GROWING PATTERNS AS EXAMPLES FOR DEVELOPING A NEW VIEW ONTO ALGEBRA AND ARITHMETIC

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Sequences of growing patterns play an increasing role in the context of introducing terms. In this paper we reflect a new view onto the role of those particular visualisations for arithmetic and as well for algebra. By using a pupil's document we illustrate in this paper the theoretical framework of our concept.

Keywords: *representation/growing pattern, pre-algebra, children's interpretation, building structures and relations into diagrams*

1 Perspectives on the Mathematical Knowledge on the Way to Algebra

On their way from arithmetic to algebra, students have to develop a new awareness for the general, for the variation and the variable. At this period a new way of thinking, a new understanding of the previously acquired mathematical concepts, symbols and operations and thus a new interpretation of old knowledge becomes necessary. Students of elementary school become acquainted with equations in arithmetic lessons primarily in the context of calculating. In a special kind of lesson culture they learn more or less subconsciously that by dealing with equations they have to calculate the part on the left of the equal sign and after that to note the result on the right ("Task-Result-Interpretation"; Winter 1982). In many cases the equal sign is interpreted as a sign demanding to calculate. In many cases its function as a symbol of equality is not spoken about or used in every day arithmetic lessons. Such restriction in the interpretation, understanding and use of arithmetic terms and symbols is an obstacle not only for the later algebraic comprehension, but also for developing successful calculation strategies for the elementary arithmetical operations in the following school years.

Today algebra is seen as the lingua franca of higher mathematics (Hefendehl-Hebeker & Oldenburg 2008). However, algebra does not obtain the meaning and power of such a superior language if its status is restricted to the transformation and calculation of terms. Algebra has to be a "system characterised by indeterminacy of objects, an analytic nature of thinking and symbolic ways of designating objects" (Cooper & Warren 2008, 24). Therefore it is indispensable for the construction of algebraic comprehension not merely to calculate terms, but increasingly to see them in their structures, in order to understand formulae and principles. "The equation (or formula) must not be perceived as a sort of calculation shorthand note but rather as a type of scheme, which can in different ways be rearranged and be filled with concrete content" (Winter 1982, 210).

Various studies are concerned with the transition from arithmetic to algebra, which is accompanied by ruptures and discontinuities from the arithmetical to the algebraical view (cf. Bednarz & Janvier 1996). In our paper we focus not only on ruptures in the transition from one view (e. g. arithmetic, geometric) to another but also on reinterpretations and developments within one view in the context of growing patterns.

2 Growing Patterns and Mathematical Visualizations as Mediators between old and new Mathematical Knowledge

If the substance of algebra is seen in the way it represents the principles and structures of mathematics and not in terms of the "behaviours" of algebra (such as simplification and factorisation) (…) (cf. Cooper &Warren 2008, 24), then it is important for the introduction to algebra to make meaningful learning possible for the students, which at the same time constructs basic ideas that are sustainable in the long term. That means that such learning and exploring of algebraic ideas is always situated in the difficult balance between a rather empirical view on concrete objects and actions on the one hand and a certainly more challenging but in the long run necessary and profitable view on relations and structures on the other hand.

On their way to algebra it is necessary especially for young students to open a learning arrangement and an exploring field in which they can move between these poles of an empirical view on concrete objects and actions and a more abstract view on relations and structures. Structured mathematical visualization and growing patterns constitute such a learning environment, which merges those poles in a natural way.

Mathematical visualization and growing patterns - as a special type of mathematical visualization (for example to represent mathematical principles) - can mediate between the mathematical structure and the student's thinking because of their special "double nature" (they are on the one hand concrete objects, which can be dealt with, which can be pointed at and counted, which can be manipulatively changed, and at the same time they are symbolic representatives of abstract mathematical ideas).

