COMMUNICATING A SENSE OF ELEMENTARY ALGEBRA TO PRESERVICE PRIMARY TEACHERS

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This article reports on a university course for preservice primary teachers on 'patterns and structures in primary school to prepare algebraic thinking'. We believe, if arithmetic is taught with an algebraic awareness, e.g. looking for patterns within arithmetic problems, algebraic thinking could be enhanced in primary school and the 'cognitive gap' between arithmetic and algebra would be reduced. In order to teach with an algebraic awareness the teachers must have developed such awareness themselves. We present the design of a course with which we contributed to this. The course serves us as a pilot experience for gaining hypotheses on the needs of teacher students and on good teaching interventions. We conclude the article with research questions in this field of teacher education.

THEORETICAL FRAMEWORK AND FOCUS OF THE PAPER

It is well known that there are many-facetted difficulties in learning algebra (see for example the contributions in Bednarz et al., 1996). Also the working group on algebraic thinking of CERME 5 has considered many features constituting elementary algebra and problems of learners. Some of the contributions are concerned with problems of constructing new mathematical objects (as formal or as abstract, cognitive objects) when dealing with algebraic expressions (e.g. Dörfler, 2007; Fischer, 2007a; Lagrange, 2007). Others point to students' often limited or inappropriate ways of interpreting symbolic arithmetic or algebraic expressions (e.g. Alexandrou-Leonidou and Philippou, 2007; Molina et al., 2007; Papaieronymou, 2007). What do these many-faced difficulties have in common with the learning of algebra? The working group agreed on one central theme of algebra underlying all other aspects discussed: 'expressing generality' (Puig et al., 2007). However, students often do not experience this feature in their algebra classes.

One reason for these difficulties is the so-called 'cognitive gap' between arithmetic and algebra. Herscovics and Linchevski (1994) highlight some aspects of it. Features like the manipulation of variables occurring twice or more in a formal expression demand truly new cognitive abilities or constructions as compared to an arithmetic viewpoint. Similarly, they suggest a new viewpoint is required to comprehend formal arithmetic expressions as entities in their own right, or to look for patterns and structures in arithmetic problems. As a consequence of the observed gap, students have to cope with several changes to their habit of solving problems, their ways of interpreting signs, their ideas on what mathematics is about.

In this article we propose that some of the features of this gap between arithmetic and algebra are not so much due to the given characteristics of the two areas of mathematics, but to a tradition of teaching arithmetic common to many countries. This tradition focuses on ways of interpreting arithmetic expressions and treating them, which cannot be extended to the algebraic sign system. What is more, the tradition of teaching arithmetic narrows the focus of mathematics to calculations and results, giving little scope for the search for general patterns and the discussion of structures. Things can be done differently. The way formal expressions are interpreted in algebra can also be used for interpreting arithmetic expressions. For example the expression 3+4 need not only be understood as a description of an activity but also as a sign for a number. Many other characteristics of algebra could effectively first be established within arithmetic contexts. A lot of research exists on including algebraic activities in mathematical learning environments for primary school children. For example several studies (e.g. Carraher et al., 2008; Fischer, 2007b; Söbbeke, 2005) report on the understanding of arithmetic or geometric patterns by young children who are not yet familiar with the conventions of the formal algebraic sign system. When they become familiar with activities of this kind in primary school children might be better prepared for the step to algebra.

But how can primary school teachers be persuaded to teach these issues? For a pilot experience we designed a university course aimed at preparing (future) primary teachers for integrating algebraic aspects in the math classes. In this article we will explain our grounds for the design of the course and report on our experiences. At the end we suggest ideas for further research to help evaluate the course and develop it further.

A central issue for our course was how to persuade primary school teachers to engage in algebraic ideas. Understandably, primary school teachers tend to focus on the goals set by curricula for the first school years. Often they are not aware of the consequences of their attitudes for the children's learning of further mathematical concepts. Moreover, many of them do not see a connection between learning mathematics in primary school and algebra in secondary schools. And those who do are not aware of different ways of dealing with arithmetic. Therefore, we consider it a necessary prerequisite to help (future) primary teachers look at the mathematics in primary school from an algebraic perspective and to show them how they can integrate pre-algebraic thinking without loosing track of their primary goals.

Mason (2007) gives some ideas on how teachers can learn to deal with the subject of expressing generality. One central point is the highlighting of typical mathematical processes involved in the search for general patterns and in their representation and use. This is one important connection between the general goals of mathematics and our specific interest in advancing algebraic thinking in primary school. We recognised different though interwoven aspects of 'algebraic awareness':

- *Experience with problem solving activities*, e.g. analysing and describing patterns and structures, continuing patterns, using structures for calculations and problem solving,

- *Knowledge of different mode of representations and structures* of problems, solution methods and solutions,
- The *disposition* to look for patterns and structures in arithmetic problems and to argue with them and to perceive arithmetic expressions as processes and as objects.

