "what changes and what does not change?" seems to be fruitful in a meta-cognitive discourse that concerns problem-solving activity (Mason, 2007; Mevarech & Kramarski, 2003). We propose to apply this question at the very early stages of mathematical learning as a possible tool to connect primary school mathematics with algebra.

WHY VARIANCE AND INVARIANCE?

The two notions of *variance* and *invariance* are strongly linked, since "invariance only makes sense and is only detectable when there is variation" (Mason, 2007). Mason claims that "invariance in the midst of change" is one of three pervasive mathematical themes. Watson and Mason (2005) have elaborated the theory of possible variation and permissible change for the needs of mathematical pedagogy. The use of the concept of variance and invariance with pre-service teachers can develop their algebraic thinking and provide them with tools to construct examples.

The issue of learning processes is related to the human ability to associate and to distinguish between different characteristics of the same object. Research (Stavy & Tirosh, 2000; Stavy, Tsamir, & Tirosh, 2002) shows that reasoning patterns "same A then same B" and "more A then more B" are prevalent among students, and direct analogy causes deep misconceptions in the learning of mathematics. Refining comprehension of various types of interconnections between change and invariance may be fruitful for improving cognitive schemes of students.

Starting from secondary school, students systematically face algebraic notation and formalism. The most significant feature of algebra for students is manipulating with letters. It seems to them (and to their teachers) as a switch from four arithmetic operations with numeric operands into *terra incognita* of some quantities that are both unknown and tend to change.

Although the abilities to deal with varying objects, to explain, and to formulate are the very essence of secondary school algebra, students are expected to grapple with these based on their experience in primary school. In the framework of systematic construction of formal algebraic concepts, pre-algebra is responsible for the development of pre-abstract apprehensions of algebra (Linchevski, 1995).

In this paper, we bring up some issues from primary school mathematics and observe these problems in terms of change and invariance. We refer, at a non-formal level, to the main components of school algebra mentioned by Linchevski, i.e. using variables and algebraic transformations, generalization, structuring, and equations.

We proposed related mathematical activities for pre-service primary school mathematics teachers, and discuss some relevant classroom findings in the last paragraph and in the appendix.

VARIANCE AND INVARIANCE INTERPLAY IN PRIMARY SCHOOL

Word problems and algorithms of school algebra often have an origin, or an analogy, in primary school mathematics. Despite the concrete numerical form of arithmetic problems, they usually enable some algebraic generalizations into patterns for several number sets with suitable restrictions. For example, the property of being divisible by 9 is invariant in relation to any change in order of digits. Analysis of mathematical problems of primary school from the point of view of algebraic concepts may be fruitful for students as a step to constructing their algebraic thinking.

A consideration of variation, change, and invariance may help to provide a non-formal algebraic vision of arithmetic issues. Every mathematical situation provides a variety of variance–invariance links. Moreover, a suitable set of variations and related invariants that describe a task may provide a way to solve it. We illustrate the appearance and application of the "change and invariance" concept in a number of topics from primary school mathematics.

Quantities and numbers

The most fundamental example of invariant is human ability to count (Invariant, n. d.). It starts with the transition from objects to quantities and develops through numerous activities of counting objects of different nature. At this stage, quantity is invariant of physical properties of specific objects. Children also learn to count a given set of objects in different ways, and discover that the result is invariant of various (correct) counting procedures.

Thus, a basic conception of equality of quantities arises: the equality represents the fact that the same quantity is obtained or described in two different ways. There is also the possibility of inverting the problem: which changes are allowed within a given quantity? This question seems concerns a misconception of equality. Linchevski and Herscovics (1996) have connected cognitive difficulties in the transition from arithmetic to algebra to dual procedural-structural algebraic thinking. A well-known example of such difficulties is the comprehension of the expression $34+7=$ as a command to carry out an action (Gray & Tall, 1991). Accordingly, in the equation $8+4=\Delta+5$ the unknown is interpreted by students as the result of adding $8+4$. In contrast, the idea of equality as an idiom of invariance invites possible changes.

An appropriate didactical scheme for primary school students is to focus on problems of decomposition of given number into a sum of two addends. Typical questions require producing additional presentations based on a given one as demonstrated in this activity:

- $8=3+5$ How can you split the same number 8 into another sum of two addends?
- How does a change in the first addend influence the second one?

- How does the change of addends of two “adjacent” decompositions vary? (At a higher level this leads to a conclusion on invariance of parity for differences of addends for several decompositions of the same number)
- For a given odd (or even) number, what can you say about the parity of addends in each decomposition?

This activity invites students to discover the role of invariant quantities in a game of changing in.

In discussions with pre-service teachers, the same questions were followed by further generalizations. For instance, the last question on parity leads to a conclusion on the invariance of parity of algebraic sums of numbers, with arbitrary distribution of +/- signs, through an analogy to the arithmetic expression. A choice of signs +/- does not influence the parity of the expression $a_1 \pm a_2 \pm \ldots \pm a_k$ (for integers $a_1, a_2, \ldots a_k$).At an advanced level, the same mathematical situation leads to combinatorial tasks, such as:

- In how many ways can we split a given natural number into the sum of equal addends?
- Can you arrange any presentation of an arbitrary multiple of three as a sum of consecutive addends by first splitting it into a sum of equal addends?
- In how many ways can we split a given natural number into sum of consecutive addends?

In the appendix, we present examples of pre-service primary school math teachers’ response to some of these questions.

With this cluster of problems, we explored the concept of permissible changes within a given invariant in a variety of mathematical questions and levels.

Comparison of quantities in terms of change and invariance

In addition to invariance, the very basic process of counting deals with *variation* of quantity. Adding each new object to a given set of objects generates a new quantity that is greater than the given one. These examples are taken from the Curricula for Primary School in Israel (Curriculum, 2006): the sum 5+1 is greater than 5, and the sum 67+2 is less by1 than the sum 67+3.

From the point of view of invariance and change, students try “to find the same” in a pair of arithmetic expressions. The same operand plays a role of a parameter, i.e. arbitrary but the same number. The only cause for different values of given expressions is the difference in second operands. Therefore, to compare two quantities one looks at them in a structural manner: namely, noting the similarity and the difference between them. For example, comparing the results of other arithmetic operations when one of the operands is the same for both expressions:

- Which one of the differences is greater: 856 – 47 or 856 – 44?
- What is the difference between the two products: 84 X 123 and 83 X 123?

- Shirli arranged dolls in nine rows with the same number in every row. She added two dolls to each row. By how many dolls did the total number of dolls increase?

In school algebra, the presence of an unknown quantity typically turns the simple problem of comparing two similar expressions into a difficult one for students. For example, the comparing the pair *a-7* and *a+7* as opposed to the pair *7-a* and *7+a*.

Further, in order to compare more "remote" arithmetic expressions, one can try to interpret them as a different change of the same connecting expression. When pre-service teachers discussed how to compare two differences, i.e. 1234-528 and1243-516, they constructed intermediate expressions, 1234-516 or 1243-528. In a similar way, they proposed using the product 83×123 for comparison of products 84×123 and 83×124. This method of comparison is also an algebraic one: two expressions *a*b* and *c*d* are interpreted as changes of the same basic structure *a*d* or *c*b*.