Mathematical visualizations and growing patterns are well-known to elementary and secondary school children from their daily mathematics classes. Geometrical patterns, which must be interpreted arithmetically, are used in class for various purposes. Steinweg (2002) notes that in text books dot patterns appear to practice calculating skills and thus function as visualizations, while sequences of dot patterns are to be explored as a separate and independent subject (cf. Steinweg 2002, 129-151). It is obvious that in everyday mathematics lessons dot patterns have predominantly the function of a methodological-didactical aid. Here is a parallel to the restricted view on equations and the equal sign mentioned above. Only in rare and isolated instances the structures incorporated in mathematical visualizations and growing patterns as well as equations are being purposefully explored and mentioned by the children. Against this backdrop Schwank and Novinska (2008) complain that didactic materials must be rescued from their shadow existence as mere aids and acquire a role as playing fields, in which genuine thinking processes can develop. Central questions such as "How many" and "if … then" in dealing with this type of materials open a smooth transition to algebraic thinking - at first based on representations which become accessible through interaction, speech and graphics (cf. Schwank und Novinska 2007, 121).

3 Features in the exploration of growing patterns on the way to Algebra

If sequences of patterns support this new view – not only to figure out arithmetic terms, but to notice the underlying structure, transpose, re-organize and reinterpret them in a positive manner, then the following five aspects seem to be of particular importance. These categories were developed by connecting first results of a case study in progress (cf. Böttinger 2007) and the results of a completed case study (cf. Söbbeke 2005). In order to interpret representations more and more in the function as a representative of relations and structures and thus to focus on the abstract and generalizable "pre-algebraic aspects" it was necessary to connect in this paper two analysis instruments and to use them both to analyse the interpretations of student Ron. In order to describe the interplay between the geometrical, the arithmetical and the algebraic view it was necessary to develop an analysis instrument (cf. Böttinger 2007) by analysing the transcriptions of the interviews. While the analysis instrument "*Four levels of VISA*" (cf. 3.5) combines various aspects of structuring and interpreting a visual representation, in the analysis instrument "*Model of categories*" (cf. 3.1-3.4) these particular features were separated, adapted to sequences of growing patterns and the gradation was worked out by analysing the interviews.

Model of categories

3.1 Structuring a single pattern

- No subdivision
- Not intended subdivision
- Intended substructure
- Examination of several substructures

3.2 Flexibility

- No change of view
- Change of view without new structuring
- Change of view with new structuring

3.3 Relation geometry - arithmetic

- Pure geometric view
- Pure arithmetic view
- Relation is established by a number of points
- Additive relation
- More complex structural relation

3.4 Relations within the series

- No relations

Fig. 1

The aim of the first case study (cf. Böttinger 2007) is to describe more precise on the basis of 20 interviews with 4th-grade children, in which way children translate geometrical relations in a sequence of growing patterns into arithmetic terms and in which way generalisations are carried out. The hypothesis is that there is no direct way from the geometrical representation to an arithmetical one and finally to an algebraic view. Instead there will be an interplay between these different views. In order to describe this interplay an analysis instrument (cf. Model of categories, Fig. 1; cf. Böttinger 2007) had been developed on the basis of the interview data.

3.1 Features concerning the structuring of single patterns

In order to continue and examine the sequence a single pattern has to be structured. A subdivision can correspond to the *intended* structure of that person who composed the assignment on the one hand. On the other hand it can be an *individual* one, which does not correspond to a priori intended ideas.

3.2 Features concerning the flexible re-organisation of single patterns

In order to generate the idea of an equation one must be aware of different perceptions of a single pattern in the sequence. The aim is to identify the equality of arithmetic or algebraic expressions on the basis of the corresponding underlying geometric structure. Closely connected to this view is that transformations of equations correspond to changing the view on geometric structures. In analysing the children's interpretation one has to consider the flexibility during the process of work. It is essential to draw a comparison to the preceding interpretations of the child and to verify, to what extent a change of view occurs. This can be without new arrangement within the single pattern, e. g. when the number of dots is solely calculated in different ways. On the other hand a proper structural reinterpretation and re-organization exists, when the child builds fundamentally different structures into the diagram as in the step before.