All of these aspects can be provoked within arithmetic and geometric contexts in primary school (grade 1 to 4).

CONCEPTUAL DESIGN OF THE COURSE

In the course we had four main goals:

- The students experience algebraic thinking within arithmetic and geometric contexts. They are encouraged by personal success and gain a broadened view on mathematical tasks.
- The students understand challenges of (pre)algebraic thinking as part of mathematics fitting in the goals of primary school.
- The students design and analyse mathematical problems concerning arithmetic or geometric patterns in a context of primary school either within a case study or while analysing schoolbooks.
- The students reflect upon learning mathematics themselves and by children.

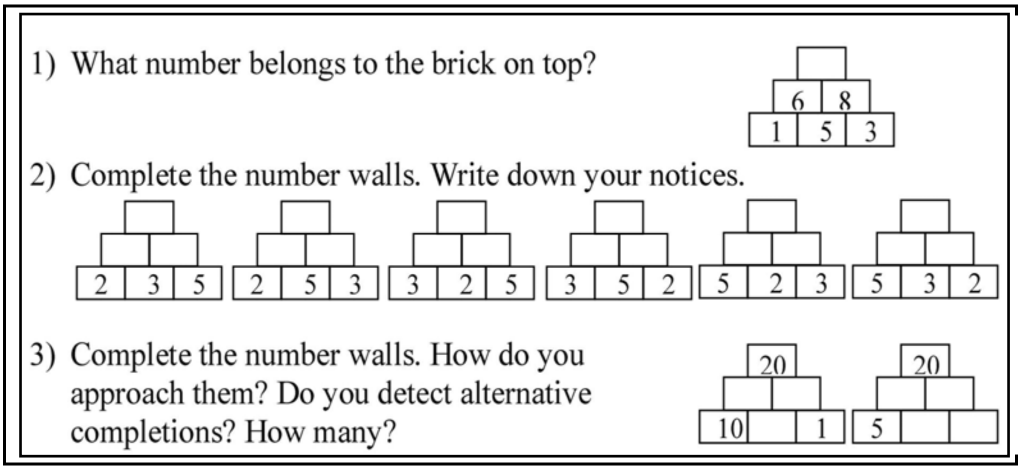
Organisational frame

The class met three hours each week for one semester (14 weeks) and was open for advanced students who had already taken some mathematics and mathematics education for primary school. Twenty three students attended the course. To obtain credits each student had either to undertake and write a report of a short empirical study with one or more children, or write a theoretical theses comparing two series of schoolbooks.

Progression

1. Introducing the course subject

During the first weeks of the course the students were presented with mathematical problems, which comprised different aspects of algebra and algebraic thinking. With this activate approach the students experienced algebraic thinking instead of dealing with a theoretical definition. We chose problems which highlighted characteristic aspects of algebraic thinking. Quite a number of these problems dealt with the discovery and expression of patterns. The students had to solve them with their preferred problem solving strategy and with at least one strategy that children in primary school might use. The class reflected upon the solutions, the solution methods and different ways of presenting both. Furthermore, problem solving strategies were elaborated and differences were highlighted between problems which appeared to be very similar at first sight but turned out to have very different algebraic potentials.



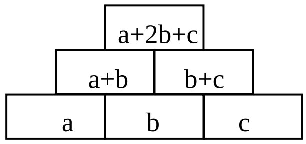


figure 1 **figure 2**

Figure 1 shows problems from a worksheet on “number walls”. Three-layer number walls involving additive structures within integers are an often used format in German school books. They are constructed as indicated in figure 2 (where a, b, and c are integers).

The first task on the worksheet presents a typical arithmetic task: the sum of integers has to be calculated. Note, however, that if used to *introduce* number walls, this already demands some degree of structural analysis. The second task also starts of with the calculation of sums. But the request to write down observations leads to a closer examination; the different walls have to be compared. Describing differences and commonalities of the six walls with the same integers in the bottom bricks demands a careful study of the walls. Verbalising the observation and explaining the findings helps the discovery of a mathematical pattern. Finally, the number walls of the third task cannot be worked out in the same straightforward way. They present disconnected problems (one of them is not solvable within integers) which can be tackled in different ways. Asking for the approach implies an explicit reflection on it; asking for other solutions and for the number of other solutions guides students towards a structural approach to the task.