Computational algorithms and techniques

In school algebra, most procedures cause changes in algebraic expressions yet preserve equality or inequality. This issue is not new for students. Almost every process of computation includes some transformation of a given arithmetic expression to another one. The transformation is valid provided it keeps invariant the value of the expression. In fact, both the rules of arithmetic operations and standard computational algorithms preserve the invariants:

- To calculate the sum 123+456, one groups similar units of addends, 123+456=(100+400)+(20+50)+(3+6) – this is a direct analogy of gathering similar terms in algebraic expressions.
- The difference 123-49 can be replaced by a new expression that retains the value of the given one: 123–49=124–50.

Fraction reduction and expansion are additional examples in elementary school of variation that preserves value.

The ability to find a suitable variation of a given expression that preserves its value is a useful starting point for oral calculations. A necessary condition to apply is the invariance of the value under the change of form of the calculated expression.

We have studied the strategies pre-service primary school math teachers apply to calculate sums of arithmetic progressions (Sinitsky & Ilany, 2008). Only 5% of the students succeeded in recalling a suitable formula and applying it correctly. After taking part in series of assignments concerning interplay of change and invariance, the students were given similar tasks. They tried to calculate sums by reducing them to

known series in various ways
(2+3+...+26→1+2+...+25; 3+6+9+...+60=[2+4+...+40]+[1+2+...+20]).

Number properties and range of generalization

When students manipulate algebraic expressions, the application of natural intuitive reasoning schemes "same A then same B" or "more A then more B" leads them to false reasoning: *"$x^2 = y^2$ implies $x = y$", " $-x > 2$, therefore $x > -2$"*. In terms of change and invariance, this is a problem of connection between different invariants.

There are numerous examples of correct ways of reasoning when letters A and B stand for the property of numbers. Examples of correct propositions concerning squares of natural numbers: "If the unit digits of two numbers are the same their squares have the same unit digit"; "The squares of numbers with the same parity are also of the same parity"`; "As natural numbers increase so do their squares".

Such a convenient tie between invariants and changes invites a wide generalizing. Accordingly, questions that lead to counter examples and determination of range of possible changes or invariants are crucial: "Does changing the order of a sum change the result?"; "Does equal square/rectangle/parallelogram area imply the same perimeter?"; "Does multiplying a number by 2 increase the number of its divisors?"

Generalizing regularities and solving problems without algebraic formalism

An equation composed to solve a word problem algebraically expresses an invariance of some (typically unknown) value. For example, in problems that concerns motion, the same distance that two vehicles cover in different manners is the invariant of the two processes involved. Hence, the ability to identify invariance through some changes is useful for solving mathematical problems.

At primary school level, the search for invariance is an effective tool to discover regularities in numerical tables and in tables of arithmetic operations. For example, in the hundred table (see appendix, example 2) numbers increase constantly, but the change between adjacent cells in any row or column is invariant of the cell position. Similarly, the difference of products of diagonals of any 2×2 square is an invariant of the choice of square.

The next stage of proving those propositions typically involves some algebraic manipulation. Detecting a proper invariant for the problem can help avoiding formal algebra and provide a transparent proof with a generic example (Mason & Pimm, 1984). This type of reasoning is presented in the appendix.

Coming back to word problems and relevant equations, we illustrate another aspect of interaction between variation and invariance in pre-algebra mathematics. This interplay may provide non-algebraic solutions for some word problems. For example:

> John bought two kinds of items: pencils that cost 30 cents each and pens that cost 50 cents each. He paid 6.20 euro for 16 items. How many pencils and how many pens did John buy?

We restate here a well-known arithmetic solution of the problem with an emphasis on variation and invariance. We start with the possibility that John bought 16 pencils at a cost of 4.80 euro. Now we need *to vary* the cost, keeping *invariant* the number of items. The answer to the question "How many pencils do we need to exchange for pens to increase the total price by 1.40?" provides the solution of the problem. In this approach, the total number of items is an invariant of the process. An alternative method of solution starts from any combination of items that provides the desirable cost (for example, 10 pens and 4 pencils). The next step is to vary the number of items keeping the total cost invariant.

A taxonomy of change and invariants

Due to many characteristics of each object or process, every variance results in several changes and introduces invariants as well. Alternatively, preserving some invariant permits variances of other properties. Thus, there are many possibilities of interrelation between change and invariance. The same sort of connection can occur in various mathematical problems and topics.

From the above and other examples, we have derived a suggested taxonomy for change, variance, and invariance:

- An invariant is given *a priori*, and the focus is on possible changes and related invariants.
- To understand the action of *prescribed change*, we look for imposed variations and for given invariants.
- To solve a problem, it is necessary to *find some key invariant* of all the procedures involved.
- To treat a mathematical situation, we *introduce a suitable variation* or a sequence of variations.

Within this classification, the two latter cases seem to be more complicated since they involve *construction* of relevant objects or procedures. On the other hand, a specific kind of relation between variation and invariance is connected more with the method of solving the problem than with the problem itself. Thus, various solutions of the same problem may bring into play different kinds of interaction of change and invariance or even a combination of those interactions.

PEDAGOGICAL ASPECTS OF THE APPROACH

We require that primary school mathematics teachers be competent to recognize relevant kinds of variations and invariants in various issues and problems of elementary mathematics. We need to start introducing this concept in teachers' education to ensure that they can construct an additional didactical tool for mathematical discourse in a classroom.

To test the influence of discourse in terms of interplay between variance and invariance on algebraic thinking of students, we designed an experimental study. The research involved future and current teachers of mathematics at elementary school. We tried to learn if, and to what extent, discourse on variance and invariance influenced beliefs and knowledge on the ability of further application of non-formal algebraic reasoning. In addition to checking the validity of our conjectures, we would like to improve the awareness of school educators about the use of variation and invariance at primary school level.

So far, pre-service teachers have participated in the study through problem solving activities in the framework of their courses in pedagogical colleges. Throughout these activities, they have discussed the ideas of variance and invariance with specific mathematical issues. We have found that future teachers have begun to construct examples for teaching in elementary school that invite algebraic thinking and argumentation in terms of change, comparison and invariants (Sinitsky & Ilany, 2008).

To promote this concept, we designed additional mathematical assignments. Each task includes a cluster of math problems on different issues at various levels of difficulty *united by the same relation of variance and invariance*. The starting point is part of the school curriculum, should be familiar to every pre-service teacher, and is a basis for further generalizations and analogies. The style of all the assignments is that of open problems in order to stimulate various approaches and strategies.

CONCLUSION

In this paper, we discussed applications of conception of changes and invariants in primary school mathematics. We looked at numerical problems from a point of view that is general and in many cases algebraic. The same types of connection can be detected in different mathematical issues. The ability to recognize variation and invariants may be an effective tool in constructing non-formal algebraic thinking of students. However, as a necessary stage, it requires the awareness of teachers on the subject. Some preliminary evidence on pre-service teachers' activities seems encouraging and invites further wide-scale research.

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APPENDIX: IT LOOKS LIKE ALGEBRA

Two samples of reasoning involving variance and invariance interplay are presented.

1. Representations of natural number as a sum of consequent addends – fragment of transcript of discussion with pre-service primary school mathematics teachers

Students wrote down all the pairs with the given product, 30, and constructed sample sums of equal addends.