3.3 Features concerning the relation between geometric and arithmetic structures.

Within her study Steinweg (2002) has worked out by what criteria children continue sequences of growing patterns. She distinguishes between a continuation by a figural aspect or by an arithmetical aspect. The figural aspect is concerned with the location of the dots and the external form built by the dots and the arithmetical aspect with the total number of dots in a single pattern. Steinweg accents that only the combination of figural and cardinal aspects lead to the intended continuation. Besides the distinction between a pure geometric view and a pure arithmetic view one has to regard the possible connections between both parameters. This can happen by a number of points, but also additive or more complex relations (e. g. multiplicative ones) can be identified.

3.4 Features concerning relations within the patterns

If sequences of patterns are used for algebraic investigation, one has to distinguish two totally different views. While the explicit formula uses the inner structure of a single figure, which must be suitable for all following figures, a recursive formula uses relations between consecutive patterns (cf. Carraher & Schliemann 2006). With the help of recursive formulas it is described, how the number of points changes from one pattern to the next. This view can be a great obstruction if the number of points in the 10th pattern is to be figured out. The student has to calculate step by step each particular pattern and simultaneously he has to control the number of steps. In addition, the indication of the recursion alone is incomplete to describe the building principle, because an initial condition is needed (Carraher, Schliemann, 2007, 697). From the union of both perspectives interesting formulas can arise. Furthermore a dependence e. g. between the width and the height of a figure leads to dependent variables that describe exactly these features of the pattern.

3.5 Features concerning the interpretation visualizations (VISA)

In the second study (cf. Söbbeke) on the basis of detailed case studies with children of elementary school four levels of children's ability to build structures into mathematical representation (ViSA) had been distinguished. The underlying assumption of the study was that learning of mathematics has to be understood as a process of the children's more and more differentiated way of understanding and interpreting abstract patterns and structures (cf. Steinbring 2005). Visual representations are a tool to represent abstract mathematical concepts as well as to think about them or to talk about these with children. Growing patterns, as a special type of visualization, are often used to represent structures and relations in order to understand elementary mathematical principles (for example triangle numbers as an example to explore sums of odd numbers, etc.). The important information is not based in the concrete features of the material, but on the abstract, the relations and the structures within the material. Thus, what is decisive for a mathematical cognition in the figures is not the colours or the number of points; it is rather the function, which the concrete feature of the material takes for something. This means, the structure of the representation makes the understanding of a mathematical legality possible, but it cannot be read directly or immediately perceived with one's senses; it must be actively *interpreted* into the representation. In the empirical study (cf. Söbbeke 2005) it had been analyzed in how far the learning child succeeded in building such abstract structures and relations into the diagram. On this basis *Four Levels of Visual Structurizing Ability* had been distinguished. These four levels characterize the children's interpretations in a spread of concrete and empirical interpretations on the one hand (cf. level one, left pole of the spread) and relational und structural interpretations on the other hand (cf. level IV, right pole of the spread) (cf. Söbbeke 2005).

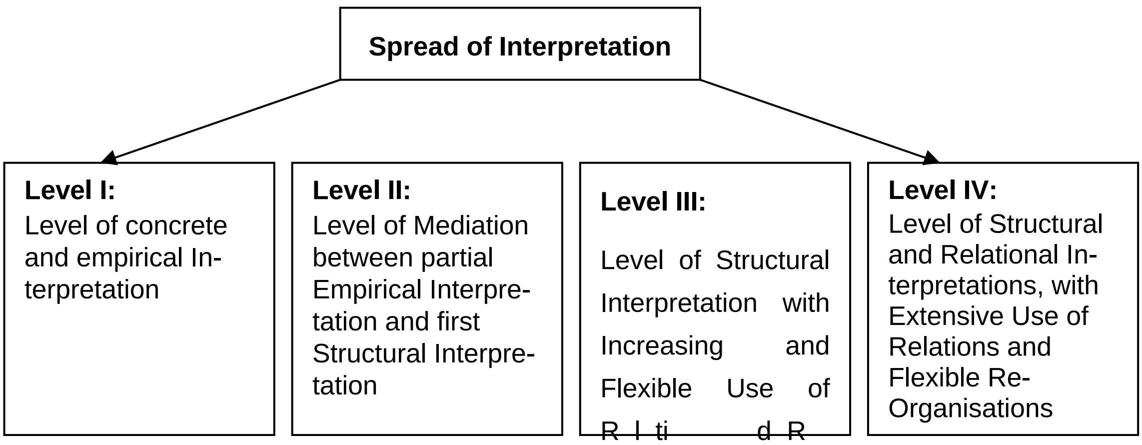

Fig. 2: Four Levels of Visual Structurizing Ability (ViSA).