Other problems given to the students offer different views of symbolical terms like the equal sign and expressions like the sums of two numbers. Given “3+4=”, say, whereas one view sees the equal sign as an instruction to calculate (3+4 adds up to 7), another promotes the view of the equal sign as a balance and of the sum as being a number (3+4 is the same number as 2+5). Cognitively the latter demands a view of an arithmetical expression as a number as well as a process (cf. Gray and Tall, 1994). Furthermore, the students were given problems on number sequences, geometric visualisations of such, arithmetic laws and (dis)connected arithmetic word problems.

Although the problems were basically taken from German schoolbooks for classes 1 to 4, the students had numerous difficulties solving them. Many of them made very formal use of variables, often with little or no understanding of the meaning. This caused mistakes on the one hand and impeded discussion of mathematical relations on the other hand. Moreover, the students frequently had difficulties to think of strategies without using variables. Often they thought of only one alternative strategy: systematic trial and improvement. Yet, they did not always acknowledge this as a valuable mathematical strategy.

Working on the given problems, the students were surprised by their experiences:

8. There are mathematical tasks with different ways of solving them, some problems can even have different solutions.
9. Strategies can be found which do not involve the formal algebraic sign system are possible. But to find such strategies requires insight into the structure.
10. The inherent structure of similar looking problems can be very different.
11. These problems offer challenges on different levels. Some of these challenges are revealed to the students only when working on them.

These experiences were facilitated by questions attached to the mathematical problems, which emphasised mathematical activities like visualising, comparing and arguing.

Besides solving the problems the students reflected upon the mathematical activities required by the children. Through this, we raised ideas of what algebraic thinking is about.

We concluded the introductory unit by taking a more theoretical standpoint. In class we discussed the paper of Lorenz (2006) on possibilities and challenges in using geometric representations of arithmetic patterns for illuminating the structure and solving problems about them. The claims of the text could well be investigated through some of the examples the class had worked on in the previous weeks.

The class then developed a notion of ‘good’ mathematical problems in general and in respect to algebraic thinking. The class agreed on the following features to constitute ‘good’ problems:

A ‘good’ mathematical problem must be

9. open to different approaches or different solutions,
10. given with a mathematical goal,
11. easy enough for every child in class to start solving the problem and to obtain a (partial) result, but also
12. challenging even for high achieving children.

The feature specifically relevant for the course is the encouragement of algebraic thinking. We listed the following characteristics of algebraic thought which can be found within arithmetic or geometric contexts:

- unknowns not only at the end of an expression,
- equal sign as balance sign,
- arithmetic expressions as representations of numbers,
- describing patterns,
- calculating big numbers effectively using structures instead of extensive calculations.

These criteria are neither original or exhaustive. But they reflect the views the students had developed at this point on the course and used as basis for their own work. Throughout the rest of the course these criteria served as an orientation for the students when developing and evaluating mathematical problems for primary school.

2. Preparing and realizing the individual projects

The students then started with their own projects. Seven carried out a case study with a child in primary school. Each of them prepared a short sequence of problems he or she was going to use in the interview. This sequence had to be analysed with respect to its algebraic potential. There was opportunity in class to have these sequences discussed in small groups and to work them through before they were used in the interviews.

After the interviews were accomplished the students had to transcribe interesting parts and analyse the children's performance. The students in Frankfurt have plenty of experience with carrying out interviews and analysing them with respect to interaction. Therefore we decided not to elaborate on these issues. Nevertheless we devoted one lesson to tools for analysing transcripts. We focused on gaining mathematical knowledge through working on representations. For this we read a paper on the epistemological triangle of Steinbring (2000). In this text two analyses are presented in which students explain and develop ideas on a mathematical problem. However this text turned out to be very difficult. It is too theory laden for our students to enable them to extract general principles and apply them for their own analyses.

Students who aimed for a theoretical thesis each had to analyse two series of schoolbooks for classes 1 to 4. Each student had to select two formats of problems like a sequence of problems with a common pattern or number walls recurring in his or her schoolbooks in different classes. He or she had to give an analysis of these formats pointing to their algebraic potential. On the ground of this analysis he or she had to evaluate the way the schoolbook makes use of these formats and compare the two series of schoolbooks. The students of this group, too, were given the opportunity to have some examples from their schoolbooks discussed in class. In addition, throughout the whole course such formats served as examples for different aspects.

The individual projects were mainly worked on at home. Meanwhile, we were able to introduce several theoretical articles on mathematics education which discuss issues related to our subject. Our main focus was to interrelate educational theories with the students' own mathematical activities as well as with their design and analysis of problems. Through this, we also deepened the students' algebraic understanding.