Student A: *“I start with equal addends. Now, for 30=10+10+10, I keep the total sum but vary the addends:* (she moves a finger from the first term to the third one and has marked it with an arrow) *30=10+10+10. We get 30=9+10+11, and it is possible to do this for each of these sums of equal addends! For example, I can derive from this sum* (she points 30=6+6+6+6+6) *another sum of consequent addends: 30=4+5+6+7+8 and... No, it does not work with 30=15+15: we need the sum to be invariant but also keep a middle term, and there is no middle addend here. Ah, I can try to split each one of 15s, but it changes the number of addends...”*

Students also obtained representation of 30 as a sum of four consequent addends: 30=6+7+8+9, and tried to derive sum of consequent addends from the sum of fifteen equal ones.

Student B: *“But we need negative numbers. Aha, after the cancellation we get exactly the same sum! It means that for every presentation of natural number as a sum of consequent natural numbers we can make more sums if we use integer numbers that will be cancelled after that, for example, 12=3+4+5 and also 12=(-2)+(-1)+0+1+2+3+4+5, because (-2)+(-1)+0+1+2=0”*

2. Divisibility of differences of two-digit numbers with “inverted” digits – sample proof

Conjecture: The difference of two two-digit numbers, where the second number has the same digits as the first one but in inverted order, is a multiple of 9.

How can we introduce the justification of this proposition without algebraic formalism in the framework of discussion with the students?

Let us check, what is the same in each pair of these numbers? They have the same digits, therefore also the same sum of digits. Now, let us mark an arbitrary pair of these numbers in a hundred table, for instance, 62 and 26. Their difference is just a distance between cells. Can we construct the route from 26 to 62 that keeps **invariant** the sum of digits? The route passes through 35, 44 and 53 before reaching 62. Each step increases the number by 9 (see “decomposition” of one of the steps in the table), therefore the total difference is a multiple of 9. Moreover, the difference between inverted two-digit numbers equals the number of such steps multiplied by 9.

1	2	3	4	5	6	...
11	12	13	14	15	16	...
21	22	23	24	25	26	...
31	32	33	34	35	36	...
41	42	43	44	45	46	...
51	52	53	54	55	56	...
61	62	63	64	65	66	...
...	...	...	...	...	...	...

GROWING PATTERNS AS EXAMPLES FOR DEVELOPING A NEW VIEW ONTO ALGEBRA AND ARITHMETIC

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Sequences of growing patterns play an increasing role in the context of introducing terms. In this paper we reflect a new view onto the role of those particular visualisations for arithmetic and as well for algebra. By using a pupil's document we illustrate in this paper the theoretical framework of our concept.

Keywords: *representation/growing pattern, pre-algebra, children's interpretation, building structures and relations into diagrams*

1 Perspectives on the Mathematical Knowledge on the Way to Algebra

On their way from arithmetic to algebra, students have to develop a new awareness for the general, for the variation and the variable. At this period a new way of thinking, a new understanding of the previously acquired mathematical concepts, symbols and operations and thus a new interpretation of old knowledge becomes necessary. Students of elementary school become acquainted with equations in arithmetic lessons primarily in the context of calculating. In a special kind of lesson culture they learn more or less subconsciously that by dealing with equations they have to calculate the part on the left of the equal sign and after that to note the result on the right ("Task-Result-Interpretation"; Winter 1982). In many cases the equal sign is interpreted as a sign demanding to calculate. In many cases its function as a symbol of equality is not spoken about or used in every day arithmetic lessons. Such restriction in the interpretation, understanding and use of arithmetic terms and symbols is an obstacle not only for the later algebraic comprehension, but also for developing successful calculation strategies for the elementary arithmetical operations in the following school years.

Today algebra is seen as the lingua franca of higher mathematics (Hefendehl-Hebeker & Oldenburg 2008). However, algebra does not obtain the meaning and power of such a superior language if its status is restricted to the transformation and calculation of terms. Algebra has to be a "system characterised by indeterminacy of objects, an analytic nature of thinking and symbolic ways of designating objects" (Cooper & Warren 2008, 24). Therefore it is indispensable for the construction of algebraic comprehension not merely to calculate terms, but increasingly to see them in their structures, in order to understand formulae and principles. "The equation (or formula) must not be perceived as a sort of calculation shorthand note but rather as a type of scheme, which can in different ways be rearranged and be filled with concrete content" (Winter 1982, 210).

Various studies are concerned with the transition from arithmetic to algebra, which is accompanied by ruptures and discontinuities from the arithmetical to the algebraical view (cf. Bednarz & Janvier 1996). In our paper we focus not only on ruptures in the transition from one view (e. g. arithmetic, geometric) to another but also on reinterpretations and developments within one view in the context of growing patterns.

2 Growing Patterns and Mathematical Visualizations as Mediators between old and new Mathematical Knowledge

If the substance of algebra is seen in the way it represents the principles and structures of mathematics and not in terms of the "behaviours" of algebra (such as simplification and factorisation) (…) (cf. Cooper &Warren 2008, 24), then it is important for the introduction to algebra to make meaningful learning possible for the students, which at the same time constructs basic ideas that are sustainable in the long term. That means that such learning and exploring of algebraic ideas is always situated in the difficult balance between a rather empirical view on concrete objects and actions on the one hand and a certainly more challenging but in the long run necessary and profitable view on relations and structures on the other hand.

On their way to algebra it is necessary especially for young students to open a learning arrangement and an exploring field in which they can move between these poles of an empirical view on concrete objects and actions and a more abstract view on relations and structures. Structured mathematical visualization and growing patterns constitute such a learning environment, which merges those poles in a natural way.

Mathematical visualization and growing patterns - as a special type of mathematical visualization (for example to represent mathematical principles) - can mediate between the mathematical structure and the student's thinking because of their special "double nature" (they are on the one hand concrete objects, which can be dealt with, which can be pointed at and counted, which can be manipulatively changed, and at the same time they are symbolic representatives of abstract mathematical ideas).

Mathematical visualizations and growing patterns are well-known to elementary and secondary school children from their daily mathematics classes. Geometrical patterns, which must be interpreted arithmetically, are used in class for various purposes. Steinweg (2002) notes that in text books dot patterns appear to practice calculating skills and thus function as visualizations, while sequences of dot patterns are to be explored as a separate and independent subject (cf. Steinweg 2002, 129-151). It is obvious that in everyday mathematics lessons dot patterns have predominantly the function of a methodological-didactical aid. Here is a parallel to the restricted view on equations and the equal sign mentioned above. Only in rare and isolated instances the structures incorporated in mathematical visualizations and growing patterns as well as equations are being purposefully explored and mentioned by the children. Against this backdrop Schwank and Novinska (2008) complain that didactic materials must be rescued from their shadow existence as mere aids and acquire a role as playing fields, in which genuine thinking processes can develop. Central questions such as "How many" and "if … then" in dealing with this type of materials open a smooth transition to algebraic thinking - at first based on representations which become accessible through interaction, speech and graphics (cf. Schwank und Novinska 2007, 121).