4 Using Growing patterns to Support Students' Way to Algebra

- Ron on his Way to an Abstract and Multi-relational View of the Pattern -

The following examples are to show how the student Ron (4th grade) deals with the challenge to use growing patterns and to interpret them more and more in the function as a representative of relations and structures and thus to focus on the abstract and generalizable "pre-algebraic aspects" in the representation. For this we connect in this paper for the first time two different analysis instruments and use them both to analyse the interpretations of student Ron. The scenes presented are not to deliver a thorough methodical analysis. Instead the analyse in this paper can be seen as a first approximation to grasp and to describe the fundamental elements of the children's way to algebra by using growing patterns, which had been pointed out in 3.1 to 3.5. The analysis is not extracted from a finalized study, but it is an example of a new approach to the theme, to the underling structure and to a more detailed view onto sequences of growing patterns. In the *first* part of the different interview phases (*beginning*, *in course*, *end*) the elements of the aspects 3.1 to 3.4 had been described with the instrument "*Model of Categories of Changing Modes of Representation*" (see fig. 1). In the second part of the interview phases Ron`s interpretations had been assigned to the "*Four Levels of Visual Structurizing Ability (ViSA)*" (cf. 3.5, fig. 2).

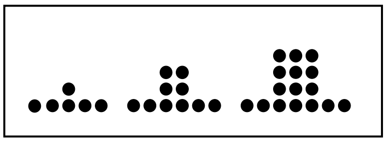

Fig.

At the ***beginning of this interview*** scene, Ron is presented the first three figures of the growing pattern and he is asked to describe what he can see (Fig. 3)

Ron	(16 seconds break) Mhm. (5 seconds break) Mhm (laughing). (10 seconds break) There at the bottom there is always one more (he points to lowest the row of dots in the first, the second, the third pattern). Five, six, seven (he touches the lower part of the first, the second, the third pattern) This next row. There are always some more.
Ron	Here there are, there are three more (he touches with his pencil the upper part of the second pattern). Here there are five more (he touches the third pattern with his pencil). (..) Since those I can remove (he puts his forefinger onto the third pattern), I can take away, because these are still there (he touches with the pencil the second pattern, afterwards he points to the not covered points of the third pattern). (…) Three, five. (6 sec. break, he moves the left forefinger to both left points of the bottom row in the third pattern, stops for a moment and takes the finger away from the paper) Mhm.

After 30 seconds reflecting about this task Ron starts to compare the three patterns. He structures the three figures into two parts: the horizontal row of dots at the bottom of the pattern and the field of dots placed at the top. In his first approach Ron does not pay attention to the part at the top of the pattern, but describes that the row of dots increases from one figure to the next and names the numbers "five", "six", "seven". In the analysis, considering the aspects 3.1 - 3.4, Ron shows that at the beginning of the interview he had developed an idea of the structure of the lower part of the pattern. Ron determines the number of dots in this part of the pattern and finds a recursive relation between the figures: "five, six, seven. … There are always some more". He builds a relation between the geometrical figure and the arithmetic in finding out the number of dots in the lower part of the pattern. Ron does not make it explicit, but his repetition of the number series can be seen as an indication that the number series and in association the structure of the lower part could always go on in this way. Against the background of his first interpretations, the number series can be understood as a preliminary stage of a recursive building principle: from one figure to the next you always have to add one point. Already at this early stage of the examination of the pattern you can see a first level of generalization.