We covered topics like learning, practising and problem categories. In particular, we compared learning mathematics via instruction to learning via discovery (cp. Wittmann, 1994) and related the findings to previous class sessions. Practising – not only algorithms of calculation but also mathematical processes like problem solving, representing mathematical ideas, argumentation – was connected to the different learning theories (cp. Winter, 1984) and discussed for one specific problem. The task of determining whether problems are open (for different solutions and solution methods) informative (regarding the learner's thinking) and process-oriented (which means, if they support mathematical activities like discovering, arguing and further elaborations; Sundermann and Selter, 2006), leads to reflecting on problems, varying and exploring them.

These articles addressed general principles of teaching mathematics in primary school. We found plenty of opportunities to interpret and understand them in respect to our subject of inducing algebraic thinking. Thus this subject appeared in the general context of teaching mathematics in primary school not as an exotic theme but as one way of complying with these general goals that are commonly shared.

3. Presenting the students' projects

In the last unit of the course the students presented some of their results. Those writing a theoretical thesis chose examples of their analytical work and some theoretical aspects related to it. Those doing an empirical analyses presented crucial aspects of their interview analyses. All of them were asked to look for ways of presentation that would actively involve the class.

The students who analysed schoolbooks had to think of criteria for their analysis first. It turned out that they used the criteria listed in the introduction only as a starting point. In order to build their criteria most of them chose one or more topics on learning mathematics we discussed during the second part of the course. It is pleasant to see that they altogether made careful analyses covering important aspects of algebraic thinking which proved a good insight into the formats.

For example one student gave an overview on which pages the formats occur in the schoolbooks before she went into quantitative and qualitative analyses. She did not only list the pages but stated the type of task linked to it, like discussing calculation rules, completing the format and comparing numbers of neighboured formats. This affected her quantitative analysis: She put the frequency of a format into perspective with the aligned task. While she noted that in one book the format was used more often she also claimed that a lot of the tasks merely practise calculating.

At the beginning of the term another student commented on a schoolbook she had seen in use in primary school. She reported that the school children would love to work on the book and do their work autonomously. Her submitted analysis of this schoolbook shows that she gained a broadened view on mathematics teaching. She stated that this particular schoolbook is based on a theory of mathematics education of tiny steps but little structural understanding of mathematics problems.

CONCLUSION AND FUTURE PROSPECTIVES

Overall we are satisfied with this course since we met our goals for most part. The students gained (more) competencies solving mathematical problems with an algebraic notion. They intend to integrate (pre-)algebraic thinking in their mathematics classes through designing adequate mathematical tasks and an appropriate attitude. They gained competencies in judging maths problems in school books and their own, as well as reflecting on their interventions. Our evaluation corresponds well with the students' feedback.

It turned out that the aspects of algebraic thinking were best understood when they were directly linked to their own experiences – and more than once – and reflected upon afterwards. For example the students had to solve a variety of problems with patterns during the first sessions which were originally designed for primary school. We reflected upon them: The students had to present their results, find different solution methods, vary the tasks, compare it with other tasks, etc. The attitude to look for patterns became an important issue for the group and the focus on patterns can be traced to the students' projects. In contrast some algebraic characteristics were not understood quite as well, like the notion of the equal sign as a balance sign. This is perhaps because we did not mention those characteristics quite as often, or because we looked at them from a more theoretical perspective.

We believe that it was not only the students who learnt a lot about (pre-)algebraic thinking: we also benefited from this course. We learnt something about the thinking of university students, gained perspectives on teaching them and at the same time got deeper insight of the potential of mathematical tasks for teaching algebraic thinking.

This teaching experience serves as a pilot study for us. On the basis of this experience we see several research questions that would be worth following up.

- The course seems to indicate that student teachers do need help to get an algebraic awareness, even though they have used much algebra in their own time at school. A quantitative empirical study of teachers' performances in observing patterns and structures in geometric or arithmetic contexts should give hard evidence on this issue. One could also investigate how, during a course like ours, students' ideas about arithmetic lessons change.

- We do not know very much about the inner representations student teachers have of principles of algebraic notation and algebraic argumentation. A qualitative empirical investigation on this issue might help us to better understand some of the underlying difficulties. In connection with this, the effects of some of the principles we applied during the course should be evaluated by empirical studies. The results of these studies might inform the development of curricula for teacher education.
- An underlying assumption of our course is that children who work on describing and using patterns in the context of arithmetic problems will be better prepared for algebra than students who only do calculations in their arithmetic classes in primary school. This conforms with theoretical positions on the nature of algebraic thinking in scientific literature. However, more empirical evidence is needed to investigate this claim.

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