3 Features in the exploration of growing patterns on the way to Algebra

If sequences of patterns support this new view – not only to figure out arithmetic terms, but to notice the underlying structure, transpose, re-organize and reinterpret them in a positive manner, then the following five aspects seem to be of particular importance. These categories were developed by connecting first results of a case study in progress (cf. Böttinger 2007) and the results of a completed case study (cf. Söbbeke 2005). In order to interpret representations more and more in the function as a representative of relations and structures and thus to focus on the abstract and generalizable "pre-algebraic aspects" it was necessary to connect in this paper two analysis instruments and to use them both to analyse the interpretations of student Ron. In order to describe the interplay between the geometrical, the arithmetical and the algebraic view it was necessary to develop an analysis instrument (cf. Böttinger 2007) by analysing the transcriptions of the interviews. While the analysis instrument "*Four levels of VISA*" (cf. 3.5) combines various aspects of structuring and interpreting a visual representation, in the analysis instrument "*Model of categories*" (cf. 3.1-3.4) these particular features were separated, adapted to sequences of growing patterns and the gradation was worked out by analysing the interviews.

Model of categories

3.1 Structuring a single pattern

- No subdivision
- Not intended subdivision
- Intended substructure
- Examination of several substructures

3.2 Flexibility

- No change of view
- Change of view without new structuring
- Change of view with new structuring

3.3 Relation geometry - arithmetic

- Pure geometric view
- Pure arithmetic view
- Relation is established by a number of points
- Additive relation
- More complex structural relation

3.4 Relations within the series

- No relations

Fig. 1

The aim of the first case study (cf. Böttinger 2007) is to describe more precise on the basis of 20 interviews with 4th-grade children, in which way children translate geometrical relations in a sequence of growing patterns into arithmetic terms and in which way generalisations are carried out. The hypothesis is that there is no direct way from the geometrical representation to an arithmetical one and finally to an algebraic view. Instead there will be an interplay between these different views. In order to describe this interplay an analysis instrument (cf. Model of categories, Fig. 1; cf. Böttinger 2007) had been developed on the basis of the interview data.

3.1 Features concerning the structuring of single patterns

In order to continue and examine the sequence a single pattern has to be structured. A subdivision can correspond to the *intended* structure of that person who composed the assignment on the one hand. On the other hand it can be an *individual* one, which does not correspond to a priori intended ideas.

3.2 Features concerning the flexible re-organisation of single patterns

In order to generate the idea of an equation one must be aware of different perceptions of a single pattern in the sequence. The aim is to identify the equality of arithmetic or algebraic expressions on the basis of the corresponding underlying geometric structure. Closely connected to this view is that transformations of equations correspond to changing the view on geometric structures. In analysing the children's interpretation one has to consider the flexibility during the process of work. It is essential to draw a comparison to the preceding interpretations of the child and to verify, to what extent a change of view occurs. This can be without new arrangement within the single pattern, e. g. when the number of dots is solely calculated in different ways. On the other hand a proper structural reinterpretation and re-organization exists, when the child builds fundamentally different structures into the diagram as in the step before.

3.3 Features concerning the relation between geometric and arithmetic structures.

Within her study Steinweg (2002) has worked out by what criteria children continue sequences of growing patterns. She distinguishes between a continuation by a figural aspect or by an arithmetical aspect. The figural aspect is concerned with the location of the dots and the external form built by the dots and the arithmetical aspect with the total number of dots in a single pattern. Steinweg accents that only the combination of figural and cardinal aspects lead to the intended continuation. Besides the distinction between a pure geometric view and a pure arithmetic view one has to regard the possible connections between both parameters. This can happen by a number of points, but also additive or more complex relations (e. g. multiplicative ones) can be identified.

3.4 Features concerning relations within the patterns

If sequences of patterns are used for algebraic investigation, one has to distinguish two totally different views. While the explicit formula uses the inner structure of a single figure, which must be suitable for all following figures, a recursive formula uses relations between consecutive patterns (cf. Carraher & Schliemann 2006). With the help of recursive formulas it is described, how the number of points changes from one pattern to the next. This view can be a great obstruction if the number of points in the 10^{th} pattern is to be figured out. The student has to calculate step by step each particular pattern and simultaneously he has to control the number of steps. In addition, the indication of the recursion alone is incomplete to describe the building principle, because an initial condition is needed (Carraher, Schliemann, 2007, 697). From the union of both perspectives interesting formulas can arise. Furthermore a dependence e. g. between the width and the height of a figure leads to dependent variables that describe exactly these features of the pattern.

3.5 Features concerning the interpretation visualizations (VISA)

In the second study (cf. Söbbeke) on the basis of detailed case studies with children of elementary school four levels of children's ability to build structures into mathematical representation (ViSA) had been distinguished. The underlying assumption of the study was that learning of mathematics has to be understood as a process of the children's more and more differentiated way of understanding and interpreting abstract patterns and structures (cf. Steinbring 2005). Visual representations are a tool to represent abstract mathematical concepts as well as to think about them or to talk about these with children. Growing patterns, as a special type of visualization, are often used to represent structures and relations in order to understand elementary mathematical principles (for example triangle numbers as an example to explore sums of odd numbers, etc.). The important information is not based in the concrete features of the material, but on the abstract, the relations and the structures within the material. Thus, what is decisive for a mathematical cognition in the figures is not the colours or the number of points; it is rather the function, which the concrete feature of the material takes for something. This means, the structure of the representation makes the understanding of a mathematical legality possible, but it cannot be read directly or immediately perceived with one's senses; it must be actively *interpreted* into the representation. In the empirical study (cf. Söbbeke 2005) it had been analyzed in how far the learning child succeeded in building such abstract structures and relations into the diagram. On this basis *Four Levels of Visual Structurizing Ability* had been distinguished. These four levels characterize the children's interpretations in a spread of concrete and empirical interpretations on the one hand (cf. level one, left pole of the spread) and relational und structural interpretations on the other hand (cf. level IV, right pole of the spread) (cf. Söbbeke 2005).

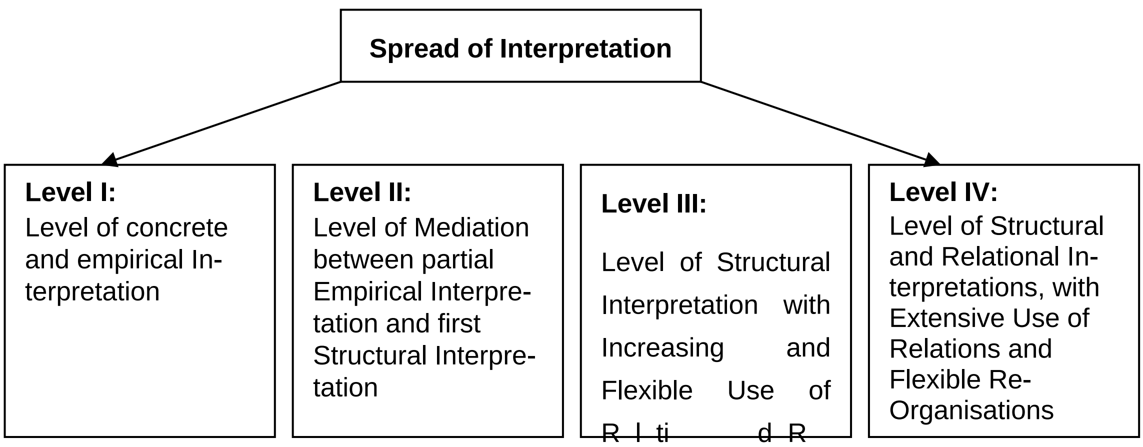

Fig. 2: Four Levels of Visual Structurizing Ability (ViSA).