After reflecting about 30 seconds about the upper part of the figures, Ron starts to describe the increasing of dots from the second to the third pattern. Ron structures the upper part into two groups: on the one hand, he sees the group of dots that had been seen in the previous figure, and on the other hand those, that had been added in the new following one: "Since those I can remove (he puts his forefinger onto the third pattern), I can take away, because these are still there". In his approaches to understand the structure of the upper part, Ron shows a first re-organization of the pattern. He does not analyse the two parts of the figures separate, but tries to understand in what way the first pattern could be identified in the second one and the second one in the third one. In the meantime he points with his finger on special areas of the lower part of the pattern, which he had described before in his first analysis of the pattern (the vertical row of dots). The numbers "three" and "five", he denominates, correspond presumably to the numbers of dots in the upper part of the pattern, marked for a better understanding here in white colour (see Fig. 4). Ron uses the numbers of dots and structures and builds first elemental relations between the different patterns into the diagram (he covers with his hands parts of the previous patterns etc.). As a kind of arithmetical information, Ron determines the number of dots in the particular figures. At the beginning of this interview the analyse shows a first recursive view on the pattern; however, Ron does not generalize this recursive view further, but applies it solely to the partly figures.

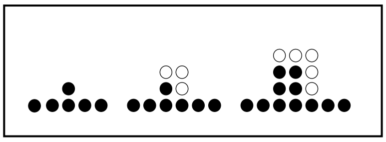

Fig.

Altogether Ron's interpretation of the pattern could be attributed to the 2nd level of *ViSA* (cf. 3.5). The child moves away from the concrete aspects of the representation (numbers of dots) and focuses increasingly on abstract relations and structures (two parts of the pattern; angle-structure of the added dots in the new figure). But the elements of interpretation often stand isolated as concrete objects, without building *rich* relations between them (for example relations between the structure of the part at the bottom and at the top of the pattern; relations between the different figures). Sometimes only sections of the diagram are taken into consideration. In interpretations on this level there is a typical mediation between partial empirical interpretations and first structural interpretations. But often the children's interpretations are still inflexible and they do not look at the representation as a multi-faceted structural diagram.

In the ***course of the interview***, Ron notices that he had always forgotten to pay attention to one point in the lower part of the pattern, while analysing the increasing of the patterns:

Ron	O no, I didn’t count those (he taps the bottom row of points in the second pattern). That means, there would be four new ones (he touches the second pattern) and here there would be six new ones (he touches the third pattern).

After that Ron constructs a recursive geometrical building principle into the growing pattern and tries to translate it into an arithmetical building principle. In the course of the interview Ron has been asked to find an arithmetical task, which corresponds to the given pattern. For this he finds calculation tasks, which correspond with the result (“16”) to the number of given dots in the third pattern. Ron interpretes and explains the proposal of the potential task “3·3+7“, given by the interviewer, solely against the background of the calculating result und does not indicate a relation between the structure of the arithmetic task and the structur of the pattern. For Ron it is crucial that the number of the dots corresponds with the result of the calucating task.

He finds the calculating task “10·3+4” in the *5^{th} pattern*, that can be seen als an analogon to the proposal of the interviewer in the 3^{th} pattern (“3·3+7“). Presumably Ron takes the aspect “number of dots” on and tries to build an analog construction (second factor of multiplication is “3” or a task with a multiplative term) like in the task of the interviewer. Finally, at the end of the interview Ron is asked to determine the number of dots in the sixth pattern. He starts to draw the *sixth* pattern onto the interview sheet.

Ron	Five, six, seven, eight, nine, ten (in the meantime he draws 10 points in a row beside the 5^{th} pattern). The first new points, this would be here, one, two, three, four, five, six (while speaking he draws a row of 6 points directly over the row of 10 points; cf. Fig 5). One, two, three, four five, six, (he draws - always counting until six - four further rows consisting of 6 points). One, two, three, four, five (with his pencil he touches the dots of the first column, but omits the corresponding dot at the bottom). Now another one (he draws a further row consisting of 6 points over the 5^{th} 6-row). Six. Ready. One, two, three, four, five, six, seven (he touches the dots of the first column including the corresponding dot at the bottom), seven. Six times seven is 42 plus four, 44.