4 Using Growing patterns to Support Students' Way to Algebra

- Ron on his Way to an Abstract and Multi-relational View of the Pattern -

The following examples are to show how the student Ron (4th grade) deals with the challenge to use growing patterns and to interpret them more and more in the function as a representative of relations and structures and thus to focus on the abstract and generalizable "pre-algebraic aspects" in the representation. For this we connect in this paper for the first time two different analysis instruments and use them both to analyse the interpretations of student Ron. The scenes presented are not to deliver a thorough methodical analysis. Instead the analyse in this paper can be seen as a first approximation to grasp and to describe the fundamental elements of the children's way to algebra by using growing patterns, which had been pointed out in 3.1 to 3.5. The analysis is not extracted from a finalized study, but it is an example of a new approach to the theme, to the underling structure and to a more detailed view onto sequences of growing patterns. In the *first* part of the different interview phases (*beginning*, *in course*, *end*) the elements of the aspects 3.1 to 3.4 had been described with the instrument "*Model of Categories of Changing Modes of Representation*" (see fig. 1). In the second part of the interview phases Ron`s interpretations had been assigned to the "*Four Levels of Visual Structurizing Ability (ViSA)*" (cf. 3.5, fig. 2).

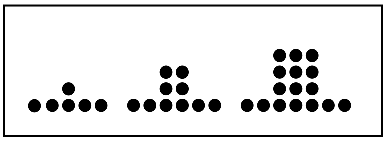

Fig.

At the ***beginning of this interview*** scene, Ron is presented the first three figures of the growing pattern and he is asked to describe what he can see (Fig. 3)

Ron	(16 seconds break) Mhm. (5 seconds break) Mhm (laughing). (10 seconds break) There at the bottom there is always one more (he points to lowest the row of dots in the first, the second, the third pattern). Five, six, seven (he touches the lower part of the first, the second, the third pattern) This next row. There are always some more.
Ron	Here there are, there are three more (he touches with his pencil the upper part of the second pattern). Here there are five more (he touches the third pattern with his pencil). (..) Since those I can remove (he puts his forefinger onto the third pattern), I can take away, because these are still there (he touches with the pencil the second pattern, afterwards he points to the not covered points of the third pattern). (…) Three, five. (6 sec. break, he moves the left forefinger to both left points of the bottom row in the third pattern, stops for a moment and takes the finger away from the paper) Mhm.

After 30 seconds reflecting about this task Ron starts to compare the three patterns. He structures the three figures into two parts: the horizontal row of dots at the bottom of the pattern and the field of dots placed at the top. In his first approach Ron does not pay attention to the part at the top of the pattern, but describes that the row of dots increases from one figure to the next and names the numbers "five", "six", "seven". In the analysis, considering the aspects 3.1 - 3.4, Ron shows that at the beginning of the interview he had developed an idea of the structure of the lower part of the pattern. Ron determines the number of dots in this part of the pattern and finds a recursive relation between the figures: "five, six, seven. … There are always some more". He builds a relation between the geometrical figure and the arithmetic in finding out the number of dots in the lower part of the pattern. Ron does not make it explicit, but his repetition of the number series can be seen as an indication that the number series and in association the structure of the lower part could always go on in this way. Against the background of his first interpretations, the number series can be understood as a preliminary stage of a recursive building principle: from one figure to the next you always have to add one point. Already at this early stage of the examination of the pattern you can see a first level of generalization.

After reflecting about 30 seconds about the upper part of the figures, Ron starts to describe the increasing of dots from the second to the third pattern. Ron structures the upper part into two groups: on the one hand, he sees the group of dots that had been seen in the previous figure, and on the other hand those, that had been added in the new following one: "Since those I can remove (he puts his forefinger onto the third pattern), I can take away, because these are still there". In his approaches to understand the structure of the upper part, Ron shows a first re-organization of the pattern. He does not analyse the two parts of the figures separate, but tries to understand in what way the first pattern could be identified in the second one and the second one in the third one. In the meantime he points with his finger on special areas of the lower part of the pattern, which he had described before in his first analysis of the pattern (the vertical row of dots). The numbers "three" and "five", he denominates, correspond presumably to the numbers of dots in the upper part of the pattern, marked for a better understanding here in white colour (see Fig. 4). Ron uses the numbers of dots and structures and builds first elemental relations between the different patterns into the diagram (he covers with his hands parts of the previous patterns etc.). As a kind of arithmetical information, Ron determines the number of dots in the particular figures. At the beginning of this interview the analyse shows a first recursive view on the pattern; however, Ron does not generalize this recursive view further, but applies it solely to the partly figures.

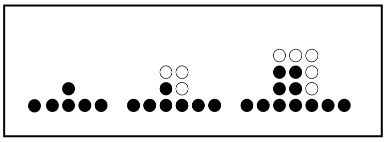

Fig.

Altogether Ron's interpretation of the pattern could be attributed to the 2nd level of *ViSA* (cf. 3.5). The child moves away from the concrete aspects of the representation (numbers of dots) and focuses increasingly on abstract relations and structures (two parts of the pattern; angle-structure of the added dots in the new figure). But the elements of interpretation often stand isolated as concrete objects, without building *rich* relations between them (for example relations between the structure of the part at the bottom and at the top of the pattern; relations between the different figures). Sometimes only sections of the diagram are taken into consideration. In interpretations on this level there is a typical mediation between partial empirical interpretations and first structural interpretations. But often the children's interpretations are still inflexible and they do not look at the representation as a multi-faceted structural diagram.

In the ***course of the interview***, Ron notices that he had always forgotten to pay attention to one point in the lower part of the pattern, while analysing the increasing of the patterns:

Ron	O no, I didn't count those (he taps the bottom row of points in the second pattern). That means, there would be four new ones (he touches the second pattern) and here there would be six new ones (he touches the third pattern).

After that Ron constructs a recursive geometrical building principle into the growing pattern and tries to translate it into an arithmetical building principle. In the course of the interview Ron has been asked to find an arithmetical task, which corresponds to the given pattern. For this he finds calculation tasks, which correspond with the result ("16") to the number of given dots in the third pattern. Ron interpretes and explains the proposal of the potential task "3·3+7", given by the interviewer, solely against the background of the calculating result und does not indicate a relation between the structure of the arithmetic task and the structur of the pattern. For Ron it is crucial that the number of the dots corresponds with the result of the calucating task.

He finds the calculating task "10·3+4" in the *5th pattern*, that can be seen als an analogon to the proposal of the interviewer in the 3th pattern ("3·3+7"). Presumably Ron takes the aspect "number of dots" on and tries to build an analog construction (second factor of multiplication is "3" or a task with a multiplative term) like in the task of the interviewer. Finally, at the end of the interview Ron is asked to determine the number of dots in the sixth pattern. He starts to draw the *sixth* pattern onto the interview sheet.

Ron	Five, six, seven, eight, nine, ten (in the meantime he draws 10 points in a row beside the 5th pattern). The first new points, this would be here, one, two, three, four, five, six (while speaking he draws a row of 6 points directly over the row of 10 points; cf. Fig 5). One, two, three, four five, six, (he draws - always counting until six - four further rows consisting of 6 points). One, two, three, four, five (with his pencil he touches the dots of the first column, but omits the corresponding dot at the bottom). Now another one (he draws a further row consisting of 6 points over the 5th 6-row). Six. Ready. One, two, three, four, five, six, seven (he touches the dots of the first column including the corresponding dot at the bottom), seven. Six times seven is 42 plus four, 44.

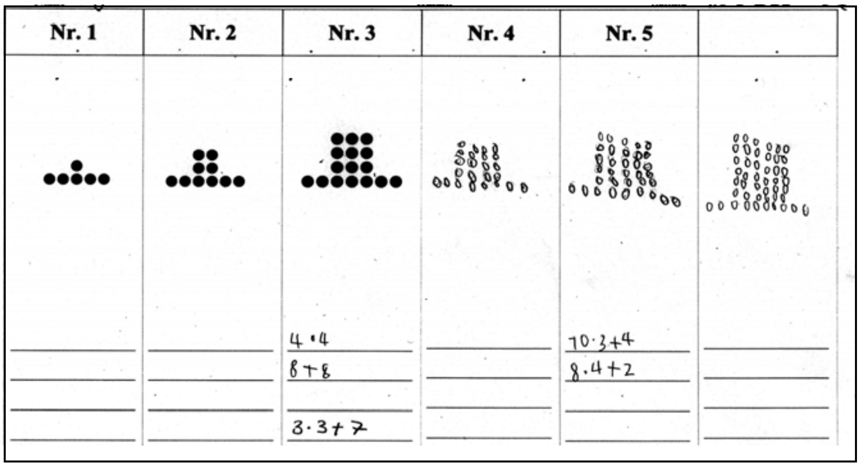

Fig. 5

At first Ron divides the 6th pattern into two parts: At the bottom he builds a long horizontal row consisting of 10 dots, in the upper part a rectangular field consisting of six rows of six dots. Subsequently he carries out an interesting new interpretation of the pattern. He structures it into a rectangle of seven rows of six dots, which reaches into the horizontal line at the bottom. Beside this 6x7-field of dots he regards two points at the left and two at the right-hand side – at whole 4 points. To figure out the total amount of numbers in the 6^{th} pattern Ron uses for the first time the inner structure of a single pattern. In comparison to his proceeding before this represents a change of view in connection with a new structuring. The relation between the geometric arrangement of the dots is no longer determined by the cardinality of a set of points but by a complex structural relation – namely a multiplicative one. By that Ron changes from his formerly recursive view onto the sequence and considers a single pattern in an explicit manner. The structure he uses is an intended one and in principle it is applicable to all patterns. But at this stage of the interview Ron does not express or indicate this generalisation.

Ron's interpretation of the pattern could be attributed to the *third level of ViSA*. In interpretations on this level *intended structures* and *relations* can be identified (for example relation between the part of the bottom and at the top of the figure; field of 6x7 dots; constancy of 4 dots in the part at the bottom). On this occasion different and multi-faceted aspects of the representation are recognised. In comparison to level II, the structures are *manifoldly coordinated* and more *flexibly re-organised*. The structures are no longer isolated, but seen as part of the whole and separated and put together in a structural way. You always find the use of *structural relations, coordination* and *re-organisation* of elements. In all, this level III of ViSA can be characterized by the combination of building structures with the increasing use of relations and re-organisations.

5 Conclusion

For a fundamental pre-algebraic comprehension it is indispensable to focus on structures, on the abstract and the general, right from the start of children's mathematics education. In this paper, growing patterns have been discussed and analysed as exploring fields on the way to focus on structures and relations. Structure sense seems to be a fundamental requirement to interpret sequences of growing patterns in an algebraical manner. Both analysing instruments examine in different ways how young children deal with the challenge to interpret this special visualization in a more structured, generalized and elementary "algebraic" way.

The examples of Ron indicate that this kind of structuring, translation and generalization does not take place in a direct and straight way. The child can partly understand the geometrical structures, translate them into arithmetic ones. It can change the view back to the geometric pattern and re-organise and re-structure the diagram. It seems that generalization is not always the "end" of this process; in fact ideas of generalization can be developed before comprehending the whole structure of the patterns.

An analysis of selected parts of the interview shows that in the *process of the examination* and the *interaction* between the student and the interviewer the child gradually develops a more differentiated, relational and generalized view onto the used diagrams, which can be described in detail by the system of categories and in a more summarising manner by means of ViSA (see e.g. the development of Ron's interpretation from level II to level III). Altogether the excerpts of the interview with Ron serve to demonstrate the change in children's interpretations in a exemplary way and to accompany and better understand their way – to an increasingly open, general and flexible view onto relations and structures within diagrams.

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STEPS TOWARDS A STRUCTURAL CONCEPTION OF THE NOTION OF VARIABLE

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If students acquire a new mathematical notion, according to Sfard (1991), they pass through different phases: an operational and a structural phase. At a grammar school in Bremen, Germany, students of age 12 to 14 first came into contact with the notion of variable using a simple programming language without a computer. As a part of the learning environment the students wrote imaginary dialogues in which they let two protagonists talk about different tasks. The imaginary dialogues of the students are analysed against the background of Sfard's theory of the dual nature of mathematical conception. In particular, the different steps towards a structural conception of the notion of variable in the context of the programming learning environment are elaborated.

INTRODUCTION

If we look at a mathematical notion, we can think about what it is in the mathematical world, how it is defined, which properties it has, and how it relates to other parts of mathematics or we can consider how a human being thinks about it and what kind of inner picture has been built. Anna Sfard (1991) distinguishes here between the word *notion* or *concept* on the one hand and *conception* on the other hand.

> The whole cluster of internal representations and associations evoked by the concept - the concept's counterpart in the internal, subjective "universe of human knowing" - will be referred to as a "conception". (Sfard, 1991, p. 3)

According to Sfard, a conception of a mathematical notion has two complementary sides, an operational and a structural one, in which a learner first passes through operational phases until a structural conception can be developed. She also points out that

> without the abstract objects all our mental activity would be more difficult. (Sfard, 1991, p. 28)

In this article the development of the conception of variable is considered. The underlying question of the presented analysis is: what are steps towards a structural conception of the notion of variable? To approach an answer the findings of a qualitative analysis of imaginary dialogues written by students of age 12 to 14 from one class will be presented.