At first Ron divides the 6^{th} pattern into two parts: At the bottom he builds a long horizontal row consisting of 10 dots, in the upper part a rectangular field consisting of six rows of six dots. Subsequently he carries out an interesting new interpretation of the pattern. He structures it into a rectangle of seven rows of six dots, which reaches into the horizontal line at the bottom. Beside this 6x7-field of dots he regards two points at the left and two at the right-hand side – at whole 4

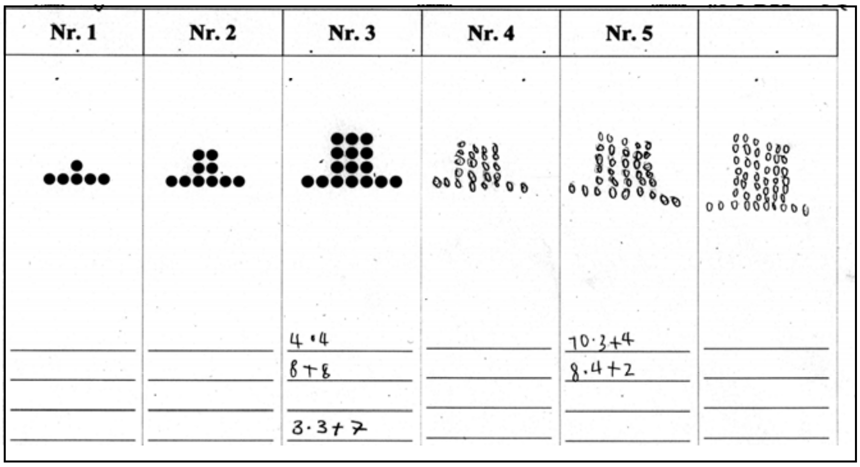

Fig. 5

points. To figure out the total amount of numbers in the 6th pattern Ron uses for the first time the inner structure of a single pattern. In comparison to his proceeding before this represents a change of view in connection with a new structuring. The relation between the geometric arrangement of the dots is no longer determined by the cardinality of a set of points but by a complex structural relation – namely a multiplicative one. By that Ron changes from his formerly recursive view onto the sequence and considers a single pattern in an explicit manner. The structure he uses is an intended one and in principle it is applicable to all patterns. But at this stage of the interview Ron does not express or indicate this generalisation.

Ron's interpretation of the pattern could be attributed to the *third level of ViSA*. In interpretations on this level *intended structures* and *relations* can be identified (for example relation between the part of the bottom and at the top of the figure; field of 6x7 dots; constancy of 4 dots in the part at the bottom). On this occasion different and multi-faceted aspects of the representation are recognised. In comparison to level II, the structures are *manifoldly coordinated* and more *flexibly re-organised*. The structures are no longer isolated, but seen as part of the whole and separated and put together in a structural way. You always find the use of *structural relations*, *coordination* and *re-organisation* of elements. In all, this level III of ViSA can be characterized by the combination of building structures with the increasing use of relations and re-organisations.

5 Conclusion

For a fundamental pre-algebraic comprehension it is indispensable to focus on structures, on the abstract and the general, right from the start of children's mathematics education. In this paper, growing patterns have been discussed and analysed as exploring fields on the way to focus on structures and relations. Structure sense seems to be a fundamental requirement to interpret sequences of growing patterns in an algebraical manner. Both analysing instruments examine in different ways how young children deal with the challenge to interpret this special visualization in a more structured, generalized and elementary "algebraic" way.

The examples of Ron indicate that this kind of structuring, translation and generalization does not take place in a direct and straight way. The child can partly understand the geometrical structures, translate them into arithmetic ones. It can change the view back to the geometric pattern and re-organise and re-structure the diagram. It seems that generalization is not always the "end" of this process; in fact ideas of generalization can be developed before comprehending the whole structure of the patterns.

An analysis of selected parts of the interview shows that in the *process of the examination* and the *interaction* between the student and the interviewer the child gradually develops a more differentiated, relational and generalized view onto the used diagrams, which can be described in detail by the system of categories and in a more summarising manner by means of ViSA (see e.g. the development of Ron's interpretation from level II to level III). Altogether the excerpts of the interview with Ron serve to demonstrate the change in children's interpretations in a exemplary way and to accompany and better understand their way – to an increasingly open, general and flexible view onto relations and structures within diagrams.

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