THEORETICAL FRAMEWORK

The theory of reification

Sfard (1991) presents a theoretical framework for the acquisition of a mathematical notion. She distinguishes between an *operational* and a *structural conception* of the same mathematical notion. If a learner has acquired an operational conception, she or he will know how to operate with the notion, i.e. with algorithms, processes and actions. For a structural conception it is necessary to recognise the notion as a mathematical object. Sfard expects that the operational conception precedes the structural. In this process from operational to structural three steps occur: *interiorization*, a process with familiar objects, *condensation*, where the former processes become separate entities and *reification*:

> to see this new entity as an integrated, object-like whole. (Sfard, 1991, p. 18)

While a learner can come gradually from interiorization to condensation, Sfard speaks of a *leap* when it comes to reification:

> "Reification (...) is defined as an ontological shift – a sudden ability to see something familiar in a totally new light. Thus, whereas interiorization and condensation are gradual, quantitative rather than qualitative changes, reification is an instantaneous quantum leap: a process solidifies into object, into a static structure." (Sfard, 1991, p. 19-20)

Sfard & Linchevski (1994) used the framework of the theory of reification to study the case of algebra. In particular, they focused on the transition from operational to structural regarding a variable as a fixed unknown on the one hand and in a functional context on the other hand. Sfard (1991) asks the question how to diagnose the stages towards a conceptual development and proposes:

> "It seems that we have no choice but to describe each phase in the formation of abstract objects in terms of such external characteristics as student's behaviour, attitudes and skills." (Sfard, 1991, p. 18)

Mathematical writing

Mathematical writing by students has been the issue of several studies, compare Borasi & Rose (1989), Clarke, Waywood & Stephens (1993), Gallin & Ruf (1998), and Shield & Galbraith (1998). Gallin & Ruf investigated the use of *journals* (in German: Reisetagebücher) in order to establish a written dialogue between the students and the teacher. While writing their journals the students can approach the regular mathematics in their singular way.

Imaginary dialogues are a different type of mathematical writing (Wille, 2008). In an imaginary dialogue the student lets two protagonists solve a mathematical task or talk about a mathematical question. Usually one protagonist understands the task better than the other. In this way the student can decide what particular themes she or he addresses. Unlike in journal writing, in an imaginary dialogue, one finds a lot of exploratory writing. On the other hand, in contrast to pure exploratory writing, like writing a letter to someone and explaining something, in imaginary dialogues the protagonists can develop a solution of a task and the protagonists can point at possible learning difficulties.

LEARNING ENVIRONMENT

The learning environment is designed for first experiences with the notion of variable. The students do not start with a single variable as a fixed unknown. Instead, they get to know a simple programming language which is executed by the students without a computer but with a little wooden robot on a sheet of paper with a coordinate grid. The programming language has similarities to LOGO (Papert, 1980). Here, as a "memory" each robot needs matchboxes on which letters for the names of variables like "a" and "b" are written. These matchboxes serve as *preset reifications* of the notion of variable, which the students fill by hand instead of assigning a number to a symbolic variable. For example to move three steps forward, the program will look like this

a ← 3

forward(a)

While executing the first line it must be assured that exactly three matches are in the matchbox named "a". In the second line, the robot will be moved into the direction it faces. The matchboxes must be used in order to move a robot, since the direct command "forward(3)" is not part of the programming language. Next to these commands there is also the command "turnaround()", which lets the robot turn by 180°. Furthermore there are a right and a left turn, commands to place the robot on a certain intersection point on the coordinate grid and different command loops. That way students can write and execute programs in order to move their robot on the grid while assigning variable by filling matchboxes with matches.

In the learning environment the programming of the robot can be combined with writing imaginary dialogues. One of the first tasks can be the following: The students get a sheet of paper with "a ← " and "b ← " on top and "turnaround()" in the middle. On another sheet of paper eight paper commands "forward(a)" and eight paper commands "forward(b)" can be cut out. The students get the following exercise with the name "cut out and explore":

> On the next sheet of paper you see a program that is not finished yet. You can use commands out of a construction kit and put them above and below the command "turnaround()". 1. Cut out as many commands as you need and write a program with them. 2. Execute your program with the matchboxes and the robot. 3. Try to write such a program that the robot comes back to his starting point. 4. For which values a and b does your program function? Are there different possible values? 5. Write your favourite program and name many values with which it works.

Right after this lesson the students get the following homework (*dialogue A*):

> Two students talk about the last task “cut out and explore”. One of the students can do it easily, the other has more difficulties. Write a dialogue in which the two students talk about the task. Write at least one page.

In the next task a simple program is presented, where over the turnaround command there are two commands “forward(a)” and under it one command “forward(b)”. There is also a table given for a and b with values (1,2), (2,4), (3,6) and (4,7). A beginning of a dialogue is also part of the task where two students talk about whether the numbers in the table should be switched. One protagonist draws also the following picture:

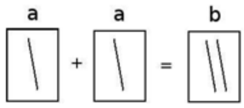


Figure 1

The students are asked to work with the program first, decide, if the table is correct and finish the dialogue (*dialogue B*). After further tasks with the robot a third imaginary dialogue task (*dialogue C*) is given. The students get the following picture:

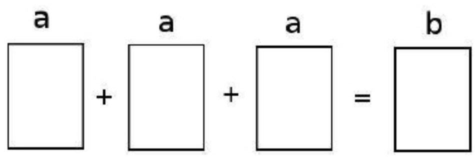
Programm

a ← ____
b ← ____
vorwärts(a)
vorwärts(a)
vorwärts(a)
umdrehen()
vorwärts(b)

Tabelle

a	b
1	3
2	6
3	9
4	12

Schachteldiagramm



Gleichung

$3 \cdot a = b$

Figure 2

Now the students are asked to think of an interesting program of a similar form, find the proper presentations like in Figure 2 and write an imaginary dialogue about it.

METHOD

The study was carried out in a class of a grammar school (Gymnasium) in Bremen, Germany, in 2008 with the above mentioned learning environment. The students wrote three different dialogues A, B and C. Dialogue A was written after the second lesson, dialogue B after three more days and dialogue C after about three weeks. The imaginary dialogues A and B were given as homework, dialogue C was written in the classroom. Since not all students did their homework or some let the protagonists talk about only non-mathematical tasks, for the analysis 16 A-dialogues, 15 B-dialogues and 22 C-dialogues could be used. For the qualitative analysis of the imaginary dialogues the framework of Sfard's theory of reification was used. The analysis was carried out in three steps:

13. examination by *four criteria*: recognised structures, occurring aspects of the notion of variable, phase in which the student is (i.e. interiorization, condensation, mixed form/indistinct, or reification), mentioned preset reification
14. creation of a *mind map* of the seen structures for each dialogue A, B, and C
15. creation of *tables* that includes the information of the mind maps and the phases

In order to examine by the four criteria, most dialogues were first transcribed and than interpreted in detail. The students' development was classified according to the phases according to these criteria:

- *interiorization*: the student can handle the program: processing the program, filling matchboxes with matches, etc.
- *condensation*: the student deals with variables as with objects but does not see them as objects, the input and output is more important than the process itself
- *mixed form/indistinct*: it cannot be decided if the student already reificated the notion of variable, variables are used in a tight relation to preset reifications
- *reification*: variables are seen as independent objects

FINDINGS

All imaginary dialogues mentioned here were written in German and translated by the author.

Mini-statistics

We can observe a shift of the students of this class from interiorization to reification as Sfard predicted. It must be mentioned that the tasks for the dialogues A, B and C were similar, but different. Thus, there is the possibility that the observed shift also depends on the different tasks. In the following table, the number of students in a certain phase of a certain dialogue is denoted:

	i	**c**	**m**	**r**	**Total**
A	11	2	2	1	16
B	5	5	3	2	15
C	4	5	6	7	22

Table 1: number of students in a certain phase

Structures recognised by the students

The structures that were recognised by the students are shown in the tables of the Figures 3 and 4. The tables should be read like a tree from left to right where each row is a branch. It is also listed which phase is assigned to the specific imaginary dialogue, in which the student recognised the structure. The letters i, c, m and r stand for the phases interiorization, condensation, mixed form/indistinct and reification. There are several crosses, if several students see the same structure. Some of the structures that can be seen as examples of *preliminary steps of reification* are discussed below. In the following, for example "Figure 3, structures in A, 7" refers to the seen structure in A written in row 7 which is here "segmentation of the distance – in segments a and b".

structures in A

		i	c	m	r
comparison of the number of steps		x			
independence of the notation	name of variable is free to choose	x			
segementation of the commands	search for a segementation	x			
	same amount of comands with a and b			x	
	with an example				x
segmentation of the distance	in steps without a and b	x			
	in segments a and b		x		
preform of abstraction	forward(a) and box content synonymical			x	
preform of substitution	preform of "a is 1"			x	
preform of reification	abstraction of matchboxes		x		
	paper commands "a's" and "b's"			x	
	relationship between the number of "a's" and "b's"			x	
	a and b as steps			x	
abstraction	a as a value in different contexts				x

structures in B

		i	c	m	r
correlation of the values in the table		x			
pre-understanding of equations	preform of symmetry in equations	x			
	correlation between program and substitution in equation	x			
	preform of the understanding of equations		x		
structure of explanation	abstract, concrete, example				x
correlation between term and distance			x		x
correlation between a and b	comparison of undefined step sizes		x		
	with the notion "a" and "b"			x	
	b is the double of a			xx	
abstraction	variables as independent objects				x

Figure 3: structures in A and B

structures in C

			i	c	m	r
1	commutativity of the commands				x	
2	different substitutions possible					x
3	structure in table	multiple addition		x		
4		as the times table			x	
5		proportion between a- and b-values			x	
6		correlation between values in the table			x	
7	variable as common name for something	“>a<” as name for a mathbox, column and field			x	
8		generic term for values, commands and distances			x	
9	correlation between a and b	as values				x
10		b is the double of a			x	xxx
11		it is “the same” with other values				x
12		how many of “a” yield “b”				x
13		concrete assignment unimportant			x	x
14	letters a and b	instead of a number of intersection points			x	
15		as a fixed unknown			x	
16		stand for a number of steps				x
17		stand for numbers in the language use		x		
18		stand for numbers		xx		x
19	step size as a scale					xx
20	preform of reification	“a)” and “b)” instead of concrete numbers in the language use		x		
21		notation “a=7” for values of the table			x	
22		a and b as generic term for values in the table			x	
23	understanding of equations	symmetry			x	
24		cancellation				x
25		equation as a summary of examples				x
26	expansion of the number of variables					x
27	abstraction	variables as independent objects				xxxx
28		terms as independent objects				x

Figure 4: structures in C

Independence of the notion

In the imaginary dialogue of a student (Figure 3, structures in A, 2) we can read that for him the name of the matchbox is free to choose. One of his protagonists explains:

> “You put arbitrarily many matches of the 16 and label the matchbox with a letter, let me say an example: “N”. You position the robot on the sea bottom and now you must give commands to the robot: for example: forward (for example N). Hence, he goes forward as much as you have put matches into the matchbox.”

The students writes “forward(for example N)” which shows that he points out that he could have chosen another name for the matchbox. If we transfer this to variables, we can call it an aspect of the independence of the name of variable. This aspect has its relevance, if we think about students who might know for example the binomial formulas with a and b, but have difficulties, when different variable names are used.

Name of variable as a generic term for multiple objects

Variables can simultaneously represent multiple values and can be abstracted from multiple real objects, like distances or the quantity of something. Hence, a preliminary step for this abstraction is to use different objects synonymously or to use a variable as a generic term for multiple objects. We can see the use of different objects synonymously in a dialogue by a student (Figure 3, structures in A, 8) who first wrote:

> “because (a) and (b) are most probable of different size.”

After this she inserted the words “forward” from above, such that the sentence looks like this:

> “because forward(a) and forward(b) are most probable of different size.”

We do not know, if she means by “(a)” the box content or a value of an abstract a, but we might consider that she uses the command “forward(a)” and whatever she thinks of as “(a)” synonymously.

The next step is to use a variable as a generic term for multiple objects as in the following dialogue (Figure 4, structures in C, 7). Here, the protagonists are named “S” and “D”.

S: Well, the table has two columns. A+b. As the two matchboxes. In >a< are two matches, and in b 8. In column >a< 2 are added in each row. In column >b< it is the same.

D: Like a times table? Where in each row it increases by 2 or 8 respectively?

S: Yes! Precisely. Now to the matchbox diagram. The field >a< stands for the number >2<. The field >b< stands for >8<. That way the diagram is eventually: 2+2+2+2=8.

When the student mentions her notation “>a<” the first time it means a matchbox. After this it is a column and the end a field which can be substituted. We can also observe that the student does not use the letter a without relating it to an object. It does not appear in a complete abstract manner.

A different student (Figure 4, structures in C, 8) uses variables as a generic term for commands,

> “We have the commands A, B, & turnaround.”

values,

> “But how do I know, what is the value of A & B?”

and distances:

> “If you go the distance a() + b(), then it makes no difference, if you go back a() + b() or b() + a().”

Talking about a and b as talking about objects

A student talks in his dialogue (Figure 3, structures in A, 11) about a and b as if they were objects. Possibly he thinks about the paper commands while talking about them.

> “If a is equal to 1 and b is equal to 2: First you must (you can) go with all a’s forward and with the half of the b’s backward and you are again on the same point.”

Since he says “with the half of the b’s”, the “b’s” are some kind of objects to him.

Correlation of different variables

Several students discuss the correlation between different variables (compare Figure 3, structures in B, 7-9 and Figure 4, structures in C, 9-13). One example is where the student recognises that b must be the double of a (Figure 3, structures in B, 9):

> “If the robot moves two half steps (a) and he must go back steps which are bigger, then b must have the double, thus an entire step.”

A different student formulates the correlation by fitting a number of a into b (Figure 4, structures in C, 12):

S2: Well, if a and b stand for the number of steps and you can turnaround only once, then you must find out how many of a yield b.

S1: Thus, if a is 1 and b 4 then one must find out how often a fits in b.

S2: Exactly!

What are a and b?

Some students discussed the topic of what the letters a and b are. Most often they used the words “stands for” instead of “is”. We find passages, all in dialogue C, saying for example that a or b stand for a number of steps (compare the preceding example), or for numbers (Figure 4, structures in C, 18):

2: Exactly and for the equation you must do this in a multiplication exercise.

1: Without numbers?

2: The letters stand for numbers, for example out of the table.

1: But there are multiple numbers. Which ones do I take?

2: That is easy. You can take every number you like. Just make sure that a has the double value.

SUMMARY

The analysis of the imaginary dialogues written by the students indicates the process from the phase interiorization, passing condensation to reification, as predicted by Sfard (1991). In the tables we see all structures that were recognised by the students. Among those structures we can also identify several preliminary steps toward a structural conception of the notion of variable: the independence of the notion, using the name of variable as a generic term for multiple objects, talking about variables as about objects, recognising correlations between different variables, and actually discussing what a letter stands for. Whether these preliminary steps eventually lead to a complete reification or not, we cannot predict. But we can observe that several students in dialogue A are tight to the preset reification of the notion of variable in form of the matchboxes or paper commands, while reading the dialogues B and C, the preset reifications disappear in many writings and the language use becomes more and more regular.

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